University of Saskatchewan
Department of Mathematics and Statistics

Numerical Analysis I
(MATH 211)
Instructor: Dr. Raymond J. Spiteri

ASSIGNMENT 04
Due: 8:30 a.m. Tuesday, March 19, 2013

1. [**15 marks**] Prove **carefully** that the function $f(x) = x^3 + 4x^2 - 10$ has a root in the interval $[1, 2]$. Starting from this interval, perform 5 iterations of the bisection method to approximate the root.

   Show your results in a table with entries for the iteration number n, current left endpoint a_n, current right endpoint b_n, current midpoint c_n, and function value $f(c_n)$.

   (**Hint:** The first iteration is $c_1 = 1.5$.)

   Give an error bound for your approximation. How many iterations will guarantee an accuracy of 10^{-5}?

2. [**15 marks**] The probability that the Yankees will win a game is

$$P(p) = \frac{1+p}{2}\left(\frac{p}{1-p+p^2}\right)^9,$$

   where p is the probability that their star pitcher eats a Twinkie in the dugout at some point during the game. Use the secant method to answer the question: If you were a Twinkie, what is the probability to 2 decimal places that you will be eaten by the star pitcher in a game that the Yankees have a 0.50 chance to win? Justify the choice of your initial guesses.

   Verify your answer using `fzero`.

   Name an advantage that the secant method would have over Newton's method in this case? Explain your reasoning.

3. [**15 marks**]

   (a) Starting from an initial guess of $x_1 = \frac{\pi}{2}$, use Newton's method to find the root of $f(x) = \sin^2 x - x \sin x + \frac{x^2}{4}$. Does the convergence rate appear to be quadratic? If not, explain why not.

   (b) With the aid of a graph, explain why Newton's method will be unable to find the root of $f(x) = xe^{-x}$ starting from $x_1 = 2$. Suggest a simple way to help Newton's method solve this problem successfully.

4. **[15 marks]** Use Newton's method to solve the following system of nonlinear equations with 6 variables $w_1,\ w_2,\ w_3,\ y_1,\ y_2$, and y_3:

$$
\begin{aligned}
w_1 + w_2 + w_3 &= 2, \\
w_1 y_1 + w_2 y_2 + w_3 y_3 &= 0, \\
w_1 y_1^2 + w_2 y_2^2 + w_3 y_3^2 &= \frac{2}{3}, \\
w_1 y_1^3 + w_2 y_2^3 + w_3 y_3^3 &= 0, \\
w_1 y_1^4 + w_2 y_2^4 + w_3 y_3^4 &= \frac{2}{5}, \\
w_1 y_1^5 + w_2 y_2^5 + w_3 y_3^5 &= 0,
\end{aligned}
$$

starting with the guess $\mathbf{x}_0 = (w_1,\ w_2,\ w_3,\ y_1,\ y_2,\ y_3)^T = (2/3, 2/3, 2/3, -1, 0, 1)^T$. Iterate until convergence of at least 10^{-10} in the infinity norm of the stopping criterion. Include a way to stop if the iteration diverges!

Bonus: [10 marks] Use the demo (free) version of `pythNon` to solve the following system of nonlinear equations:

$$
\begin{aligned}
16x^4 + 16y^4 + z^4 &= 16, \\
x^2 + y^2 + z^2 &= 3, \\
x^3 - y &= 1,
\end{aligned}
$$

starting with the guess $\mathbf{x}_0 = (x, y, z)^T = (1, 1, 1)^T$.

Constructive feedback on what you liked or did not like about `pythNon` is welcome.

5. **[20 marks]** Apply $S_{\pi/2}(0, \pi/2)$ and $S_{\pi/4}(0, \pi/2)$ to

$$\int_0^{\frac{\pi}{2}} \sin x\, dx,$$

where $S_h(a, b)$ is Simpson's rule applied to the interval $[a, b]$ with $h = b - a$. Use $S_{\pi/2}(0, \pi/2)$ and $S_{\pi/4}(0, \pi/2)$ to compute an error estimate for $S_{\pi/4}(0, \pi/2)$. Comment on the quality of the error estimate.

6. **[20 marks]** Estimate the value of

$$\int_0^{\frac{\pi}{2}} \sin x\, dx$$

by using the trapezoidal rule with $h = \pi/2,\ \pi/4$. Call the approximations $T_{\pi/2}(0, \pi/2)$ and $T_{\pi/4}(0, \pi/2)$, respectively. Comment on the accuracy of the two estimates, specifically noting the number of correct decimal places.

Based on the known order of trapezoidal rule, derive a new rule $R_{\pi/2,\pi/4}(0, \pi/2)$ that takes a linear combination of $T_{\pi/2}(0, \pi/2)$ and $T_{\pi/4}(0, \pi/2)$ to produce a more accurate estimate. Comment on the accuracy of $R_{\pi/2,\pi/4}(0, \pi/2)$.