University of Saskatchewan
Department of Mathematics and Statistics

Numerical Analysis III
(MATH 314)
Instructor: Dr. Raymond J. Spiteri

ASSIGNMENT 01
Due: 10:00 a.m. Tuesday, September 29, 2009

1. [**10 marks**]

   The program `dfs.m` that accompanies the course text provides a modest capability for computing a direction field and solutions of a scalar ODE, $\dot{y} = f(t, y)$. The first argument of `dfs.m` is a string defining $f(t, y)$. In this the independent variable must be called t and the dependent variable must be called y. The second argument is an array `[ wL wR wB wT ]` specifying a plot window. Specifically, solutions are plotted for values $y(t)$ with $wL \leq t \leq wR,\ wB \leq y \leq wT$.The program first plots a direction field. If you then indicate a point in the plot window by placing the cursor there and clicking, it computes and plots the solution of the ODE through this point. Clicking at a point outside the window terminates the run. For example, Figure 1.4 of the text can be reproduced with the command

   ```
   >> dfs('cos(t)*y', [ 0 12 -6 6 ]) ;
   ```

   and clicking at the appropriate points in the window.

   Use `dfs.m` to study graphically the stability of the ODE

   $$\dot{y} = 5(y - t^2). \tag{1}$$

   An appropriate plot window is given by `[ 0 5 -2 20 ]`.

2. [**10 marks**]

   Compare local and global errors as in Figure 1.5 of the text when solving equation (1) with $y(0) = 0.08$ and using Euler's method with the constant step size $\Delta t = 0.1$ to integrate from $t = 0$ to $t = 2$.

   The stability of this problem is studied analytically in the text and numerically in the previous question. With this in mind, discuss the behaviour of the global errors.

3. **[10 marks]**

Murphy (1965) extends the classical Falkner–Skan similarity solutions $f(\eta)$ for laminar incompressible boundary layer flow over curved surfaces. He derives a BVP consisting of the ODE

$$f'''' + (\Omega + f)f''' + \Omega f f'' - (2\beta - 1)[f'f'' + \Omega(f')^2] = 0$$

to be solved on $0 \le \eta \le b$ with boundary conditions

$$f(0) = f'(0) = 0, \quad f'(b) = e^{-\Omega b}, \quad f''(b) = -\Omega e^{-\Omega b}$$

Here Ω is a curvature parameter, β is a pressure-gradient parameter, and b is large enough that the exponential terms in the boundary conditions describe the correct asymptotic behavior. Physically significant quantities are the displacement thickness

$$\Delta^* = \int_0^b [1 - f'(\eta)e^{\Omega\eta}]d\eta$$

and the momentum thickness

$$\theta = \int_0^b f'(\eta)e^{\Omega\eta}[1 - f'(\eta)e^{\Omega\eta}]d\eta.$$

Formulate the BVP in terms of a system of first-order equations.

Add equations and initial values so that the displacement thickness and the momentum thickness can each be computed along with the solution $f(\eta)$.

4. **[10 marks]**

Caughy (1970) describes the large-amplitude whirling of an elastic string by a BVP consisting of the ODE

$$\mu'' + \omega^2\left(\frac{1-\alpha^2}{H}\frac{1}{\sqrt{1+\mu^2}} + \alpha^2\right)\mu = 0$$

and boundary conditions

$$\mu'(0) = 0, \quad \mu'(1) = 0.$$

Here α is a physical constant with $0 < \alpha < 1$. Because the whirling frequency ω is to be determined as part of solving the BVP, there must be another boundary condition. Caughy specifies the amplitude ε of the solution at the origin:

$$\mu(0) = \varepsilon.$$

An unusual aspect of this problem is that an important constant H is defined in terms of the solution $\mu(x)$ throughout the interval of integration:

$$H = \frac{1}{\alpha^2}\left[1 - (1-\alpha^2)\int_0^1 \frac{dx}{\sqrt{1+\mu^2(x)}}\right].$$

Formulate this BVP in the standard form. As in the Sturm–Liouville example, you can introduce a new variable $y_3(x)$, a first-order ODE, and a boundary condition to deal with the integral term in the definition of H.

The trick to dealing with H is to let it be a new variable $y_4(x)$. It is a constant, so this new variable satisfies the first-order differential equation $y_4' = 0$. It is given the correct constant value by the boundary condition resulting from the definition of H:

$$y_4(1) = \frac{1}{\alpha^2}[1 - (1 - \alpha^2)y_3(1)].$$

5. **[10 marks]**

Volterra's model of predator-prey interaction can be formulated as

$$\begin{aligned} x' &= a(x - xy), \\ y' &= -c(y - xy). \end{aligned}$$

(a) Show that solutions of this system of ODEs satisfy the nonlinear conservation law

$$G(t, x, y) = x^{-c}y^{-a}e^{cx+ay} = \text{ constant.}$$

(b) Write a MATLAB program to integrate the differential equations with Euler's method and constant step size Δt. Using parameter values $a = 2$ and $c = 1$ and initial values $x(0) = 1$ and $y(0) = 3$, integrate the IVP for $0 \leq t \leq 10$.

Plot the solution in the phase plane; that is, plot $(x(t), y(t))$.

Also, calculate and plot the conserved quantity $G(t, x(t), y(t))$.

The theory says that G is constant and the solution is periodic, hence the curve plotted in the phase plane is closed.

Experiment with the step size Δt to find a value for which G is approximately constant and the curve you compute appears to be closed.

After you have learned to use the MATLAB IVP solvers in the next chapter, you may want to revisit this problem and solve it with `ode45` instead of Euler's method.