University of Saskatchewan
Department of Mathematics and Statistics

**Numerical Analysis III**
(MATH 314)
Instructor: Dr. Raymond J. Spiteri

ASSIGNMENT 02
**Due: 10:00 a.m. Tuesday, October 20, 2009**

1. **[10 marks]**

   To understand better the equations of order, derive the three equations for a formula of the form
   $$y_{n+1} = y_n + \Delta t[b_1 f_{n,1} + b_2 f_{n,2}],$$
   where
   $$\begin{aligned} f_{n,1} &= f(t_n, y_n) \\ f_{n,2} &= f(t_n + c_2\Delta t, y_n + \Delta t a_{2,1} f_{n,1}), \end{aligned}$$
   to be of second order.

   In the step from $(t_n, y_n)$, the result $y_{n+1}$ of the formula is to approximate the solution of
   $$u' = f(t, u), \quad u(t_n) = y_n$$
   at $t_{n+1} = t_n + \Delta t$.

   Expand $u(t_{n+1})$ and $y_{n+1}$ about $t_n$ in powers of $\Delta t$ and equate terms to compute the equations of condition. To simplify the expansions, do this for a scalar function $f(t, u)$.

2. **[10 marks]**

   The explicit Runge–Kutta formulas
   $$\begin{aligned} y_{n+1} &= y_n + \Delta t f_{n,2} \\ y^*_{n+1} &= y_n + \frac{\Delta t}{9}[2f_{n,1} + 3f_{n,2} + 4f_{n,3}] \end{aligned}$$
   of three stages
   $$\begin{aligned} f_{n,1} &= f(t_n, y_n) \\ f_{n,2} &= f\left(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} f_{n,1}\right) \\ f_{n,3} &= f\left(t_n + \frac{3}{4}\Delta t, y_n + \frac{3}{4}\Delta t f_{n,2}\right) \end{aligned}$$
   are of order 2 and 3, respectively .

   Express this (2,3) pair as an extended Butcher tableau.

The local error of the lower-order formula is estimated by $est = y_{n+1}^* - y_{n+1}$. Suppose that the integration is to be advanced with the higher-order result (local extrapolation) and that you are given a relative error tolerance $\tau_r$, and absolute error tolerance $\tau_a$. This means that you will accept the step if

$$|est| \leq \tau_r |y_{n+1}^*| + \tau_a.$$

What step size $\Delta t_{new} = \sigma \Delta t$ should you use if you must repeat the step because the estimated local error is too large?

What $\Delta t_{new}$ should you use for the next step if the estimated local error is acceptable?

Some solvers measure the error relative to $0.5(|y_n| + |y_{n+1}^*|)$ instead of $|y_{n+1}^*|$. Why might this be a good idea?

3. **[10 marks]** Use both the Butcher and the Albrecht forms of the order conditions to verify that the embedded Runge–Kutta pair in the previous question consists of schemes having orders 2 and 3.

   Explain what is meant by a Runge–Kutta method having the "first same as last" FSAL property. Does the embedded Runge–Kutta pair from the previous question have this property?

4. **[10 marks]** Work out the details in deriving AB2 and BDF2; i.e., explicitly construct the appropriate interpolating polynomials and integrate them to obtain the corresponding numerical method.

5. **[10 marks]**

$$y_{n+1} + \frac{3}{2} y_n - 3 y_{n-1} + \frac{1}{2} y_{n-2} - 3 \Delta t f(t_n, y_n) = 0 \tag{1}$$

   Verify that the formula (1) is of order 3.

   Perform the numerical experiment described in the text that resulted the Table 1:

Table 1: Maximum error when $\Delta t = 2^{-i}$.

| $i$ | AB3 | Formula(1) |
|---|---|---|
| 2 | $1.34e-003$ | $9.68e-004$ |
| 3 | $2.31e-004$ | $6.16e-003$ |
| 4 | $3.15e-005$ | $1.27e+000$ |
| 5 | $4.08e-006$ | $6.43e+005$ |
| 6 | $5.18e-007$ | $2.27e+018$ |
| 7 | $6.53e-008$ | $4.23e+044$ |
| 8 | $8.19e-009$ | $2.27e+098$ |
| 9 | $1.03e-009$ | $1.03e+207$ |
| 10 | $1.28e-010$ | Inf |