University of Saskatchewan
Department of Mathematics and Statistics

Numerical Analysis III
(MATH 314)
Instructor: Dr. Raymond J. Spiteri

ASSIGNMENT 04
Due: 10:00 a.m. Tuesday, November 24, 2009

1. [**20 marks**]

   A resistively shunted junction model of a linear array of N capacitance-free Josephson weak links is given by

   $$\dot{\phi}_i = R_i[(I_i - I_{c_i}\sin\phi_i) + \alpha(I_{i-1} - I_{c_{i-1}}\sin\phi_{i-1}) + \alpha(I_{i+1} - I_{c_{i+1}}\sin\phi_{i+1})],\ i = 1, 2, \ldots, N,$$

   where R_i is the resistance of junction i, I_{c_1} is its critical current, ϕ_i is its phase, and I_i is the applied DC bias current. Quantities with subscript 0 or $N+1$ are taken to be 0. With the settings $I_i \equiv 2$, R_i, $I_{c_i} = 1 + 0.02u_i$, where u_i is a random number chosen from the uniform distribution on $[-1, 1]$.

   Starting with initial conditions $\phi_i(0) = 0$, $i = 1, 2, \ldots, N$, plot the Poincaré section ϕ_1 vs. ϕ_2 each time ϕ_3 goes through a multiple of 2π for 500 points when $N = 3$ and $N = 500$. Comment on the regularity of the plots for $\alpha = 0.01$ and $\alpha = 0.1$.

2. [**25 marks**]

   In this question, we verify some of the details of the Galerkin method described in Example 2.3.11 to solve the PDE

   $$u_t = u_{xx}, \quad 0 \le x \le 1,\ t \ge 0,$$

   subject to the boundary conditions

   $$u(0, t) = 0, \quad u(1, t) = 1,$$

   and initial conditions

   $$u(x, 0) = x + \sin(\pi x).$$

   (a) Give formulas for the piecewise linear basis functions $S_j(x)$, $j = 0, 1, \ldots, m+1$, that satisfy

   $$S_j(x) = \begin{cases} 1 & \text{if } x = x_j, \\ 0 & \text{otherwise.} \end{cases}$$

   (b) Assuming $v(x, t) = \sum_{j=1}^{m} S_j(x)v_j(t)$, show that the Galerkin condition

   $$(R, S_i) = (v_t, S_i) - (v_{xx}, S_i) = 0, \quad i = 1, 2, \ldots, m, \tag{1}$$

   where the inner product (f, g) of two functions $f,\ g \in C[0, 1]$ is defined by

   $$(f, g) := \int_0^1 f(x)g(x)\,dx,$$

and

$$R(x,t) = v_t(x,t) - v_{xx}(x,t),$$

leads to the set of coupled ODEs

$$\sum_{j=1}^{m} (S_j, S_i)\dot{v}_j + \sum_{j=1}^{m} \left(\frac{dS_j}{dx}, \frac{dS_i}{dx} \right) v_j = 0.$$

(c) Show that if we assume a uniform mesh spacing of Δx, the ODEs reduce to

$$\frac{1}{6}\dot{v}_{j-1} + \frac{4}{6}\dot{v}_j + \frac{1}{6}\dot{v}_{j+1} = \frac{v_{j-1} - 2v_j + v_{j+1}}{(\Delta x)^2}, \quad j = 1, 2, \ldots, m.$$

3. **[10 marks]** Use `bvp4c` to solve the BVP

$$yy'' = -1, \quad y(0) = 0, \quad y'(0.5) = 0$$

The text shows that $y(x) \sim x\sqrt{-2\log(x)}$ as $x \to 0$. It will be convenient to code a sub-function to evaluate the approximation $v(x) = x\sqrt{-2\log(x)}$. Move the BC from the singular point at the origin to, say, $d = 0.001$ by imposing $y(d) = v(d)$.

Compute $y(x)$ on $[d, 0.5]$ using `bvp4c` with default tolerances and guesses of $y(x) \approx x(1-x)$ and $y'(x) \approx 1 - 2x$. Corresponding to Figure 4.11 of the famous book by Bender & Orszag (1999), plot both $y(x)$ and $v(x)$ with `axis ([0 0.5 0 0.5])`. You should augment the array for $y(x)$ with $y(0) = 0$ and the array for $v(x)$ with $v(0) = 0$.

4. **[20 marks]** Consider a shock wave in a one-dimensional nozzle. The steady-state Navier–Stokes equations give

$$\epsilon A(x)uu'' - \left[1 + \frac{\gamma}{2} - \epsilon A'(x)\right] uu' + \frac{u'}{u} + \frac{A'(x)}{A(x)}\left(1 - \frac{\gamma - 1}{2}\right) u^2 = 0, \quad 0 < x < 1,$$

where x is the normalized downstream distance from the inlet, $u = u(x)$ is a normalized velocity, $A(x) = 1 + x^2$ is the cross-sectional area of the nozzle at position x, $\gamma = 1.4$ is the ratio of specific heats for constant pressure and volume of an ideal gas, and $\epsilon = 4.792 \times 10^{-8}$ is (essentially) the viscosity. The boundary conditions are

$$u(0) = 0.9129, \qquad u(1) = 0.375.$$

Find and plot $u(x)$ and determine the location of the shock.

Hint: Use continuation in ϵ.

5. **[25 marks]** In this exercise we compare the performance of `bvp4c` to `bvp5c` on Example 3.5.6 (fluid flow in long vertical channel with fluid injection).

The problem statement is

$$
\begin{aligned}
f''' - R[(f')^2 - ff''] + RA = 0, \\
h'' + Rfh' + 1 = 0, \\
\theta'' + P_e f\theta' = 0,
\end{aligned}
$$

where R is the Reynolds number, $P_e = 0.7R$ is the Peclet number, and A is unknown, subject to the boundary conditions

$$
\begin{aligned}
f(0) = f'(0) = 0,\ f(1) = 1,\ f'(1) = 0, \\
h(0) = h(1) = 0,\ \theta(0) = 0,\ \theta(1) = 1.
\end{aligned}
$$

(a) Use continuation to solve the problem for $R = 10^n$, $n = 2, 3, 4, 5, 6$.

(b) Compare the run times for `bvp4c` and `bvp5c` to solve the problem for $R = 10^6$ using the default settings. You may need to increase the default mesh size. If so, report this. Compare the final mesh sizes as well for $R = 10^6$. Explain your observations for both final run times and mesh sizes.

(c) Starting with the solution and mesh for $R = 10^5$ as the initial guess, compare the run times for `bvp4c` and `bvp5c` to solve the problem for $R = 10^6$ using both vectorization and analytical partial derivatives.