University of Saskatchewan
Department of Mathematics and Statistics

Numerical Analysis III
(MATH 314)
Instructor: Dr. Raymond J. Spiteri

ASSIGNMENT 04
Due: 10:00 a.m. Tuesday, November 26, 2013

1. **[25 marks]** Consider the predator-prey model

$$\dot{y}_1 = \alpha y_1 + \beta y_1 y_2,$$
$$\dot{y}_2 = \gamma y_2 + \delta y_1 y_2,$$

where $\alpha = 0.25$, $\beta = -0.01$, $\gamma = -1$, and $\delta = 0.01$.

Give a boundary-value formulation to find the period of the cycle corresponding to the initial conditions $y_1(0) = 80,\ y_2(0) = 30$, i.e., the length of time it takes for these initial conditions to repeat.

Use MATLAB's `bvp5c` to find correct to 5 decimal places the period of the predator-prey model above.

2. **[25 marks]** Consider the nonlinear problem

$$v'' + \frac{4}{x}v' + (xv - 1)v = 0, \qquad 0 < x < \infty,$$

$$v'(0) = 0, \qquad v(\infty) = 0.$$

This problem arises in electromagnetic self-interaction theory. It is a well-behaved problem with a smooth, nontrivial solution[1]. One way to solve it numerically is to replace $[0, \infty)$ with a large, finite interval $[0, L]$ and require $v(L) = 0$. For large x, the solution to this problem is expected to decay like $e^{-\alpha x}$ for some $\alpha > 0$.

(a) Determine the asymptotic behaviour of $v(x)$; i.e., find α.
Hint: Ignore the (two) smallest terms in the ODE as $x \to \infty$ to obtain an ODE that is easy to solve. Don't forget the BC at infinity!

(b) Verify that another valid BC to impose at infinity is $\alpha v(L) + v'(L) = 0$.

(c) Describe a strategy that would guarantee L was chosen to be large enough.

3. **[25 marks]** Consider the boundary-value problem

$$y'' + ee^{y} = 0, \qquad 0 < x < 1,$$

$$y(0) = y(1) = 0.$$

[1]Of course, it also has a trivial solution $v(x) \equiv 0$.

(a) Show that a function of the form

$$y(x) = -2 \ln \left\{ \frac{\cosh[(x - 1/2)\theta/2]}{\cosh(\theta/4)} \right\}$$

is a solution, provided θ satisfies

$$\theta = \sqrt{2e} \cosh(\theta/4). \tag{1}$$

(b) Prove that (1) has two solutions θ and find them.
Hint: Convert (1) to the form $f(\theta) = 0$, use the Intermediate Value Theorem to show there exists at least one root of (1), and then show that $f'(\theta) = 0$ has exactly *one* root.

(c) Use MATLAB's `bvp5c` with suitable initial guesses of the form

$$y(x) = cx(1 - x)$$

to find both solutions to the boundary-value problem.

4. **[25 marks]** A simple pendulum consists of a unit mass attached to a massless rod. In the absence of friction, the motion of the simple pendulum is governed by the differential-algebraic equations

$$\begin{aligned}
\dot{x} &= u, \\
\dot{u} &= -\lambda x, \\
\dot{y} &= v, \\
\dot{v} &= -\lambda y - g, \\
0 &= x^2 + y^2 - 1,
\end{aligned}$$

where (x, y) is the position of the mass, $g = 13.750371633$, and λ is known as a *Lagrange multiplier.* (It represents a reaction force that the rod exerts on the mass to keep it from flying off.)

(a) What is the index of this problem?

(b) What would the algebraic equation (constraint) be if the index were reduced by one?

(c) Solve the index-2 formulation of this problem using MATLAB's `ode15s` from $t = 0$ until $t = 2$ starting with the initial condition $x(0) = 1,\ u(0) = y(0) = v(0) = 0$. Also take $\lambda(0) = 1$. What is fishy about this approach?

(d) Solve the index-1 formulation of the problem as described in part (c).

(e) Does the DAE require specification of $\lambda(0)$ for the problem to be well-posed? Explain.

5. **[0 marks]** Provide a brief progress report for your project, re-iterating the goals for the project (as stated in your interim report), giving any revisions to these goals, and assessing the success in achieving the goals.