CHAPTER 1: *ODEs*

- Very common for ODEs to be present in mathematical models in science, engineering, finance, etc.

- Very common for "real" ODEs to not have analytical solutions. $\rightarrow$ *We need numerical methods!*

  Algorithms / software must be

  – efficient (time *and* memory)
  – reliable
  – robust

- Broadly classify ODEs with respect to side conditions.
  e.g.,
  $$\ddot{u}(t) + u(t) = 0, \quad 0 \leq t \leq t_f.$$

  Note: $\dot{u}(t) := \frac{d}{dt}u(t)$

Solution:

$$u(t) = \alpha \sin(t + \beta),$$

where α, β are arbitrary constants.

Exercise: Verify $u(t)$ satisfies the ODE.

$$\begin{array}{c}\text{2 arbitrary constants} \leftrightarrow \text{2 different conditions} \\ \updownarrow \\ \text{we can impose 2 side conditions}\end{array}$$

How we do this determines the nature of the ODE.
e.g., IVP:

$$\begin{array}{ccc} u(0) = c_1, & & \dot{u}(0) = c_2 \\ \downarrow & & \downarrow \\ \alpha \sin \beta = c_1 & \rightarrow\leftarrow & \alpha \cos \beta = c_2 \\ & \downarrow & \\ \beta = \arctan \frac{c_1}{c_2}, & & \alpha = \frac{c_1}{\sin \beta} \quad (= \frac{c_2}{\cos \beta}) \end{array}$$

$\rightarrow$ Solution is unique for all $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

e.g., BVP:

$$u(0) = c_1, \quad u(t_f) = c_2.$$

Let $t_f = \pi,\ c_1 = 0$*.*
Then

$$u(0) = \alpha \sin \beta = 0,$$
$$u(\pi) = \alpha \sin(\beta + \pi) = c_2.$$

But

$$\sin \beta \equiv -\sin(\beta + \pi).$$

$\therefore$ if $c_2 \neq 0$, there is no solution.

If $c_2 = 0$ and $\beta = 0$, α is arbitrary
$\Longrightarrow$ an infinite number of solutions.

If $t_f \neq \pi$, it is possible to have a unique solution.

With a BVP, anything is possible !

1.1 IVPs

Standard form

$$\begin{aligned} \dot{\mathbf{y}} &= \mathbf{f}(t, \mathbf{y}), \qquad 0 \le t \le t_f, \\ \mathbf{y}(0) &= \mathbf{y}_0. \end{aligned}$$

BEWARE ! $\mathbf{y}, \mathbf{y}_0$ are vectors;
$\mathbf{f}$ is a vector-valued function.

Note 1. *When $\mathbf{f} = \mathbf{f}(\mathbf{y})$, the ODE is autonomous.*

Non-autonomous ODEs can be transformed to autonomous ODEs by introducing a new variable

$$\mathbf{Y} = \begin{pmatrix} \mathbf{y} \\ t \end{pmatrix}$$

and a new right-hand side

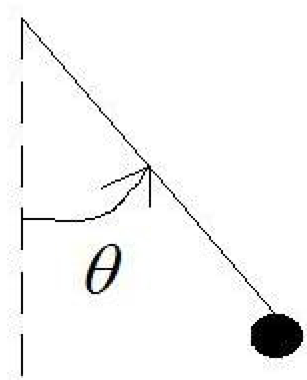
$$\mathbf{F} = \begin{pmatrix} \mathbf{f} \\ 1 \end{pmatrix};$$

then

$$\dot{\mathbf{Y}} = \mathbf{F}(\mathbf{Y}).$$

We will often write $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ without loss of generality.

Example 1. *(Simple pendulum)*



Newton's law:

$$m\ddot{\theta} = -g \sin\theta \quad \textit{(ignore friction)}$$

Convert to first-order system. Let

$$\begin{aligned} Y_1 &= \theta, \\ Y_2 &= \dot{\theta}. \end{aligned}$$

Then

$$\begin{aligned}\dot{Y}_1 &= Y_2,\\ \dot{Y}_2 &= -\frac{g}{m}\sin Y_1.\end{aligned}$$

Initial conditions: $\theta(0) = \theta_0, \quad \dot{\theta}(0) = \omega_0.$

Exercise: *Let* $m = 1,\ g = 9.8,\ \theta_0 = 1,\ \omega_0 = 1.$
Solve and visualize using Matlab's ode45.

Example 2. *(Predator-Prey model)*
$\rightarrow$ *population biology*

$y_1(t)$	*Prey population at time* t	
$y_2(t)$	*Predator population at time* t	
α	*Prey's net growth rate (birth – death)*	$\alpha > 0$
β	*Probability of interaction*	$\beta > 0$
γ	*Predator's growth rate without prey*	$\gamma < 0$
δ	*Predator growth rate when meeting prey*	$\delta > 0$

$$\begin{aligned} \dot{y_1} &= \alpha y_1 - \beta y_1 y_2 \\ \dot{y_2} &= \gamma y_2 + \delta y_1 y_2 \end{aligned}$$

Typical values:

$$\alpha = 0.25, \beta = 0.01, \gamma = -1, \delta = 0.01.$$

Starting from $y(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 80 \\ 30 \end{pmatrix}$, *model possesses a periodic solution* $y(T) = y(0)$ *for a* $T > 0$.

Exercise: *Use ode45 to estimate T.*
What happens if you take different ICs ?
(Plot y_2 *vs* y_1*.)*

Example 3. *(Diffusion problem)*

$$u(x,t) = \text{temperature in metal rod}$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right) + g(x,u)$$

$$u = u(x,t) \text{ is unknown}$$

$$0 \leq x \leq 1, \quad t \geq 0.$$

For simplicity, let $p \equiv 1$.

Initial data

$$u(x,0) = u_0(x).$$

Boundary data

$$u(0,t) = \alpha(t), \quad u(1,t) = \beta(t).$$

- *Divide up* $[0,1]$ *into* $m+1$ *equal subintervals:*

$$\Delta x = \frac{1}{m+1}$$

- *Let* $y_i(t) \approx u(x_i,t),\ x_i = i\Delta x,\ i = 0,1,\ldots,m+1,$
  $\rightarrow$ method of lines.

- *Let* $\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} \approx \frac{y_{i+1}-2y_i+y_{i-1}}{(\Delta x)^2}$.

Then

$$\dot{y}_i = \frac{1}{(\Delta x)^2}[y_{i+1} - 2y_i + y_{i-1}] + g(x_i, y_i),$$

$$y_0 = \alpha(t), \quad y_{m+1} = \beta(t),$$
$$y_i(0) = u_0(x_i).$$

A system of m *coupled ODEs !*

Theorem 1. *(Existence, uniqueness of IVP solutions)*

$$\begin{aligned}\dot{\mathbf{y}} &= \mathbf{f}(t,\mathbf{y}),\\ \mathbf{y}(0) &= \mathbf{y}_0.\end{aligned}$$

Let $\mathbf{f}(t,\mathbf{y})$ *be continuous for all* $(t,\mathbf{y})$ *in*

$$\boldsymbol{\mathcal{D}} = \{0 \le t \le t_f, \quad 0 \le \|\mathbf{y}\| < \infty\}.$$

Let $\mathbf{f}(t,\mathbf{y})$ *satisfy a Lipschitz condition in* $\boldsymbol{\mathcal{D}}$;

i.e.,

$$||\mathbf{f}(t,\mathbf{y}) - \mathbf{f}(t,\hat{\mathbf{y}})|| \le L||\mathbf{y} - \hat{\mathbf{y}}||$$

for some constant $0 < L < \infty$ *and all pairs* $(t,\mathbf{y})$, $(t,\hat{\mathbf{y}})$ *in* $\boldsymbol{\mathcal{D}}$.

(L can be taken as a (potentially conservative) bound on the norm of the Jacobian matrix $\partial\mathbf{f}/\partial\mathbf{y}$.*)*

Then

- *for any* $\mathbf{y}_0$*, there is a unique (and differentiable) solution to the IVP in* $[0, t_f]$.

Moreover,

- $\mathbf{y}$ *depends continuously on the data.*

- *If* $\dot{\hat{\mathbf{y}}} = \mathbf{f}(t, \hat{\mathbf{y}}) + \mathbf{r}(t, \hat{\mathbf{y}})$ *with* $||\mathbf{r}|| \leq M$ *on* $\mathcal{D}$*,*

then

$$\begin{aligned} ||\mathbf{y}(t) - \hat{\mathbf{y}}(t)|| &\leq e^{Lt}||\mathbf{y}(0) - \hat{\mathbf{y}}(0)|| + \frac{M}{L}(e^{Lt} - 1) \\ &\leq e^{Lt}||\mathbf{y}(0) - \hat{\mathbf{y}}(0)||. \end{aligned}$$

i.e., If ICs / parameters / $\mathbf{f}(t, y)$ *are changed slightly, solution changes slightly.*

Often $\mathcal{D}$ *must be restricted for these results to hold.*

e.g., if we restrict $\mathcal{D}$ *so that* $\mathbf{y}$ *satisfies* $\|\mathbf{y} - \mathbf{y}_0\| \leq \gamma$*, a finite* L *exists, and* $\|\mathbf{f}(t, \mathbf{y})\| \leq M$*, then a unique solution is guaranteed for* $0 \leq t \leq \min(t_f, \gamma/M)$.

This is the definition of a well-posed problem*:*

The solution

- *exists*
- *is unique*
- *is not sensitive to perturbation.*

We have seen

IVPs have a local nature.

- Solution marches in time
- Past or future values not needed in solution determination

BVPs have a global nature.

- Need to account for solution values everywhere!
- Existence and uniqueness much more complicated!

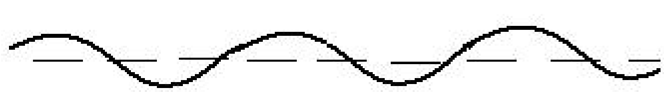
BVPs are “harder” to solve than IVPs.

1.2 BVPs

General form:

$$\begin{aligned} \mathbf{y}' &= \mathbf{f}(x, \mathbf{y}), \\ \mathbf{g}(\mathbf{y}(a), \mathbf{y}(b)) &= \mathbf{0}. \end{aligned}$$

Example 4. *(Vibrating spring)*



$u =$ *displacement from equilibrium*

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = r(x),$$
$$p(x) > 0, \quad q(x) \geq 0, \quad a \leq x \leq b.$$

Suppose one end is fixed, the other is free:

$$\rightarrow u(a) = 0, \quad u'(b) = 0.$$

More discussion about BVPs is deferred until later.

1.3 DAEs

So far our model problems look like

$$\dot{\mathbf{y}}(t) = \mathbf{f}(t, \mathbf{y}(t)).$$

$\rightarrow$ explicit ODE

More generally, however, we can have

$$\mathbf{F}(t, \mathbf{y}(t), \dot{\mathbf{y}}(t)) = \mathbf{0}.$$

$\rightarrow$ implicit ODE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{y}}}$ is nonsingular.

(Then in principle you can solve for $\dot{\mathbf{y}}$.)

Now consider explicit ODE *with constraints*:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(t,\mathbf{x},\mathbf{z}),\\ \mathbf{0} &= \mathbf{g}(t,\mathbf{x},\mathbf{z}).\end{aligned}$$

$\mathbf{x} \leftrightarrow$ differential variables
$\mathbf{z} \leftrightarrow$ algebraic variables

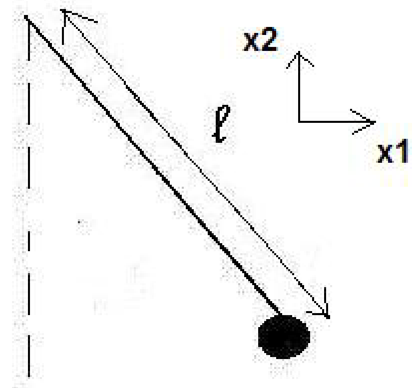
The components of $\mathbf{x}$ are not independent !
$\rightarrow$ semi-explicit DAEs

We can cast this as an implicit ODE:

$$\mathbf{y} = \begin{pmatrix} \mathbf{x} \\ \mathbf{z} \end{pmatrix}, \ \mathbf{F} = \begin{pmatrix} \dot{\mathbf{x}} - \mathbf{f} \\ \mathbf{g} \end{pmatrix} \Rightarrow \mathbf{F}(t,\mathbf{y},\dot{\mathbf{y}}) = \mathbf{0}.$$

But $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{y}}} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ is singular for all $t, \mathbf{x}, \mathbf{z}, \dot{\mathbf{x}}, \dot{\mathbf{z}}$.

Example 5. *(Simple pendulum — again)*



$$\begin{aligned} \ddot{x}_1 &= -z x_1 \\ \ddot{x}_2 &= -z x_2 - g \\ {x_1}^2 + {x_2}^2 &= l^2 \end{aligned}$$

$z =$ *Lagrange multiplier (reaction force)*
$\rightarrow$ *Simple case of a multibody system.*

Note 2. *Letting* $x_1 = l \sin\theta,\ x_2 = -l\cos\theta$*, we can eliminate* z*.* $\rightarrow$ *This takes us back to Example 1.*

Real life is rarely this convenient.

Such a transformation may not be

- *possible (complicated, discontinuous, etc.)*
- *advisable (painstaking, less efficient, etc.)*

FINAL NOTE ON DAEs:

DAEs are not ODEs!

DAEs are fundamentally different from ODEs
(even implicit ones).

$$\begin{aligned} \dot{x} &= z, \\ 0 &= x - t. \end{aligned}$$

Clearly, the solution is $x = t,\ z = 1$.

$\rightarrow$ No ICs or BCs needed!

If you try to set $x(0) = x_0$, then no solution if $x_0 \neq 0$.
($x_0 = 0$ is consistent, but not necessary.)

Much more discussion on DAEs deferred to later.