Notation

scalar	lower case Greek	α, β, σ
vector	lower case	u, v, x, y, b
matrix	upper case	A, B, C

Defining Vectors in Matlab

- Assign any expression that evaluates to a vector

```
>> v = [1 3 5 7];
>> w = [2; 4; 6; 8];
>> x = linspace(0, 10, 6);
>> y = 0:30:180 ;
>> z = sin(y*pi/180);
```

- Distinquish between row and column vectors

```
>> r = [1 2 3]    % row vector
>> s = [1 2 3]'   % column vector
>> r - s
??? Error using ==> -
Matrix dimensions must agree.
```

Altough `r` and `s` have the same elements, they are not the same vector.

Vector Addition and Subtraction

Additon and subtaction are element-by-element operatons

$$
\begin{aligned}
c = a + b &\iff c_i = a_i + b_i \quad i = 1, \ldots, n \\
d = a - b &\iff d_i = a_i - b_i \quad i = 1, \ldots, n
\end{aligned}
$$

Example:

$$
a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}
$$

$$
a + b = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \quad a - b = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}
$$

Multiplication by a Scalar

Multplication by a scalar involves multplying each element in the vector by the scalar:

$$
b = \sigma a \iff b_i = \sigma a_i \quad i = 1, \ldots, n
$$

Example:

$$
a = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} \quad b = \frac{a}{2} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}
$$

Vector Transpose

The transpose of a row vector is a column vector:

$$u = \left[\ 1, 2, 3\ \right] \quad \text{then} \quad u^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Likewise if v is the column vector

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{then} \quad v^T = \left[\ 1, 2, 3\ \right]$$

Linear Combinations

$$\alpha \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} + \beta \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} \alpha u_1 + \beta v_1 \\ \alpha u_2 + \beta v_2 \\ \vdots \\ \alpha u_m + \beta v_m \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$$

Example:

$$r = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \quad s = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$t = 2r + 3s = \begin{bmatrix} -4 \\ 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 15 \end{bmatrix}$$

Vector Inner Product

Assuming that x and y are column vectors:

$$\sigma = x \cdot y \quad \Longleftrightarrow \quad \sigma = \sum_{i=1}^{n} x_i y_i = x^T y = y^T x$$

$$x^T y = \left[\; x_1, x_2, x_3, x_4 \;\right] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4$$

Computing the Inner Product in Matlab

The $*$ operator performs the inner product if two vectors are compatible.

```
>> u = (0 : 3)';                % u and v are
>> v = (3 : -1 : 0)';           % column vectors
>> s = u*v
??? Error using ==> *
Inner matrix dimensions must agree.


>> s = u'*v
s =
    4
>> t = v'*u
t =
    4
```

Matrix Notation

The matrix A with m rows and n columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

a_{ij} = element in **row** i, and **column** j

In MATLAB we can define a matrix with

```
>> A = [ ... ; ... ; ... ]
```

where semicolons separate lists of row elements.

The $a_{2,3}$ element of the MATLAB matrix `A` is `A(2,3)`.

Matrix Operations

Addition and Subtraction

$$C = A + B$$

or

$$c_{i,j} = a_{i,j} + b_{i,j} \quad i = 1, \ldots, m; \quad j = 1, \ldots, n$$

Multiplication by a Scalar

$$B = \sigma A$$

or

$$b_{i,j} = \sigma a_{i,j} \quad i = 1, \ldots, m; \quad j = 1, \ldots, n$$

Matrix Transpose

$$B = A^T$$

or

$$b_{i,j} = a_{j,i} \quad i = 1, \ldots, m; \quad j = 1, \ldots, n$$

In MATLAB

```
>> A =[0 0 0; 0 0 0; 1 2 3; 0 0 0]
A=
   0 0 0
   0 0 0
   1 2 3
   0 0 0
>> B=A'
B=
   0 0 1 0
   0 0 2 0
   0 0 3 0
```

Matrix-Vector Multiplation

$$Ax = b$$

Row View

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad i = 1, \ldots, m$$

$$\begin{bmatrix} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \\ \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b_i \quad i = 1, \ldots, m$$

Column View

$$a_{(j)} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \quad j = 1, \ldots, n$$

$$x_1 a_{(1)} + x_2 a_{(2)} + \cdots + x_n a_{(n)} = b$$

or

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Example:

$$\begin{bmatrix} 5 & 0 & 0 & -1 \\ -3 & 4 & -7 & 1 \\ 1 & 2 & 3 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -3 \\ -1 \end{bmatrix} =$$

$$\begin{bmatrix} (5)(4) & + & (0)(2) & + & (0)(-3) & + & (-1)(-1) \\ (-3)(4) & + & (4)(2) & + & (-7)(-3) & + & (1)(-1) \\ (1)(4) & + & (2)(2) & + & (3)(-3) & + & (6)(-1) \end{bmatrix} = \begin{bmatrix} 21 \\ 16 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 0 & -1 \\ -3 & 4 & -7 & 1 \\ 1 & 2 & 3 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -3 \\ -1 \end{bmatrix} =$$

$$4 \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ -7 \\ 3 \end{bmatrix} - 1 \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 21 \\ 16 \\ -7 \end{bmatrix}$$

Vector Spaces

$\mathbf{R}^1$ = Space of all vectrs with one element These vectors define the point along a line.

$\mathbf{R}^2$ = Space of all vectors with two elements.. These vectors define the points in a plane.

$\mathbf{R}^n$ = Space of all vectors with n elements. These vectors define the points in an n-dimensional space (hyperplane).

Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \qquad I \cdot x = x \quad \text{for any } x \text{ in } \mathbf{R}^n$$

For example, in $\mathbf{R}^3$:

$$I \cdot x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x$$

Matrix Inverse A^{-1}

$$A^{-1} \cdot A = A \cdot A^{-1} = I$$

Solve the Linear System $Ax = b$

$$b = Ax \qquad A^{-1}b = A^{-1}Ax = Ix = x \quad \Longrightarrow x = A^{-1}b$$

Determinants

- Only square matrices have detrminants.
- If $det(A) = 0$, then A is singular, and A^{-1} does not exist.
- $det(I) = 1$ for any identity matrix I.
- $det(AB) = det(A)det(B)$.
- $det(A^T) = det(A)$.
- Cramer's rule uses (many!) determinants to express the solution to $Ax = b$.

The *determinant* has a number of useful properties:

- A is *singular* if and only if $det(A) = 0$.
- A is *nonsingular* if and only if $det(A) \neq 0$.
- A has an inverse A^{-1} if and only if A is nonsingular.
- $Ax = b$ has a unique solution if and only if A is nonsingular.
- If $det(A) = 0$, $Ax = b$ has a solution (not unique) if $Ax = b$ is consistent.
- $det(A)$ is not useful for numerical computation.
  - Computation of $det(A)$ is expensive
  - Computation of $det(A)$ can cause overflow
- For diagonal and triangular matrices, $det(A)$ is the product of diagonal elements
- The built-in **det** computes the determinant of a matrix by first factoring it into $A = LU$, and then computing

$$det(A) = det(L)det(U) = (l_{11}l_{22} \cdots l_{nn})(u_{11}u_{22} \cdots u_{nn})$$