On the Biplanar Crossing Number of K_n

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Abstract

The crossing number $cr(G)$ of a graph G is the minimum number of edge crossings over all drawings of G in the Euclidean plane. The k-planar crossing number $cr_k(G)$ of G is $\min\{cr(G_1) + cr(G_2) + \ldots + cr(G_k)\}$, where the minimum is taken over all possible decompositions of G into k subgraphs $G_1, G_2, \ldots, G_k$. The problem of computing the crossing number of complete graphs, $cr(K_n)$, exactly for small n and bounding its value for large n has been the subject of extensive recent research. In this paper we examine the biplanar crossing number of complete graphs, $cr_2(K_n)$. Since 1971, Owens' construction [IEEE Transactions on Circuit Theory, 18(2):277–280, 1971] has been the best known construction for biplanar drawings of K_n for large values of n. We propose an improved technique for constructing biplanar drawings of K_n, which reduces the lower order terms of Owens' upper bound. For small fixed n, we show that $cr_2(K_{10}) = 2$, $cr_2(K_{11}) \in \{4, 5, 6\}$, and for $n \geq 12$, we improve previous upper and lower bounds on $cr_2(K_n)$.

1 Introduction

A *drawing* of a graph G on $\mathbb{R}^2$ is a mapping of each vertex of G to a distinct point in $\mathbb{R}^2$ and each edge of G to a simple continuous curve between its corresponding endpoints. Throughout the paper we assume that the drawings are nice, i.e., the interiors of edges do not pass through vertices, edges may create crossings but do not touch otherwise, and finally, no three edges cross in a point. The *crossing number of* G is the smallest integer, denoted by $cr(G)$, such that G admits a drawing with $cr(G)$ edge crossings.

Determining the crossing numbers of complete graphs is one of the most studied problems in combinatorial geometry (e.g., [2, 4, 8, 15, 18, 19]). The problem of determining $cr(K_n)$, i.e., the crossing number of a complete graph with n vertices, has been studied since the early 1960s [11, 12, 21]. From that time it was known [11] that $cr(K_n)$ is bounded from above by Z_n, where $Z_n = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$. Given a complete graph of n vertices, there are several construction techniques [9] to produce a drawing of the graph with exactly Z_n crossings. In fact, it is conjectured that the equality $cr(K_n) = Z_n$ holds in general [12, 21]. Pan and Richter [17] showed that the conjecture holds for the case when $n \leq 12$.

The definition of crossing number naturally extends to an arbitrary number of planes. Given a graph $G = (V, E)$, the *k-planar crossing number* $cr_k(G)$ of G is equal to $\min\{cr(G_1) + cr(G_2) + \ldots + cr(G_k)\}$, where the minimum is taken over all possible decompositions of G into k subgraphs $G_i = (V_i, E_i), 1 \leq i \leq k$, such that $V = \{V_1 \cup \ldots \cup V_k\}$ and $E = \{E_1 \cup \ldots \cup E_k\}$. In 1971, Owens [16] showed that $cr_2(K_n)$ is bounded from above by W_n, where $W_n =$

$$\begin{cases} Z_{\lceil n/2\rceil} + Z_{\lfloor n/2\rfloor} + \frac{n^2(n-4)(n-8)}{384}, & \text{if } n = 4m. \\ Z_{\lceil n/2\rceil} + Z_{\lfloor n/2\rfloor} + \frac{(n-1)(n-3)^2(n-5)}{384}, & \text{if } n = 4m+1. \\ Z_{\lceil n/2\rceil} + Z_{\lfloor n/2\rfloor} + \frac{n(n-2)(n-4)(n-6)}{384}, & \text{if } n = 4m+2. \\ Z_{\lceil n/2\rceil} + Z_{\lfloor n/2\rfloor} + \frac{(n+1)(n-3)^2(n-7)}{384}, & \text{if } n = 4m+3. \end{cases}$$

A rich body of research examines the asymptotic behaviour of the k-planar crossing numbers of complete and complete bipartite graphs [3, 20], and there have also been significant efforts to determine tight bounds on biplanar crossing numbers for these classes of graphs [6, 7, 10]. While tight bounds for $cr(K_n)$ are known for $n \leq 12$ [11, 17], the value of $cr_2(K_n)$ is known only when $n \leq 9$, i.e., $cr_2(K_n) = 0$ if $n < 9$, and $cr_2(K_9) = 1$ [14]. In a survey on the biplanar crossing number, Czabarka et al. [6] posed an open question that asks to determine $cr_2(K_n)$ when n is small.

A 1-*page drawing* Γ of G is a drawing of G on the Euclidean plane such that all the vertices lie on a circle C in Γ and the edges that do not belong to the boundary of C lie interior to C. The 1-*page crossing number* $\nu(G)$ of G is the minimum number of crossings over all the 1-page drawings of G. The *k-page crossing number* $\nu_k(G)$ of G is $\min\{\nu(G_1)+\nu(G_2)+\ldots+\nu(G_k)\}$, where the minimum is taken over all possible decompositions of G into k subgraphs $G_1, \ldots, G_k$, and the order of vertices along C is the same for all these subgraphs. Observe that a 2-page drawing of K_n with t crossings determines a drawing of

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K_n into a single plane with exactly t crossings[1]. Recall that the currently best known upper bound on $cr(K_n)$ is Z_n. Several studies proved that $\nu_2(K_n) = Z_n$ for different values of n [5, 8, 9], and very recently Ábrego et al. [1] proved the equality for every $n \in \mathbb{Z}^+$. However, it is still unknown whether $cr(K_n)$ is strictly smaller than $\nu_2(K_n)$, i.e., we only know that $cr(K_n) \leq \nu_2(K_n) = Z_n$.

An analogous relationship between the k-planar crossing number and $2k$-page drawings of K_n is $cr_k(K_n) \leq cp_{2k}(K_n)$. Interestingly, we observe that W_n, which is the best known upper bound on $cr_2(K_n)$ for large values of n, is equal to the best known upper bound [9] on $\nu_4(K_n)$, when $n = 4m$ for some $m \in \mathbb{Z}^+$; see Section 2. However, the equality does not hold in general since $cr_2(K_9) = 1 < cp_4(K_9) = 3$ [9].

In this paper we propose an improved technique for constructing biplanar drawings, which reduces Owens' [16] upper bound on $cr_2(K_n)$. Although the improvement is obtained by a slight modification of the Owens' construction, this is interesting since no such perturbation is known that can improve the conjectured value of $cr(K_n)$. For small fixed n, we show that $cr_2(K_{10}) = 2$, $cr_2(K_{11}) \in \{4, 5, 6\}$, and for $n \geq 12$, we improve previous upper and lower bounds on $cr_2(K_n)$.

2 Technical Details

De Klerk et al. [9] gave a generalized construction for k-page drawings of complete graphs. For some cases, e.g., when $n = 4m$ and $m \in \mathbb{Z}^+$, their upper bound on 4-page crossing number (thus the biplanar crossing number) of K_n, matches exactly the upper bound obtained by Owens [16] for biplanar drawings of complete graphs. We first briefly recall the construction given by Owens [16], and then the construction given by de Klerk et al. [9].

2.1 Owens' [16] Construction

Given a complete graph K_n (assume for convenience that $n = 4m$, where $m \in \mathbb{Z}^+$), in each plane Owens constructed two vertex disjoint cycles $C = (v_1, \ldots, v_{n/2})$ and $C' = (u_1, \ldots, u_{n/2})$, each with $n/2$ vertices. He constructed the complete graph induced by the vertices on C using a 2-page drawing of $K_{n/2}$, i.e., placing the edges of the ith page, $i \in \{1, 2\}$, interior to the cycle C in the ith plane. The complete graph induced by the vertices on C' was constructed exterior to C' in a similar way. The remaining edges that form a complete bipartite graph $K_{n/2,n/2}$ connecting the vertices of C with the vertices of C', were drawn as follows: for each v_j on C, the first plane contains the edges from v_j to $n/4$ consecutive vertices on C' starting at u_j in clockwise order. The remaining edges of $K_{n/2,n/2}$ are drawn in the second plane symmetrically. Figures 1 (a)–(b) illustrate such a construction for K_{16}.

[1]Imagine the drawing on a sphere, where the first page is drawn on the upper hemisphere, and the second page is drawn on the lower hemisphere.

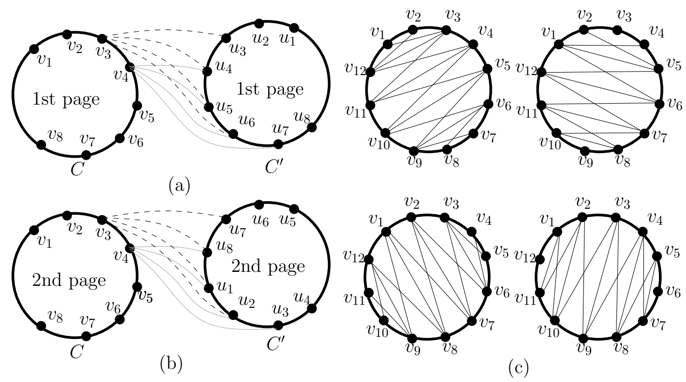

Figure 1: (a)–(b) Owens' [16] Construction. (c) De Klerk et al.'s [9] Construction.

2.2 De Klerk et al.'s [9] Construction

De Klerk et al. [9] showed that for complete graphs K_n, where $n = km$ with $m, k \in \mathbb{Z}^+$, the k-page crossing number of K_n is $\nu_k(K_n) = \frac{1}{12k^2}\left(1 - \frac{1}{2k}\right)n^4 - \frac{1}{4k}n^3 + \left(\frac{7}{24k} + \frac{1}{6}\right)n^2 - \frac{1}{4}n$. We can observe that this is equal to the Owens' [16] upper bound when $k = 4$, as shown in Appendix A.

To construct the k-page drawing, let the vertices of K_n be $v_1, \ldots, v_n$, and let M_i be the set of edges $\{(v_a, v_b) : 1 \leq a, b \leq n, \text{ and } i = (a + b - 2) \mod n\}$. Now draw the edges $M_{(j-1)n/k} \cup \ldots \cup M_{jn/k-1}$ in the jth page. Figure 1 (c) illustrates the construction for K_{12} on 4 pages. Pairing the k pages and placing them in each side of a circle yields a $\lceil k/2 \rceil$-planar drawing, which implies that $cr_k(K_n) \leq cp_{2k}(K_n)$.

3 Biplanar Crossing Number for Small Values of n

In this section we establish some tight bounds on the biplanar crossing number of K_n when n is small. It has been known for a long time that $cr_2(K_n) = 0$ if $n < 9$, and $cr_2(K_9) = 1$ [16]. We may thus assume that $n > 9$. We first prove that $cr_2(K_{10}) = 2$ and $cr_2(K_{11}) \in \{4, 5, 6\}$, and then provide a technique to compute good upper bounds on $cr_2(K_n)$, when $n > 9$.

Biplanar Crossing Numbers of K_{10} and K_{11}. We construct biplanar drawings of K_{10} and K_{11} with exactly 2 and 6 edge crossings, respectively, as shown in Figure 2. We now show that 2 and 4 edge crossings are necessary for K_{10} and K_{11}, respectively. Suppose for a contradiction that K_{10} admits a biplanar drawing with fewer than two edge crossings, and let Γ be such a biplanar drawing. Since K_{10} contains K_9 as a subgraph, Γ must contain exactly one edge crossing. Let (u, v) be

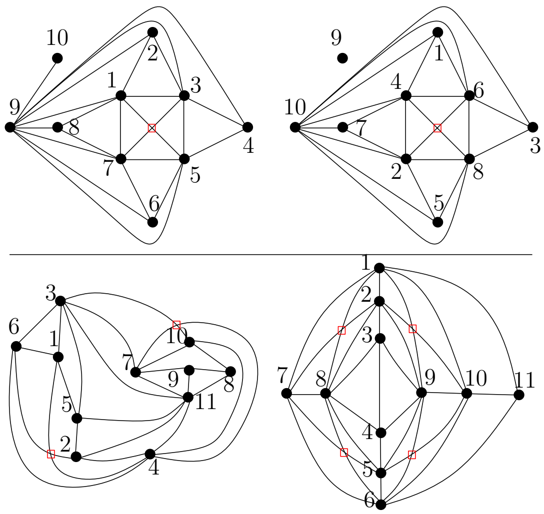

Figure 2: Biplanar drawings of K_{10} and K_{11} with two and six edge crossings, respectively.

an edge on Γ that is involved in this crossing. Then the deletion of v and its incident edges from Γ would give a biplanar drawing of K_9 without any edge crossing, which contradicts that $cr_2(K_9) = 1$.

For $cr_2(K_{11})$, we prove a lower bound of 4 as follows: Let Γ be an optimal biplanar drawing with at most 3 crossings. Observe that Γ must have at least 3 crossings, otherwise we can delete some vertex which is incident to some crossing in Γ to obtain a biplanar drawing of K_{10} with at most one edge crossing. Observe that no vertex v in Γ can be adjacent to two or more edge crossings, because otherwise deletion of v from Γ would yield a biplanar drawing of K_{10} with at most 1 crossing, which contradicts that $cr_2(K_{10}) = 2$. Since every crossing involves four distinct vertices and every vertex in Γ is incident to at most one crossing, Γ must have at least 12 distinct vertices, which is a contradiction.

Biplanar Crossing Numbers of K_n, where $n \geq 12$. Let Γ be a biplanar drawing of K_n. Observe that one can construct a biplanar drawing of K_{n+1} by executing the following steps:

S_1. Pick a vertex v in Γ and create a copy v' of v in each of the two layers of Γ.

S_2. In each layer of Γ, place v' arbitrarily close to v and add the edge (v, v') so that this edge does not introduce any new crossing.

$S3$. Let $W = \{w_1, w_2, \ldots, w_{\lfloor d_v^i/2 \rfloor}\}$ be the neighbors of v in clockwise order in the ith layer of Γ, where d_v^i denotes the degree of v in the ith layer. For each $w \in W$, we add the edge (v', w) closely following the edge (v, w) such that v' appears after v while examining the neighbors of w in clockwise order. The edges from v' to the remaining neighbors $\{w_{\lfloor d_v^i/2 \rfloor + 1}, \ldots, w_{d_v^i}\}$ of v are added symmetrically.

S_4. Remove the edge (v, v') from the second layer.

Let the resulting drawing be Γ'. It is straightforward to verify that the number of newly created crossings among the edges incident to v and v' is exactly $\sum_{i\in[1,2]} \left(\frac{\lfloor d_v^i/2 \rfloor (\lfloor d_v^i/2 \rfloor - 1) + (\lceil d_v^i/2 \rceil - 1)\lceil d_v^i/2 \rceil}{2} \right)$. Moreover, a crossing between two edges (v, w) and (x, y), where $v \notin \{x, y\}$, corresponds to a crossing between (v', w) and (x, y). Therefore, if v is adjacent to c_v^i crossings in the ith layer, then the number of crossings in Γ' is $\sum_{i\in[1,2]} \left(\frac{\lfloor d_v^i/2 \rfloor (\lfloor d_v^i/2 \rfloor - 1) + (\lceil d_v^i/2 \rceil - 1)\lceil d_v^i/2 \rceil}{2} + c_v^i \right)$ more than the number of crossings in Γ.

To obtain better drawings, we choose the vertex v that minimizes the number of newly introduced crossings (break ties arbitrarily). Table 1 shows the number of crossings obtained by the above construction technique, when $n \in [12, 30]$, and the lower bounds using the inequality $cr_2(K_n) \geq \frac{cr_2(K_{n-1})\binom{n}{4}}{\binom{n-1}{4}}$, which is widely used to establish lower bounds on crossing number [7]. Note that the upper bounds of Table 1 are significantly smaller than the values $18, 37, 53, 75, 100, 152$, for $n = 12, \ldots, 17$, obtained by Owens' construction.

4 Upper Bounds on $cr_2(K_n)$

Assume that $n = 8m+4$, where $m \in \mathbb{Z}^+$. We begin with the construction of Owens [16], and later we modify the drawing to improve the number of crossings. We use a slightly different presentation for Owens' [16] construction, which will be more convenient for the subsequent description.

4.1 Basic Construction

Let the planar layers of the drawing be $\mathcal{L}_j$, where $j \in [1, 2]$. In layer $\mathcal{L}_j$, we arrange the vertices into two circles: C_{in}^j and C_{out}^j, where each of them contains $n/2$ vertices. We then embed the cycle C_{in}^j interior to the cycle C_{out}^j such that the resulting embedding of the cycles remains crossing free, as shown in Figures 3(a)–(b). We now draw the edges that connects the vertices of C_{in}^j and C_{out}^j.

In $\mathcal{L}_1$, let the vertices on C_{in}^1 be $v_1, v_2, \ldots, v_{4m+2}$ and the vertices on C_{out}^1 be $u_1, u_2, \ldots, u_{4m+2}$ in clockwise order. For each $j \in \{1, 2, \ldots, 4m+2\}$, connect u_j to the vertices $v_{j-m}, \ldots, v_j, \ldots, v_{j+m}$. Note that the indices wrap around, i.e., for any $v_{j'}$, if $j' < 1$ (respectively, $j' > n/2$), then $v_{j'} = v_{n/2+j'}$ (respectively, $v_{j'} = v_{j'-n/2}$). In the other planar layer $\mathcal{L}_2$, let the vertices on C_{in}^2 be $u_1, u_2, \ldots, u_{2m+1}$ and the vertices on C_{out}^2 be $v_1, v_2, \ldots, v_{2m+1}$ in clockwise order. For each $j \in \{1, 2, \ldots, 4m+2\}$, connect v_j to those vertices of C_{in}^2 that are not incident to v_j in $\mathcal{L}_1$. As illustrated in

Table 1: Upper and lower bounds on $cr_2(K_n)$, where $n \in [12, 30]$.

n	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
$U.B.$	14	26	43	62	81	103	148	176	226	332	469	652	717	958	1261	1399
$L.B.$	6	9	13	19	26	35	46	60	76	95	118	145	176	212	253	299

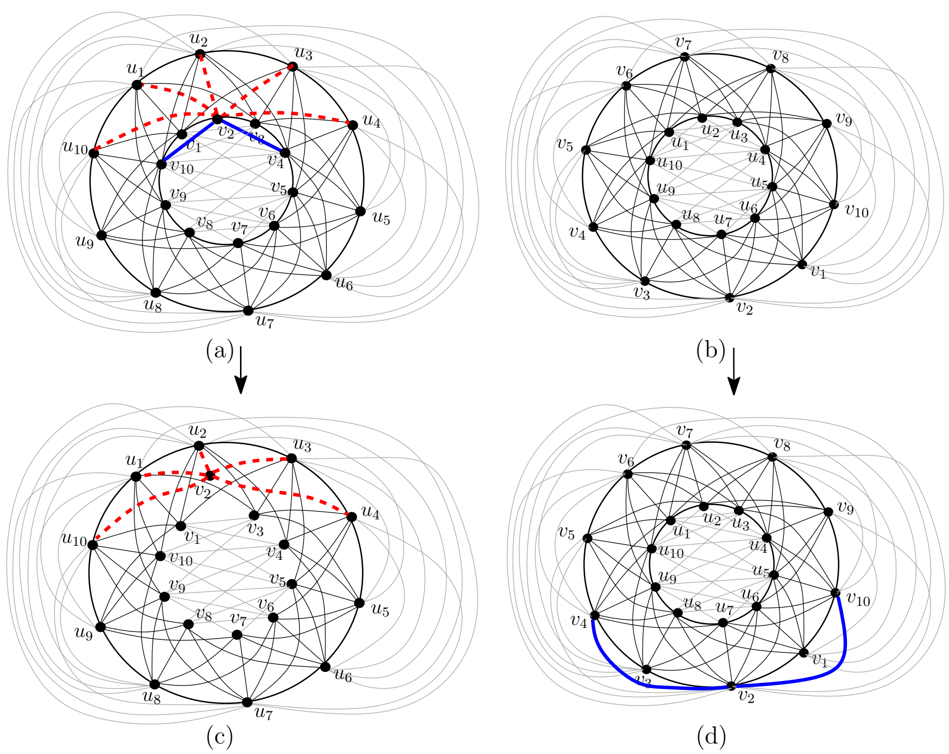

Figure 3: (a)–(b) Basic construction with 324 crossings. (c)–(d) Modification of Γ with 322 crossings. The blue and red edges are shown in bold and dashed lines, respectively.

Figures 3(a)–(b), all these edges lie in the closed region between C^j_{in} and C^j_{out}.

Note that we may now complete the drawing of K_n by adding the edges among $\{u_1, \ldots, u_{n/2}\}$ and the edges among $\{v_1, \ldots, v_{n/2}\}$. For the set $\{v_1, \ldots, v_{n/2}\}$, we construct a 2-page drawing of $K_{n/2}$, where the edges of one page lie inside C^1_{in} and the edges of the other page lie outside of C^2_{out}. Similarly, for the set $\{u_1, \ldots, u_{n/2}\}$, we construct a 2-page drawing of $K_{n/2}$, where the edges of one page lie inside C^2_{in} and the edges of the other page lie outside of C^1_{out}. Let the resulting drawing be Γ. Since this construction is equivalent to that of Owens [16], the number of crossings in Γ is W_n.

4.2 Improvement

We now modify the drawing Γ to obtain a biplanar drawing with fewer crossings, as illustrated in Figures 3(c)–(d).

We first delete the incident edges of v_2 that lie inside C^1_{in}, and then add these edges outside of C^2_{out}, as illustrated in thick lines (blue) in Figure 3. We then remove the edges that lie on the boundary of C^1_{in}, and finally, move the vertex v_2 infinitesimally close to u_2 inside the cycle u_2, v_1, v_3, as shown in dashed lines (red) in Figure 3. Let the resulting drawing be Γ', which has smaller number of crossings than Γ. We now show how to modify the drawing for larger values of n.

Let $n = 16m + 4$, $n' = n/2$, $p = \lfloor n'/4 \rfloor + 1$ and $q = \lceil p/2 \rceil$. We now choose v_q to carry out the modifications, note that for $n = 20$, we have $v_q = v_2$. Let the edges incident to v_q that lie inside C^1_{in} in Γ but moved outside of C^2_{out} in Γ', be the blue edges. Denote the incident edges of v_q that lie outside of C^1_{in} in Γ as the red edges. Let the number of edge crossings on the blue edges in Γ and Γ' be α and α', respectively. Similarly, let the number of edge crossings on the red edges in Γ and Γ' be β and β', respectively. Then the number of edge crossings in Γ' is $W_n + (\alpha' + \beta') - (\alpha + \beta)$. We now briefly describe the computation of $\alpha, \alpha', \beta, \beta'$.

Crossings on the Blue Edges in Γ ***(i.e.,*** α***)***: We partition edge crossings into the following three types.

- A denotes the number of crossings between the edges

(v_q, v_w) and (x, y), where $w \in \{q+2, \ldots, 2p-q\}$, $x \in \{v_{q+1}, \ldots, v_{w-1}\}$, and $y \in \{v_{2p}, \ldots, v_{n'}\}$. Therefore, $A = \sum_{i=q+2}^{2p-q} \sum_{j=q+1}^{i-1} (j + (q-2))$, as shown in Figure 4(a).

- B denotes the number of crossings between the edges (v_q, v_w) and (x, y), where $w \in \{q+2, \ldots, p\}$, $x \in \{v_{q+1}, \ldots, v_{w-1}\}$, and $y \in \{v_{w+1}, \ldots, v_{2p}\}$. Therefore, $B = \sum_{i=q+2}^{p} \sum_{j=q+1}^{i-1} ((2p-j) - i)$, as shown in Figure 4(b).
- C denotes the number of crossings between the edges (v_q, v_w) and (x, y), where $w \in \{p+1, \ldots, 2p-q\}$, $x \in \{v_{q+1}, \ldots, v_p\}$, and $y \in \{v_{w+1}, \ldots, v_{2p}\}$. Therefore, $C = \sum_{i=p+1}^{2p-q} \frac{(2p-q-i-1)(2p-q-i)}{2}$, as shown in Figure 4(c).

The drawing is symmetric with respect to the axis through v_q and its diametrically opposite vertex. Thus the number of crossings removed from Γ by moving the blue edges from the inner layer is exactly $\alpha = 2(A + B + C)$.

Crossings on the Blue Edges in Γ' ***(i.e.,*** α'***)***: We partition these edge crossings into the following three types.

- A' denotes the number of crossings between the edges (v_q, v_w) and (x, y), where $w \in \{q+2, \ldots, p\}$, $x \in \{v_{q+1}, \ldots, v_{w-1}\}$, and $y \in \{v_{w+1}, \ldots, v_{n'}\}$. Therefore, $A' = \sum_{i=q+2}^{p} \sum_{j=q+1}^{i-1} 2p - 1$, as shown in Figure 4(d).
- B' is an upper bound on the number of crossings between the edges (v_q, v_w) and (x, y), where $w \in \{p+1, \ldots, 2p-q\}$, $x \in \{v_{q+1}, \ldots, v_p\}$, and $y \in \{v_{w+1}, \ldots, v_{n'}\}$. Therefore, $B' = \sum_{i=p+1}^{2p-q} \left((p-q)(2p-1) - \frac{(i-p)(i-p+1)}{2} \right)$, as shown in Figure 4(e).
- C' denotes the number of crossings between the edges (v_q, v_w) and (x, y), where $w \in \{p+2, \ldots, 2p-q\}$, $x \in \{v_{p+1}, \ldots, v_{w-1}\}$, and $y \in \{v_{w+1}, \ldots, v_{n'}\}$. Therefore, $C' = \sum_{i=p+2}^{2p-q} \sum_{j=p+1}^{i-1} ((2p-1) - 2(j-p) - (i-j))$, as shown in Figure 4(f).

The drawing is symmetric with respect to the axis through v_q and its diametrically opposite vertex. Hence the number of crossings introduced in Γ' by moving the blue edges to the outer layer is at most $\alpha' = 2(A' + B' + C')$.

Crossings on the Red Edges in Γ ***(i.e.,*** β***)***: The number of crossings created by the edges (v_q, u') and (v_{q+j}, u''), where $1 \le j \le 2m-1$ and u', u'' lie on C^1_{out}, is $(2m-j)(2m-j+1)/2$. Figure 5(a) illustrates a scenario where $m = 4$. Symmetrically, the number of crossings created by the edges (v_q, u') and (v_{q-j}, u'') is

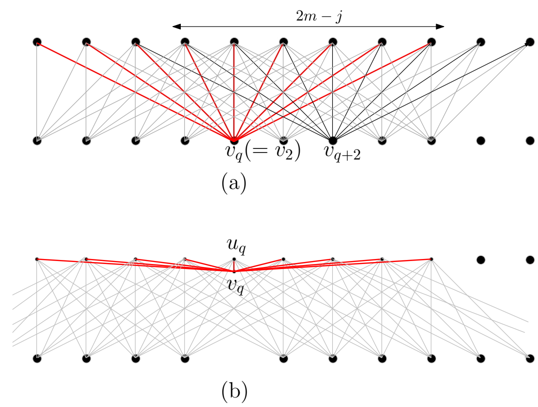

Figure 5: Crossings on the red edges: (a) Γ, and (b) Γ'.

$(2m-j)(2m-j+1)/2$. Hence the number of crossings in the red edges is $\beta = \sum_{j=1}^{2m-1} (2m-j)(2m-j+1)$.

Crossings on the Red edges in Γ' ***(i.e.,*** β'***)***: It is straightforward to observe that the number of such crossings is $\beta' = 2m + 2\sum_{i=1}^{m-1} 2mi$, as illustrated in Figure 5(b) when $m = 4$.

Now the number of crossings in Γ'' is $W_n + (\alpha' + \beta') - (\alpha + \beta)$, which can be simplified using Maple [13] to get an upper bound of $W_n - \frac{1}{384}n^3 + O(n^2)$. Since the modification we carried out for v_q can also be applied around independently to its diametrically opposite vertex, we can obtain a bound of $W_n - \frac{1}{192}n^3 + O(n^2)$. The following theorem summarizes the result of this section.

Theorem 1 *Every K_n, where $n = 16m + 4$ and $m \in \mathbb{Z}^+$, admits a biplanar drawing with at most $W_n - n^3/192 + O(n^2)$ edge crossings.*

5 Conclusion

In this paper we have given bounds on the biplanar crossing number of K_n. For small values of n, our technique for computing $cr_2(K_n)$ is incremental. Hence it is natural to ask whether every optimal biplanar drawing of K_{n+1} contains an optimal drawing of K_n. We proved that $cr_2(K_{11}) \in \{4, 5, 6\}$. It would be interesting to find an analytical argument to prove a better lower or upper bound on $cr_2(K_{11})$. Finally, given $f(n)$, how efficiently can we find k such that $cr_k(K_n) \in \Theta(f(n))$?

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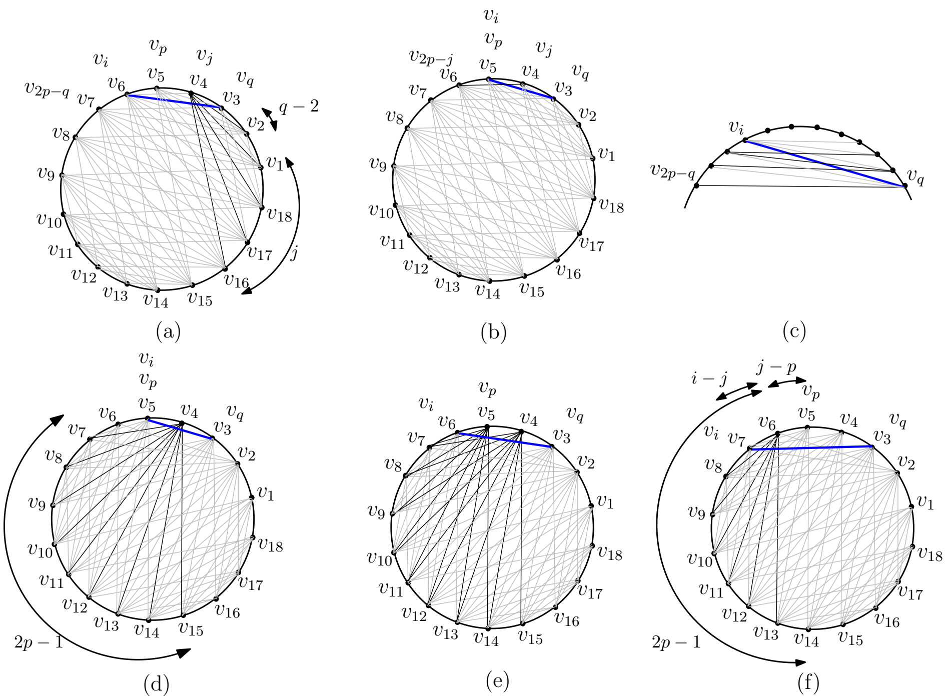

Figure 4: Computation with respect to v_q, where $n = 36$. (a)–(c) Computation of α. (d)–(f) Computation of α'.

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Appendix A

5.1 De Klerk et al.'s [9] Construction

De Klerk et al. [9] showed that for complete graphs K_n, where $n = km$ with $m, k \in \mathbb{Z}^+$, the k-page crossing number of K_n is

$$\begin{aligned}\nu_k(K_n) &= \frac{1}{12k^2}\left(1-\frac{1}{2k}\right)n^4 - \frac{1}{4k}n^3 + \left(\frac{7}{24k}+\frac{1}{6}\right)n^2 - \frac{1}{4}n \\ &= \frac{7}{1536}n^4 - \frac{1}{16}n^3 + \frac{23}{96}n^2 - \frac{1}{4}n, \text{ when } k = 4.\end{aligned}$$

We can observe that this is equal to Owens' [16] upper bound when $k = 4$, as follows. Since $n = 4m$, we may assume $n = 2q$ with $q = 2m$. Then we have

$$\begin{aligned}Z_q &= \frac{1}{4}\left\lfloor\frac{q}{2}\right\rfloor\left\lfloor\frac{q-1}{2}\right\rfloor\left\lfloor\frac{q-2}{2}\right\rfloor\left\lfloor\frac{q-3}{2}\right\rfloor \\ &= \frac{1}{4}\left(\frac{q}{2}\right)\left(\frac{q}{2}-1\right)\left(\frac{q}{2}-1\right)\left(\frac{q}{2}-2\right) \\ &= \frac{1}{1024}q(q-4)^2(q-8).\end{aligned}$$

From Owens' [16] upper bound, we have

$$\begin{aligned}W_n &= Z_{\lceil q\rceil} + Z_{\lfloor q\rfloor} + \frac{n^2(n-4)(n-8)}{384} \\ &= \frac{7}{1536}n^4 - \frac{1}{16}n^3 + \frac{23}{96}n^2 - \frac{1}{4}n \\ &= \nu_4(K_n).\end{aligned}$$