A Note on Minimum-Segment Drawings of Planar Graphs

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Abstract

A straight-line drawing of a planar graph G is a planar drawing of G, where each vertex is mapped to a point on the Euclidean plane and each edge is drawn as a straight line segment. A segment in a straight-line drawing is a maximal set of edges that form a straight line segment. A minimum-segment drawing of G is a straight-line drawing of G, where the number of segments is the minimum among all possible straight-line drawings of G. In this paper we prove that it is NP-complete to determine whether a plane graph G has a straight-line drawing with at most k segments, where $k \geq 3$. We also prove that the problem of deciding whether a given partial drawing of G can be extended to a straight-line drawing with at most k segments is NP-complete, even when G is an outerplanar graph. Finally, we investigate a worst-case lower bound on the number of segments required by straight-line drawings of arbitrary spanning trees of a given planar graph.

1 Introduction

A *planar graph* is a graph that admits a plane embedding. A *plane graph* is a fixed planar embedding of a planar graph. A *straight-line drawing* Γ of a planar graph G is an embedding of G in the Euclidean plane, in which each vertex of G is mapped to a distinct point, each edge of G is a straight line segment, and no two edges intersect except possibly at a common endpoint. A *segment* of Γ is a maximal set of edges in Γ that form a straight line segment. Γ is called a *minimum-segment drawing* of G if the number of segments in Γ is the minimum possible. Figure 1(a) depicts a plane graph G, Figure 1(b) depicts its straight-line drawing with 13 segments, and Figure 1(c) shows a minimum-segment drawing of G with 7 segments.

Dujmović *et al.* [4] showed that $\eta/2$ segments are necessary and sufficient for a straight-line drawing of a tree, where η is the number of odd degree vertices in the tree.

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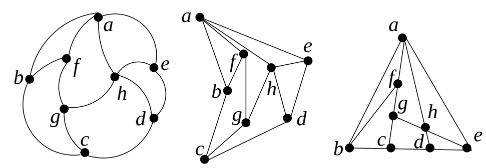

Figure 1: (a) A plane graph G. (b) A straight-line drawing of G. (c) A minimum-segment drawing of G.

They gave optimal bounds on the number of segments in straight-line drawings of outerplanar graphs, plane 2-trees and plane 3-trees, as well as algorithms for constructing straight-line drawings of planar 3-connected graphs with at most $5n/2$ segments, where n is the number of vertices. Later, Samee *et al.* [13] gave a linear-time algorithm for computing minimum-segment drawings of series-parallel graphs, where all the vertices have degree at most three. Recently, Biswas *et al.* [2] gave a linear-time algorithm to obtain minimum-segment convex drawings of 3-connected cubic plane graphs.

A natural question is: what is the time complexity of computing a minimum-segment drawing of a planar graph [2]? Dujmović *et al.* [4] posed the following related questions: (a) Is there a polynomial-time algorithm to draw a given outerplanar graph with the minimum number of segments? (b) What is the minimum c such that every n-vertex planar graph has a plane drawing with at most $cn + O(1)$ segments?

In many applications a graph is drawn emphasizing the drawing of one of its spanning trees, and the other edges are displayed on demand [5, 8, 11]. Given an arbitrary spanning tree, one may want to draw it with the minimum number of segments, where the edges that are not in the spanning tree are to be drawn with polylines or curves. Given a planar graph G, we investigate a worst-case lower bound on the number of segments required by straight-line drawings of arbitrary spanning trees of G. For this purpose, we introduce a new graph parameter for planar graphs, which we define as follows: the *spanning-tree segment complexity* of a planar graph G is the minimum positive integer C such that every spanning tree of G admits a drawing with at most C segments. Observe that any lower bound on C is a lower bound on the number of segments required by straight-line drawings of those spanning trees of G that determine the spanning-tree segment complexity of G. For simplicity, in the rest of the paper we use the term *segment complexity* instead of the term spanning-tree segment complexity.

Main results: Our main results are given below.

(1) Given an arbitrary integer $k \geq 3$, it is NP-complete to decide if a given plane graph has a straight-line drawing with at most k segments (see Section 3).

(2) It is NP-complete to determine whether a given partial drawing of an outerplanar graph G can be extended to a straight-line drawing of G with at most k segments, even when the partial drawing can be extended to a straight-line drawing of G (see Section 4).

(3) In Section 5, we derive lower bounds on segment complexities of different classes of planar graphs (see Table 1).

Graph Class	Lower Bound on C
Maximal outerplanar	$n/6$
Plane 2-tree	$n/6$
Plane 3-tree	$(2n-5)/6$
Plane 3-connected	$n/8$
Plane 4-connected	$n/5$

Table 1: **Lower Bound on Segment Complexity.** Here n denotes the number of vertices.

2 Preliminaries

Here we introduce some preliminary definitions.

Let $G = (V, E)$ be a connected simple graph with vertex set V and edge set E. Let v be a vertex in G. We denote the degree of v by $\deg(v)$. G is called *k-connected*, $k \geq 1$, if the minimum number of vertices, whose removal results in a disconnected graph or a single-vertex graph, is k. An *independent set* S is a subset of V, such that no two vertices of S are adjacent.

A plane graph partitions the plane into connected regions, called *faces*. The unbounded face is called the *outer face* and all other faces are called the *inner faces*. The vertices on the boundary of the outer face are called the *outer vertices* and all other vertices are called the *inner vertices*. A *maximal planar graph* is a planar graph, where addition of any edge results in a nonplanar graph.

An *outerplanar graph* is a planar graph that admits a plane embedding, where all its vertices are on the outer face. We call such an embedding an *outerplanar embedding*. An outerplanar graph G is called a *maximal outerplanar graph* if addition of any edge to G results in a graph that does not admit an outerplanar embedding.

An *arrangement* of a set L of n lines is the subdivision of the plane induced by L, where the vertices are the intersection points of the lines. An arrangement $A(L)$ of L is called *simple* if no three lines intersect at the same point and no two lines are parallel. In this paper we consider simple arrangements only. An *arrangement graph* $G(L)$ is the graph obtained from $A(L)$ by removing the infinite half edges (see Figure 2). The following lemma

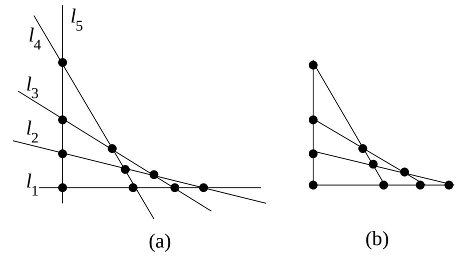

Figure 2: (a) An arrangement of 5 lines. (b) Arrangement graph.

gives some properties of arrangement graphs.

Lemma 1 [Bose *et al.* [3]] *Let G be a 2-connected graph, where each vertex has degree at most four. Then G is an arrangement graph of a set of l lines if and only if G admits a straight-line drawing Γ such that:*

1. *Each segment contains $l-2$ edges.*

2. *All the vertices of degree two and degree three in G are on the outer face of Γ.*

3. *Each vertex of degree two is the endpoint of exactly two segments and each vertex of degree three is the endpoint of exactly one segment. No segment has an endpoint that is a vertex of degree four.*

4. *The number of segments is $l = n_2 + (n_3/2)$, where n_2 and n_3 are the number of vertices of degree two and degree three, respectively.*

We call Γ an *arrangement drawing* of G.

3 Minimum-Segment Drawing

In this section we prove that it is NP-complete to decide whether a plane graph has a straight-line drawing with a given number of segments. We first need the following two lemmas.

Lemma 2 *Let G be a graph with $l(l-1)/2$ vertices and $l(l-2)$ edges, where $l \geq 3$. Let the number of degree two and degree three vertices be n_2 and n_3, respectively. Then G is an arrangement graph if and only if G admits a straight-line drawing Γ with l segments, where $l = n_2 + (n_3/2)$.*

Proof. By Lemma 1, if G is an arrangement graph, then G admits a drawing with $l = n_2 + (n_3/2)$ segments.

We thus assume that Γ is a straight-line drawing of G with l segments and then prove that Γ is an arrangement drawing of G. By Lemma 1, this will imply that G is an arrangement graph.

We first prove that Γ satisfies Property 1 of Lemma 1. Suppose for a contradiction that there exists a segment l' that contains at least $l-1$ edges. Therefore, l' is intersected by at least l other straight lines. Thus the number of segments in Γ is at least $l+1$, a contradiction. Thus each segment contains at most $l-2$ edges. Since the number of edges in Γ is $l(l-2)$ and there are l segments, therefore each segment contains exactly $l-2$ edges, which proves the property.

We next prove that Γ satisfies Property 2 of Lemma 1. Since each segment of Γ contains $l-2$ edges, it is intersected by all the other $l-1$ segments in Γ. Thus Γ contains all pairwise intersections of the l segments and any extension of the segments beyond their endpoints will not create any new crossings. We now claim that the vertices of degree two and three must lie on the outer face of Γ. Suppose for a contradiction that there exists an inner vertex v such that $\deg(v) < 4$. Then extension of the segments at v towards infinity will intersect the outer face of Γ, which contradicts the fact that Γ contains all pairwise intersections of the l segments.

Finally, we prove that Γ satisfies Property 3 of Lemma 1. The number of vertices in G is $l(l-1)/2$ and the number of segments in Γ is l. Thus, Γ contains all pairwise intersections of the l segments and each vertex v in Γ must be an intersection point of two different segments. Consequently, if $\deg(v) = 4$, then v cannot be an endpoint of any of those two different segments. Similarly, if $\deg(v) = 3$, then v is the endpoint of one of those two different segments. If $\deg(v)$ is two, then v must be the endpoints those two different segments. $\square$

Lemma 3 *An arrangement drawing of an arrangement graph G is a minimum-segment drawing of G* [1].

Proof. Let G be an arrangement graph of l lines. By Lemma 2, G admits a drawing with at most l segments. We now prove that any straight-line drawing of G contains at least l segments.

Let w be any vertex of G. Observe that if $deg(w) = 2$, then the two neighbors x and y of w are adjacent. Since wxy form a triangle, in any minimum-segment drawing of G, xw and yw must lie on different segments. Therefore, each vertex of degree two will be the endpoint of at least two segments in any minimum-segment drawing.

If $deg(w) = 3$, then let x, y and z be the three neighbors of w. Observe that at most two of the edges wx, wy and wz can lie on the same segment, which implies that w must be an endpoint of the segment that contains the remaining edge. Therefore, each vertex of degree three will be the endpoint of at least one segment in any minimum-segment drawing.

Let the number of vertices of degree two and degree three be n_2 and n_3, respectively. Then any minimum-segment drawing must contain at least $(2n_2+n_3)/2$ segments. By Lemma 2, $(2n_2 + n_3)/2 = l$. $\square$

We are now ready to prove that it is NP-complete to decide whether a plane graph admits a straight-line drawing with a given number of segments. We define the Min-Seg-Draw problem as follows:

INSTANCE : A plane graph G, where the vertices are uniquely labeled, and an integer $k \geq 3$.

QUESTION : Is there a straight-line drawing Γ of G with at most k segments?

We reduce an NP-hard problem, Arrangement-Graph-Recognition [3], to Min-Seg-Draw.

INSTANCE : A plane 2-connected graph G with $k(k-1)/2$ vertices and $k(k-2)$ edges, where the degree of each vertex of G is at most four and all the vertices of degree two and degree three are on the outer face of G.

QUESTION : Is G an arrangement graph?

We now have the following theorem.

Theorem 4 Min-Seg-Draw *is NP-Complete.*

Proof. Given a drawing Γ, we can certify whether Γ is a straight-line drawing with at most k segments in polynomial time. We can also verify in polynomial time whether Γ is a drawing of G or not as follows: We first compare the outer face of G with outer face of Γ. If they are different then Γ is not a drawing of G. Otherwise, for each vertex v, we compare the clockwise ordering of the neighbors of v in Γ with the corresponding ordering of neighbors of v in G. If for any vertex the two orderings are different, then Γ is not a drawing of G. In all other cases Γ is a drawing of G. Thus the problem is in NP.

To prove the problem is NP-hard we reduce Arrangement-Graph-Recognition to Min-Seg-Draw. Let G be an instance of Arrangement-Graph-Recognition. We assign a unique label to each vertex of G. The resulting labeled graph G' is an instance of Min-Seg-Draw.

By Lemma 2 and Lemma 3, G' is an arrangement graph if and only if G' admits a straight-line drawing with at most k segments. Therefore, the answer to the instance of Min-Seg-Draw is the answer to the instance of Arrangement-Graph-Recognition. $\square$

[1]The authors wish to thank Therese Biedl for bringing their attention to an error in the proof of this lemma. A revised proof is going to appear in the Journal version.

4 Minimum-Segment Drawing with Given Partial Drawing

Drawing a graph extending a given partial drawing is a well-studied problem [1, 7]. The problem of deciding whether a given partial drawing can be extended to a straight-line drawing of a given planar graph has been shown to be NP-complete by Patrignani [12]. We show that given a planar graph G and the drawing of a subgraph of G, determining whether the drawing can be extended to a straight-line drawing of G with at most k segments is NP-complete, even when G is outerplanar and the partial drawing can be extended to some straight-line drawing of G. A formal definition of the decision problem is as follows:

INSTANCE : An outerplanar graph G, a straight-line drawing Γ' of a subgraph G' of G such that Γ' can be extended to some straight-line drawing of G, and an integer $k \geq 1$.

QUESTION : Is there a straight-line drawing of G, which includes Γ', with at most k segments?

We call this problem PARTIAL-MIN-SEG. We prove NP-hardness by reduction from a strongly NP-complete problem 3-PARTITION [6] which is defined as follows.

INSTANCE : A set of $3m$ nonzero positive integers $S=\{a_1, a_2, \ldots, a_{3m}\}$ and an integer $B > 0$, where $a_1 + a_2 + \ldots + a_{3m} = mB$ and $B/4 < a_i < B/2$, $1 \leq i \leq 3m$.

QUESTION : Can S be partitioned into m subsets $S_1, S_2, \ldots, S_m$ such that $|S_1|=|S_2|=\ldots=|S_m|=3$ and the sum of the integers in each subset is equal to B?

Observe that the NP-completeness of 3-PARTITION holds even when each integer of S is greater than one.

A *fan* f is a maximal outerplanar graph with n vertices, where a vertex v has degree $n-1$. We call v the *apex* of f and all the other vertices the *path vertices*. We call the edges that are incident to v the *ribs* of f.

We now have the following theorem.

Theorem 5 PARTIAL-MIN-SEG *is NP-Complete.*

Proof. We can prove that the problem is in NP as in the proof of Theorem 4. We now create an instance of PARTIAL-MIN-SEG from an instance $B, S=\{a_1, \ldots, a_{3m}\}$, of 3-PARTITION, where each integer of S is greater than one.

We construct in polynomial time an outerplanar graph G as in Figure 3(a), where $3m+2$ fans have a common apex v. Each fan f_i, $1 \leq i \leq 3m$, contains exactly a_i path vertices. There are two more fans f' and f'' which contain $m+1$ path vertices and $mB+m+1$ path vertices, respectively. The size of G is polynomial since 3-PARTITION is strongly NP-complete. We denote by G' the subgraph of G induced by the vertices of f' and f''. We construct a straight-line outerplanar drawing Γ' of G' that satisfies the following (a)–(c).

(a) Let $w_1, \ldots, w_{m+1}$ be the path vertices of f' ordered clockwise around v and let $u_1, u_2, \ldots, u_{mB+m+1}$ be the path vertices of f'' ordered clockwise around v. For each j, $1 \leq j \leq m{+}1$, rib (w_j, v) of f' and rib (v, u_i) of f'' form a segment, $i{=}B(j{-}1){+}j$. These segments are shown in bold lines in Figure 3(a).

(b) The edges between path vertices of f' and f'' are drawn on two different segments. All the other edges of f'' are drawn as separate segments, which are shown as thin lines in Figure 3(a).

The gray region in Figure 3(a) shows Γ'. By construction, the number of segments in Γ' is $k' = mB+m+3$. We can observe that G admits some straight-line drawing that includes Γ'. We now ask whether G admits a straight-line drawing, including Γ', with at most $k = mB{+}m{+}3{+}3m$ segments. In the following we prove that such a drawing exists if and only if the given instance of 3-PARTITION has a positive answer.

We first assume that the 3-PARTITION we considered has a positive answer. In other words, S can be partitioned into m subsets $S_1, S_2, \ldots, S_m$ such that each S_i, $1 \leq i \leq m$, contains exactly three integers and the sum of the integers in S_i is equal to B. Since each fan f_i, $1 \leq i \leq 3m$, requires at least one new segment to draw the edges between path vertices, any extension of Γ' requires at least $k' + 3m = k$ segments. Let E' be the set of ribs of f'' that are not drawn on the same segment as any rib of f'. To obtain a straight-line drawing of G with exactly k segments, we need to draw each rib of each f_i on the same segment as one of the ribs in E'. Let e_1 and e_2 be any two consecutive ribs of f' in Γ' and let e'_1 and e'_2 be the ribs of f'' that are on the same segments as e_1 and e_2, respectively. Then the number of ribs between e'_1 and e'_2 is B. Let the integers in any S_i, $1 \leq i \leq m$, be a, b and c, where $a+b+c = B$. We place the fans that have a, b and c path vertices inside the face bounded by the ribs e_1 and e_2 in Γ' in such a way that each rib of a, b and c shares a segment with one of the ribs of f'' between e'_1 and e'_2. In this way, we place the three fans with path vertices corresponding to the integers in S_i in the face bounded by the pair of ribs e_i and e_{i+1}, where $1 \leq i \leq m$. The final drawing Γ of G that includes Γ' has exactly k segments.

We now assume that the given instance of 3-PARTITION has a negative answer and hence the set S cannot be partitioned into m subsets as described above. We prove that in that case G does not have a drawing with k or fewer segments including Γ'. Recall that any extension of Γ' to some straight-line drawing of G requires at least k segments. Suppose for a contradiction that G has a drawing Γ including Γ' with exactly k segments. Then each rib of each f_i, $1 \leq i \leq 3m$, must be drawn on the same segment as one of the ribs of E'. Since Γ is a planar drawing of G, each f_i must be placed inside a face bounded by two consecutive ribs of f'. Therefore, the fans $f_1, \ldots, f_{3m}$ are partitioned into m subsets and the total number of ribs for each set of fans must be B. Since $a_i < B/2$, no two fans can cumulatively have B ribs. Similarly, since $B/4 < a_i$, four or more fans cumulatively have more than B ribs. Therefore, each subset must contain exactly three fans. Hence each subset of fans corresponds to a subset S_i of S that contains three integers whose sum is B. This gives a solution to the given instance of 3-Partition, a contradiction. Therefore, G cannot have a drawing with at most k segments including Γ'. □

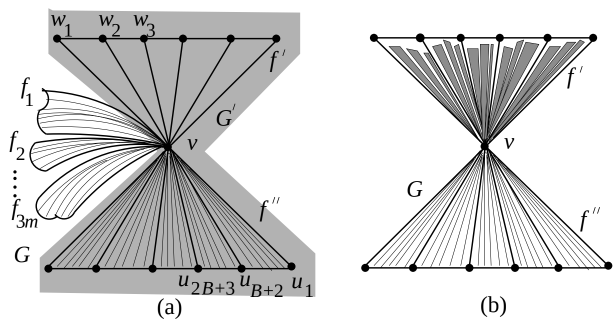

Figure 3: Illustration for the proof of Theorem 5.

5 Segment Complexity of Planar Graphs

In this section we give lower bounds on the segment complexities of different classes of planar graphs. Recall that segment complexity of a planar graph G is the minimum positive integer C, such that any spanning tree of G admits a drawing with at most C segments. Dujmović *et al.* [4] proved that if the number of odd degree vertices in a tree is η, then any straight-line drawing of the tree requires at least $\eta/2$ segments. If a spanning tree T of G has x leaves, then $x \leq \eta$ and any straight-line drawing of the tree requires at least $x/2$ segments. Thus we have the following observation.

Observation 1 *Let G be a planar graph with a spanning tree T, where the number of leaves is x. Then $x/2$ is a lower bound on the segment complexity of G.*

By Observation 1, we obtain a lower bound on the segment complexity of a planar graph by finding a spanning tree with many leaves. A *maximum-leaf spanning tree* of a graph G is a spanning tree of G, where the number of leaves is the maximum possible. It is NP-hard to find a maximum-leaf spanning tree in a graph G, even when G is a planar bipartite graph with maximum degree four [10]. In the following we obtain lower bounds on segment complexities for maximal outerplanar graphs, plane 2-trees, plane 3-trees, plane 3-connected graphs and plane 4-connected graphs.

A graph G with n vertices is a *k-tree* if G satisfies the following properties:

(a) If $n = k$, then G is the complete graph K_n.

(b) If $n > k$, then G can be constructed from a k-tree G' with $n-1$ vertices by adding a vertex adjacent to exactly k vertices of G', where the induced graph of these k-vertices is a complete graph.

Every k-tree $G = (V, E)$ admits an ordered partition $\pi = (V_1, V_2, ..., V_m)$ of V that satisfies the following:

(a) V_1 contains k vertices inducing a complete graph and every other partition contains only one vertex.

(b) Let G_k, $1 \leq k \leq m$, be the subgraph of G induced by $V_1 \cup V_2 \cup ... \cup V_k$. Then G_k, $k > 1$, is a k-tree obtained by adding V_k to G_{k-1}.

Every 2-tree is 3-colorable. The following lemma finds a spanning tree of a plane 2-tree using graph coloring.

Lemma 6 *Let G be a plane 2-tree with $n \geq 3$ vertices. Let S be a set of vertices that are assigned the same color c in a 3-coloring of G. Then $G - S$ is a tree.*

Proof. Let $\pi = (V_1, V_2, ..., V_m)$ be an ordered partition of V. We use induction on m. The case when $m = 1$ is straightforward since G_1 is K_2. We thus assume that for each G_i, $1 \leq i \leq m-1$, $G_i - S_i$ is a tree. Now consider $G_m = G$. Let z be the vertex in V_m and let x and y be its neighbors. By the definition of plane 2-tree, x and y are adjacent. We assume that G is colored with colors c_1, c_2, c_3 such that color(x)=c_1, color(y)=c_2 and color(z)=c_3. If c=c_3, then $G-S$=$G_{m-1}-S_{m-1}$ is a tree by induction. If c= c_1 or $c = c_2$, then $G-S$ is formed by connecting vertex z to $G_{m-1}-S_{m-1}$ with exactly one edge. Since $G_{m-1}-S_{m-1}$ is a tree, $G-S$ is a tree. □

We use Lemma 6 to prove the following theorem.

Theorem 7 *Let G be a maximal outerplanar graph with $n \geq 3$ vertices. Then the segment complexity of G is at least $n/6$.*

Proof. We show that every maximal outerplanar graph G with $n \geq 3$ vertices has a spanning tree T, where the number of leaves in T is at least $n/3$. By Observation 1, this will prove that the segment complexity of G is at least $n/6$.

Every maximal outerplanar graph admits a 3-coloring. Let S_i, $1 \leq i \leq 3$, be a set of vertices that are assigned color i in a 3-coloring of G. The set with the maximum cardinality among S_1, S_2 and S_3 will have at least $n/3$ vertices. Without loss of generality assume that the set with the maximum cardinality is S_1, that is, $|S_1| \geq n/3$. Every outerplanar graph is a plane 2-tree. Therefore, by Lemma 6, $G - S_1$ is a tree, which we denote by T'.

Let v be a vertex in S_1. Since S_1 is an independent set and G is connected, there exists an edge (x, v), where x is a vertex of T'. Therefore, we can connect v to x to obtain another tree that contains v as one of its leaves. By making the vertices of S_1 into leaves in T', we can obtain a spanning tree T with at least $n/3$ leaves. □

In a similar technique as we used in the proof of Theorem 7 we can prove the following theorem.

Theorem 8 *Let G be a plane 2-tree with $n \geq 3$ vertices. Then the segment complexity of G is at least $n/6$.*

Every plane 3-tree G has a spanning tree with at least $(2n-5)/3$ leaves [14]. Furthermore, Kleitman and West [9] proved that every plane 4-connected graph has a spanning tree with at least $2n/5$ leaves, and every plane 3-connected graph has a spanning tree with at least $n/4$ leaves. We combine these results with Observation 1 to obtain the following theorem.

Theorem 9 *The segment complexities of plane 3-trees, plane 4-connected graphs and plane 3-connected graphs are at least $(2n-5)/6$, $n/8$ and $n/5$, respectively.*

6 Conclusion

Among other results, we have proved that it is NP-complete to decide whether a plane graph G has a straight-line drawing with k segments. This motivates finding approximation algorithms for minimum-segment drawings of different classes of planar graphs.

A minimum-segment drawing becomes more visually coherent if we minimize the number of distinct lines that contain the segments of the drawing. We call such a drawing a *minimum-line drawing*. Figures 4(a) and (b) depict two different minimum-segment drawings of a tree, where the number of lines are 7 and 6, respectively. Since the number of distinct slopes used in both figures is two, the problem of computing a minimum-line drawing is different from the problem of minimizing the number of distinct slopes.

Open Problem: *Compute non-trivial upper bounds on the number of lines required for minimum-line drawings of different classes of planar graphs.*

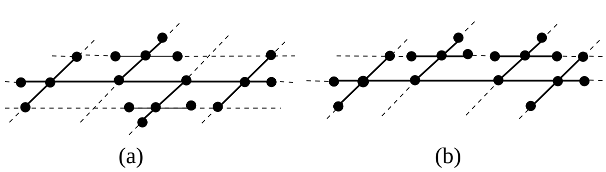

Figure 4: (a) A minimum-segment drawing. (b) A minimum-segment drawing, which is also a minimum-line drawing. Lines are shown in dotted lines.

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