Acyclic Colorings of Graph Subdivisions

Debajyoti Mondal[1], Rahnuma Islam Nishat[2],
Sue Whitesides[2], and Md. Saidur Rahman[3]

[1]Department of Computer Science, University of Manitoba
[2]Department of Computer Science, University of Victoria
[3]Department of Computer Science and Engineering,
Bangladesh University of Engineering and Technology (BUET)
debajyoti_mondal_cse@yahoo.com, rnishat@uvic.ca,
sue@uvic.ca, saidurrahman@cse.buet.ac.bd

Abstract. An acyclic coloring of a graph G is a coloring of the vertices of G, where no two adjacent vertices of G receive the same color and no cycle of G is bichromatic. An acyclic k-coloring of G is an acyclic coloring of G using at most k colors. In this paper we prove that any triangulated plane graph G with n vertices has a subdivision that is acyclically 4-colorable, where the number of division vertices is at most $2n-6$. We show that it is NP-complete to decide whether a graph with degree at most 7 is acyclically 4-colorable or not. Furthermore, we give some sufficient conditions on the number of division vertices for acyclic 3-coloring of subdivisions of partial k-trees and cubic graphs.
Keywords. Acyclic coloring, Subdivision, Triangulated plane graph.

1 Introduction

A *coloring* of a graph G is an assignment of colors to the vertices of G such that no two adjacent vertices receive the same color. A coloring of G is an *acyclic coloring* if G has no bichromatic cycle in that coloring. The *acyclic chromatic number* of G is the minimum number of colors required in any acyclic coloring of G. See Figure 1 for an example.

The large number of applications of acyclic coloring has motivated much research [4, 7]. For example, acyclic coloring of planar graphs has been used to obtain upper bounds on the volume of 3-dimensional straight-line grid drawings of planar graphs [6]. Consequently, acyclic coloring of planar graph subdivisions can give upper bounds on the volume of 3-dimensional polyline grid drawings, where the number of division vertices gives an upper bound on the number of bends sufficient to achieve that volume. As another example, solving large scale optimization problems often makes use of sparse forms of Hessian matrices; acyclic coloring provides a technique to compute these sparse forms [7].

Acyclic coloring was first studied by Grünbaum in 1973 [8]. He proved an upper bound of nine for the acyclic chromatic number of any planar graph G,

[2] Work is supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC) and University of Victoria.

with $n \geq 6$ vertices. He also conjectured that five colors are sufficient for acyclic coloring of any planar graph. His upper bound was improved many times [1, 9, 10] and at last Borodin [3] proved that five is both an upper bound and a lower bound. Testing acyclic 3-colorability is NP-complete for planar bipartite graphs with maximum degree 4, and testing acyclic 4-colorability is NP-complete for planar bipartite graphs with the maximum degree 8 [13].

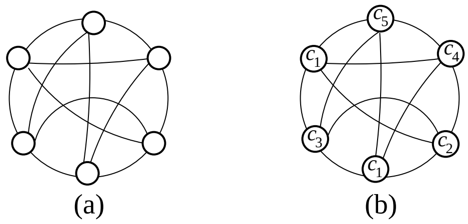

Fig. 1. (a) A graph G and (b) an acyclic coloring of G using five colors c_1–c_5.

Subdividing an edge (u, v) of a graph G is the operation of deleting the edge (u, v) and adding a path $u(= w_0), w_1, w_2, \ldots, w_k, v(= w_{k+1})$ through new vertices $w_1, w_2, \ldots, w_k$, $k \geq 1$, of degree two. A graph G' is said to be a *subdivision* of a graph G if G' is obtained from G by subdividing some of the edges of G. A vertex v of G' is called an *original vertex* if v is a vertex of G; otherwise, v is called a *division* vertex. Wood [15] observed that every graph has a subdivision with two division vertices per edge that is acyclically 3-colorable. Recently Angelini and Frati [2] proved that every plane graph has a subdivision with one division vertex per edge that is acyclically 3-colorable.

Main Results : We study acyclic colorings of subdivisions of graphs and prove the following claims.

(1) Every cubic graph with n vertices has a subdivision that is acyclically 3-colorable, where the number of division vertices is $3n/4$. Every triconnected plane cubic graph has a subdivision that is acyclically 3-colorable, where the number of division vertices is at most $n/2$. Every Hamiltonian cubic graph has a subdivision that is acyclically 3-colorable, where the number of division vertices is at most $n/2 + 1$. See Section 2.
(2) Every partial k-tree, $k \leq 8$, has a subdivision with at most one division vertex per edge that is acyclically 3-colorable. See Section 2.
(3) Every triangulated plane graph G with n vertices has a subdivision with at most one division vertex per edge that is acyclically 4-colorable, where the total number of division vertices is at most $2n - 6$. See Section 3.
(4) It is NP-complete to decide whether a graph with degree at most 7 is acyclically 4-colorable or not. See Section 4.

2 Preliminaries

In this section we present some definitions and preliminary results that are used throughout the paper. See also [12] for graph theoretic terms.

Let $G = (V, E)$ be a connected graph with vertex set V and edge set E. The degree $d(v)$ of a vertex $v \in V$ is the number of neighbors of v in G. A *subgraph* of a graph $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. If G' contains exactly the edges of G that join vertices in V', then G' is called the *subgraph induced by* V'. If $V' = V$ then G' is a *spanning subgraph* of G. A *spanning tree* is a spanning subgraph of G that is a tree. The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph. G is said to be *k-connected* if $\kappa(G) \geq k$.

Let $P = u_0, u_1, u_2, \ldots, u_{l+1}$, $l \geq 1$, be a path of G such that $d(u_0) \geq 3$, $d(u_1) = d(u_2) = \ldots = d(u_l) = 2$, and $d(u_{l+1}) \geq 3$. Then we call the subpath $P' = u_1, u_2, \ldots, u_l$ of P a *chain* of G. A *subsequence* is a sequence that can be derived from another sequence by deleting some elements without changing the order of the remaining elements. An *ear* of a graph G is a maximal path whose internal vertices have degree two in G. An *ear decomposition* of G is a decomposition $P_1, \ldots, P_k$ such that P_1 is a cycle and P_i, $2 \leq i \leq k$, is an ear of $P_1 \cup \ldots \cup P_i$.

Throughout the paper, division vertices are colored gray in all the figures. We now have the following two facts.

Fact 1. *Let G be a graph with two distinct vertices u and v and let G' be a graph obtained by adding a chain $w_1, \ldots, w_k$ between the vertices u and v of G. Let G be acyclically 3-colorable such that the colors of u and v are different. Then G' is acyclically 3-colorable.*

Proof. In an acyclic coloring of G that colors vertices u and v differently, let the colors of vertices u and v be c_1 and c_2, respectively. For each w_i, $i = 1, 2, \ldots, k$, we assign color c_3 when i is odd and color c_1 when i is even as in Figure 2(a). Clearly, no two adjacent vertices of G' have the same color. Therefore, the coloring of G' is a valid 3-coloring. Suppose for a contradiction that the coloring of G' is not acyclic. Then G' must contain a bichromatic cycle C. The cycle C either contains the chain $u, w_1, w_2, \ldots, w_k, v$ or is a cycle in G. C cannot contain the chain since the three vertices u, v and w_1 are assigned three different colors c_1, c_2 and c_3, respectively. Thus we can assume that C is a cycle in G. Since G does not contain any bichromatic cycle, C cannot be a bichromatic cycle, a contradiction. □

Fact 2. *Let G be a biconnected graph with n vertices and let $P_1 \cup \ldots \cup P_k$ be an ear decomposition of G where each ear P_i, $2 \leq i \leq k$, contains at least one internal vertex. Then G has a subdivision G', with at most $k-1$ division vertices, that is acyclically 3-colorable.*

Proof. We prove the claim by induction on k. The case $k = 1$ is trivial since P_1 is a cycle, which is acyclically 3-colorable. Therefore we assume that $k > 1$ and that the claim is true for the graphs $P_1 \cup \ldots \cup P_i$, $1 \leq i \leq k-1$. By induction, $G - P_k$ has a subdivision G'' that is acyclically 3-colorable and that has at most $k-2$ division vertices. Let the end vertices of P_k in G be u and v. If u and v have different colors in G'' then we can prove in a similar way as in the proof of Fact 1 that G has a subdivision G' that is acyclically 3-colorable and that has the same number of division vertices as G'', which is at most $k-2$. Otherwise, u and v have the same color in G''. Let the color of u and v be c_1 and let the two other colors in G'' be c_2 and c_3. If P_k contains more than one internal vertex then we assign the colors c_2 and c_3 to the vertices alternately. If P_k contains only one internal vertex v then we subdivide an edge of P_k once. We color v with c_2 and the division vertex with c_3 as shown in Figure 2(b). In both cases we can prove in a similar way as in the proof of Fact 1 that G' has no bichromatic cycle. Moreover, the number of division vertices in G' is at most $(k-2)+1 = k-1$. □

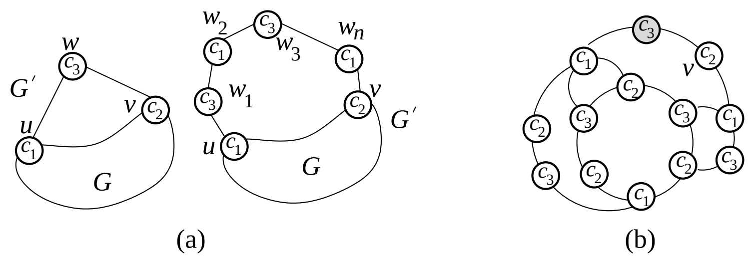

Fig. 2. Illustration for the proof of (a) Fact 1 and (b) Fact 2.

Let $G = (V, E)$ be a 3-connected plane graph and let (v_1, v_2) be an edge on the outer face of G. Let $\pi = (V_1, V_2, ..., V_l)$ be an ordered partition of V. Then we denote by G_k, $1 \leq k \leq l$, the subgraph of G induced by $V_1 \cup V_2 \cup ... \cup V_k$ and by C_k the outer cycle of G_k. We call the vertices of the outer face the *outer vertices*. An *outer chain* of G_k is a chain on C_k. We call π a *canonical decomposition* of G with an edge (v_1, v_2) on the outer face if the following conditions are satisfied [12].

(a) V_1 is the set of all vertices on the inner face that contains the edge (v_1, v_2). V_l is a singleton set containing an outer vertex v, $v \notin \{v_1, v_2\}$.
(b) For each index k, $2 \leq k \leq l-1$, all vertices in V_k are outer vertices of G_k and the following conditions hold:
  (1) if $|V_k| = 1$, then the vertex in V_k has two or more neighbors in G_{k-1} and has at least one neighbor in $G - G_k$; and
  (2) If $|V_k| > 1$, then V_k is an outer chain of G_k.

Figure 3 illustrates a canonical decomposition of a 3-connected plane graph.

A *cubic graph* G is a graph such that every vertex of G has degree exactly three. Every cubic graph has an acyclic 4-coloring [14]. We can get an acyclic

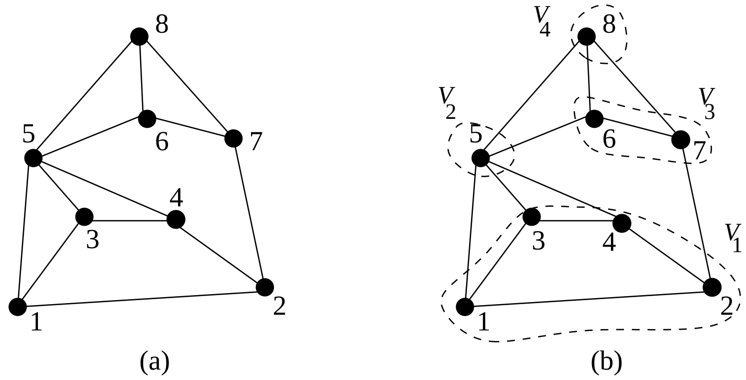

Fig. 3. (a) A 3-connected plane graph G and (b) a canonical decomposition of G.

3-coloring of a subdivision G' of G with $3n/4$ division vertices from an acyclic 4-coloring of G as follows. Let c_4 be the color of the vertices that belong to the smallest color class and let the other colors be c_1, c_2 and c_3. We first assign to each vertex v with color c_4 a different color $c \in \{c_1, c_2, c_3\}$. If all three neighbors of v have different colors, we assign any one of the three colors c_1, c_2, c_3 to v. Otherwise, we assign v the color that is not assigned to any of its neighbors. We then subdivide each of the three edges incident to v with a vertex u such that u is assigned a color c_1, c_2 or c_3, which is not assigned to the end vertices of the edge. It is now straightforward to observe that the resulting subdivision G' of G is acyclically colored with 3 colors. Since the number of vertices with color c_4 is at most $n/4$, the number of division vertices in G' is at most $3n/4$.

In the following two lemmas we show two subclasses of cubic graphs for which we can obtain acyclic 3-colorings using smaller number of division vertices.

Lemma 1. *Let G be a triconnected plane cubic graph with n vertices. Then G has a subdivision G' with at most one division vertex per edge that is acyclically 3-colorable and has at most $n/2$ division vertices.*

Proof. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a canonical decomposition of G. G_1 is a cycle, which can be colored acyclically with three colors c_1, c_2 and c_3. Since every vertex of G has degree three, each $V_i, 1 < i < k$, has exactly two neighbors in G_{i-1}. Therefore, V_i corresponds to an ear of G_i and $V_1 \cup \ldots \cup V_{k-1}$ corresponds to an ear decomposition of G_{k-1}. By Fact 2, G_{k-1} has a subdivision G'_{k-1} that is acyclically 3-colorable with at most $k-2$ division vertices. We now add the singleton set V_k to G'_{k-1}. First, suppose that all the three neighbors of V_k have the same color c_1. Then V_k is assigned color c_2 and any two edges incident to V_k are subdivided with division vertices of color c_3 as in Figure 4(a). In all other cases, at most one edge incident to V_k is subdivided. Thus any cycle that passes through the vertex V_k uses three different colors. Since G'_{k-1} has at most $k-2$ division vertices and the last partition needs at most two division vertices, the total number of division vertices in the subdivision G' of G is equal to the number of partitions in π. Note that the addition of V_k creates two inner faces and the addition of each V_i, $1 < i < k$, creates one inner face. Let the number of inner faces of G be F. Then the number of partitions is $F - 1 = n/2$ by Euler's formula. Therefore, G' has at most $n/2$ division vertices. $\square$

Lemma 2. *Any Hamiltonian cubic graph G, not necessarily planar, with n vertices has a subdivision G' that is acyclically 3-colorable and has $n/2+1$ division vertices.*

Proof. Let C be a Hamiltonian cycle in G. Since the number of vertices in G is even by the degree-sum formula, we can color the vertices on C with colors c_1 and c_2. We next subdivide an edge on C and each of the other edges in G that are not on C to get G'. We assign color c_3 to all the division vertices. See Figure 4(b). Each cycle C' in G' corresponds to a unique cycle C'' in G that contains only

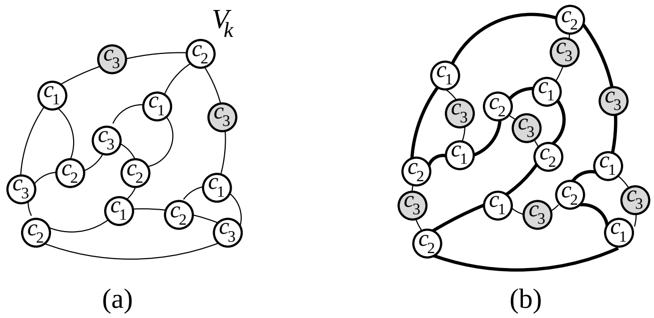

Fig. 4. Illustration for the proof of (a) Lemma 1 and (b) Lemma 2.

the original vertices of C'. If C' in G' corresponds to the Hamiltonian cycle C in G, then C' is not bichromatic. Since every vertex in G' has degree at most 3, no cycle can be formed with only edges that are not on C in G. Let C' be a cycle in G' that corresponds to a cycle C'' in G where $C'' \neq C$. Then C'' must contain at least one edge e on C and one edge e' not on C. According to the coloring of G', the end vertices of e in G must have different colors c_1, c_2 and the division vertex on the edge e' has the remaining color c_3. Therefore G' does not contain any bichromatic cycle. The total number of edges in G is $3n/2$. We have subdivided all the edges of G other than $(n-1)$ edges on C. As a result, the total number of division vertices in G' is $3n/2 - (n-1) = n/2 + 1$. $\square$

A graph G with n vertices is a *k-tree* if G satisfies the following (a)-(b).

(a) If $n = k$, then G is the complete graph with k vertices.
(b) If $n > k$, then G can be constructed from a k-tree G' with $n-1$ vertices by adding a new vertex to exactly k vertices of G', where the induced graph of these k-vertices is a complete graph.

Let G be a k-tree with vertex set V. Then by definition, there is an ordered partition $\pi = (V_1, V_2, ..., V_m)$ of V that satisfies the following:

(a) V_1 contains k vertices inducing a complete graph.
(b) Let G_k, $1 \leq k \leq m$, be the subgraph of G induced by $V_1 \cup V_2 \cup ... \cup V_k$. Then G_k, $k > 1$, is a k-tree obtained by adding V_k to G_{k-1}, where V_k is a singleton set and its neighbors in G_{k-1} form a k-clique.

A partial k-tree is a subgraph of a k-tree. It is straightforward to observe that k-trees are acyclically $(k+1)$-colorable.

Lemma 3. *For $k \leq 8$, every partial k-tree G with n vertices has a subdivision G' with at most one division vertex per edge that is acyclically 3-colorable.*

Proof. For $n \leq 3$, G is itself acyclically 3-colorable. We thus assume that $n > 3$ and that all partial k-trees with less than n vertices have a subdivision with at most one division vertex per edge that is acyclically 3-colorable. Let G be a partial k-tree obtained from a k-tree K. Let $\pi = (V_1, V_2, ..., V_m)$ be an ordered partition of the vertex set of K and let $\pi' = (V'_1, V'_2, ..., V'_{m'})$ be an ordered partition of the vertex set of G, where $V'_1 \subseteq V_1$ and $V'_2, ..., V'_{m'}$ is a subsequence of $V_2, ..., V_m$. Now we add $V'_{m'}$ to $G_{m'-1}$ to obtain G. By induction $G_{m'-1}$ has a subdivision $G'_{m'-1}$ that is acyclically 3-colorable, where the number of division vertices per edge of $G_{m'-1}$ is at most one. Let $V'_{m'} = v$. By definition of k-tree, v is connected to at most k original vertices of $G'_{m'-1}$. However, the neighbors of v may not induce a complete graph since G is a partial k-tree. Let G'' be the graph obtained by adding v to $G'_{m'-1}$. Then G'' is a subdivision of G. To get G' from G'', we consider the following three cases.

Case 1 : The neighbors of v in G'' have the same color c_1. Assign color c_2 to v and subdivide each edge (v, u), where u is a neighbor of v. Finally, assign color c_3 to all these new division vertices. See Figure 5(a). Thus any cycle that passes through v uses three different colors.

Case 2 : The neighbors of v in G'' have color c_1 and c_2. Then assign color c_3 to v. For each neighbor u of v, if u has color c_1, subdivide the edge (v, u) and assign color c_2 to the division vertex. Similarly, for each neighbor u of v, if u has color c_2, subdivide the edge (v, u) and assign color c_1 to the division vertex as in Figure 5(b). So any cycle that passes through v, uses three different colors.

Case 3 : The neighbors of v have all three colors c_1, c_2 and c_3. Since $k \leq 8$ there is at least one color assigned to less than or equal to two neighbors of v. Let the color be c_3. Assign color c_3 to v. If only one neighbor u_1 of v has color c_3, subdivide edge (v, u_1) and assign color c_1 to the division vertex. If two neighbors u_1, u_2 of v have color c_3, subdivide each of the edges (v, u_1) and (v, u_2) once. Then assign color c_1 to the division vertex of edge (u, v_1) and color c_2 to the division vertex of edge (u, v_2). For each neighbor $u \notin \{u_1, u_2\}$ of v, if u has color c_1, then subdivide the edge (v, u) and assign color c_2 to the division vertex. Similarly, for each neighbor u of v, if u has color c_2, then subdivide the edge (v, u) and assign color c_1 to the division vertex. See Figure 5(c). Note that any cycle that passes through v but does not contain both u_1 or u_2, must have three different colors. Any cycle that passes through v, u_1 and u_2 has color c_3 on v and colors c_1, c_2 on the two division vertices on the edges $(v, u_1), (v, u_2)$.

In all the three cases above, any cycle that passes through vertex v is not a bichromatic cycle. All the other cycles are cycles of $G'_{m'-1}$, thus are not bichromatic. Thus the computed coloring in each of Cases 1–3 is an acyclic 3-coloring of a subdivision G' of G. By construction, the number of division vertices on each edge of G is at most one. $\square$

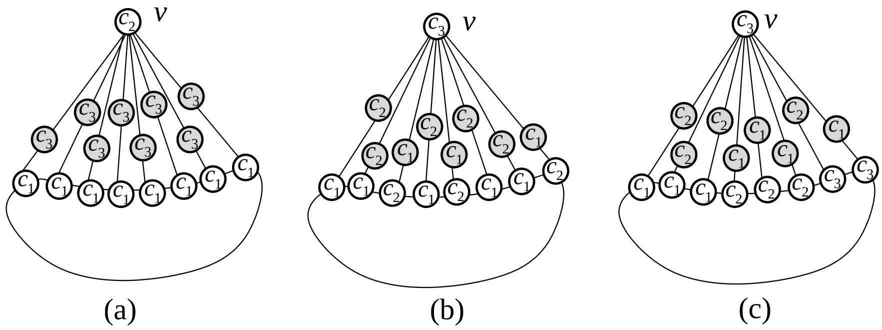

Fig. 5. Illustration for the proof of Lemma 3.

An *independent set* S of G is a set of vertices in G, where no two vertices in S are adjacent in G. The following lemma is of independent interest.

Lemma 4. *Let S be an independent set of a graph G. If $G - S$ is acyclic then G is acyclically 3-colorable.*

Proof. If $G - S$ is acyclic then $G - S$ is a tree or a forest and hence, it is 2–colorable. Color the vertices of $G - S$ with colors c_1 and c_2. Add the vertices of S to $G - S$ and assign the vertices color c_3. Since S is an independent set, a cycle in G contains at least one edge (u_1, u_2) from $G - S$ and at least one vertex u_3 from S. Since, by the coloring method given above, u_1, u_2 and u_3 have different colors, there is no bichromatic cycle in G. $\square$

3 Acyclic coloring of plane graphs

In this section we prove our results for acyclic 3 and 4-colorability of plane graph subdivisions. We first introduce "canonical ordering" of triangulated plane graphs. Let G be a triangulated plane graph on $n \geq 3$ vertices. We denote by $C_0(G)$ the outer cycle of G. Let the three vertices on $C_0(G)$ be v_1, v_2 and v_n in counterclockwise order. Let $\pi = (v_1, v_2, \ldots, v_n)$ be an ordering of all vertices in G. For each integer k, $3 \leq k \leq n$, we denote by G_k the plane subgraph of G induced by the k vertices $v_1, v_2, \ldots, v_k$. We call π a *canonical ordering* of G with respect to the outer edge (v_1, v_2) if it satisfies the following conditions [12]:

(a) G_k is 2-connected and internally triangulated for each k, $3 \le k \le n$.
(b) If $k+1 \le n$, then the vertex v_{k+1} is located on the outer face of G_k, and all neighbors of v_{k+1} in G_k appear on $C_0(G_k)$ consecutively.

Observe that the vertex partition obtained by a canonical decomposition of a triangulated plane graph G determines a vertex ordering, which corresponds to a canonical ordering of G.

Let E^* be the set of edges that do not belong to any $C_0(G_k)$, $3 \le k \le n$. We call these edges the *internal edges* of G because they never appear on the outer face of any G_k. We call all the other edges of G, the *external edges*. Let $V^* = V - \{v_1, v_2\}$ and let $G^* = (V^*, E^*)$. Now we prove that G^* is a tree.

Lemma 5. *For any triangulated plane graph G with a canonical ordering $\pi = (v_1, v_2, \ldots, v_n)$, the subgraph $G^* = (V^*, E^*)$ is a tree.*

Proof. We prove that G^* is a tree by first showing that G^* is connected and then showing that $|E^*| = |V^*| - 1$.

To show that G^* is connected, we show that each internal node v_k, $3 \le k \le n$, has a path to v_n in G^*. For a contradiction, let k be the maximum index such that v_k, $k < n$, does not have such a path to v_n. Since $v_k \in C_0(G_k)$ but $v_k \notin C_0(G)$, there exists an integer l, $k < l \le n$, such that $v_k \in C_0(G_{l-1})$ but $v_k \notin C_0(G_l)$. Hence by property (b) of π, (v_k, v_l) must be an internal edge in G. Since $l > k$, by assumption there must be a path from v_l to v_n in G^*. Therefore v_k has a path to v_n in G^* which is a contradiction.

Each v_k, $3 \le k \le n$, is connected to G_{k-1} by exactly two external edges. Since (v_1, v_2) is also an external edge, the number of external edges in G is $2(n-2)+1 = 2n-3$. By Euler's formula, G has $3n-6$ edges in total. Therefore, $|E^*| = 3n - 6 - (2n - 3) = n - 3 = |V^*| - 1$. Therefore, G^* is a tree. $\square$

We use Lemma 5 to prove the following theorem on acyclic 3-colorability of subdivisions of triangulated plane graphs. This theorem is originally proved by Angelini and Frati [2]. However, our proof is simpler and relates acyclic coloring of graph subdivisions with canonical ordering, which is an important tool for developing graph algorithms.

Theorem 1. *Any triangulated plane graph G has a subdivision G' with one division vertex per edge that is acyclically 3-colorable.*

Proof. Let $G = (V, E)$ be a triangulated plane graph and let $\pi = (v_1, v_2, ..., v_n)$ be a canonical ordering of G. Let E' be the set of external edges and let $E^* = E - E'$ be the set of internal edges of G. Let $G_s = (V, E')$. We now compute a subdivision G'_s of G_s and color G'_s acyclically with three colors as follows.

We assign colors c_1, c_2 and c_3 to the vertices v_1, v_2 and v_3, respectively. For $3 \le k \le n$, as we traverse $C_0(G_k)$ in clockwise order starting at v_1 and ending at v_2, let l_{v_k} be the first neighbor of v_k encountered and let r_{v_k} be the other neighbor of v_k on $C_0(G_k)$. Then assign v_k a color other than the colors of l_{v_k} and r_{v_k}.

We now subdivide each edge in E' with one division vertex to get G'_s. Finally, we assign each division vertex a color other than the colors of its two neighbors. It is easy to see that every edge in G_s along with its division vertex uses three different colors. Therefore, the resulting coloring of G'_s is an acyclic 3-coloring.

We now add the edges of E^* to G'_s and subdivide each of these edges with one division vertex to obtain G'. We assign each new division vertex a color other than the colors of its two neighbors. By Lemma 5, E^* is the edge set of a tree. Therefore, any cycle in G' must contain an edge from E'. Consequently, the cycle must use three different colors. Figure 6(a) shows an example of G', where the edges of E' are shown by solid lines and the edges of E^* are shown by dashed lines. □

We now extend the technique used in the proof of Theorem 1 to obtain the following theorem on acyclic 4-colorability of triangulated plane graphs.

Theorem 2. *Any triangulated plane graph G has a subdivision G' with at most one division vertex per edge that is acyclically 4-colorable, where the number of division vertices in G' is at most $2n - 6$.*

Proof. We define π and G_s as in the proof of Theorem 1. We first compute a subdivision G'_s of G_s and color G'_s acyclically with three colors as follows. We assign colors c_1, c_2 and c_3 to the vertices v_1, v_2 and v_3, respectively. For $3 \leq k \leq n$, as we traverse $C_0(G_k)$ in clockwise order starting at v_1, let l_{v_k} be the first neighbor of v_k encountered and let r_{v_k} be the other neighbor of v_k on $C_0(G_k)$. Then for each vertex v_k we consider the following two cases.

Case 1: The colors of l_{v_k} and r_{v_k} are the same. In this case we assign v_k a color other than the color of l_{v_k} and r_{v_k}. Then we subdivide edge (v_k, r_{v_k}) with one division vertex and assign the division vertex a color other than the colors of its two neighbors.

Case 2: The colors of l_{v_k} and r_{v_k} are different. In this case we assign v_k a color other than the color of l_{v_k} and r_{v_k} and do not subdivide any edge.

At each addition of v_k, Cases 1 and 2 ensure that any cycle passing through v_k has three different colors. Hence, the resulting subdivision is the required G'_s and the computed coloring of G'_s is an acyclic 3-coloring.

We now add the edges of E^* to G'_s and subdivide each of these edges with one division vertex to obtain G'. We assign each new division vertex the fourth color. Any cycle that does not contain any internal edge is contained in G'_s and hence, uses three different colors. On the other hand, any cycle that contains an internal edge must use the fourth color and two other colors from the original vertices on the cycle. Therefore, the computed coloring of G' is an acyclic 4-coloring. Figure 6(b) shows an example of G'. We have not subdivided any edges between the vertices v_1, v_2 and v_3. Moreover, for each v_k, we subdivided exactly one external edge. Therefore the number of division vertices is at most $(3n - 6) - (n - 3) - 3 = 2n - 6$. □

Observe that canonical ordering and Schnyder's realizer of a triangulated plane graph are equivalent notions [11]. Using the fact that $G^* = (V^*, E^*)$ is a

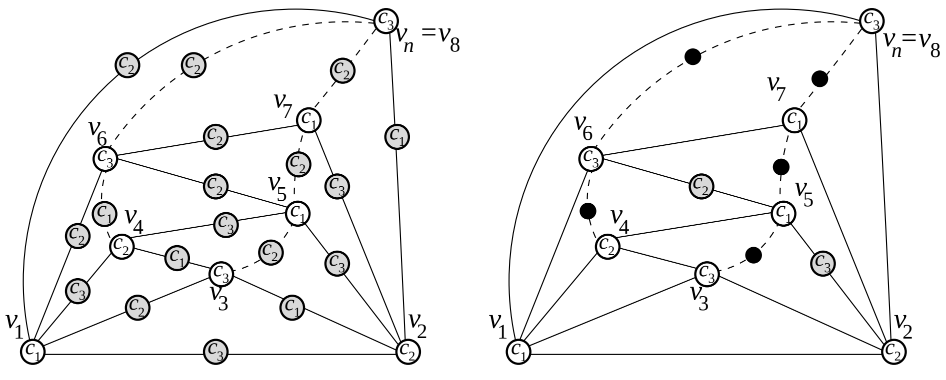

Fig. 6. Illustration for the proof of (a) Theorem 1 and (b) Theorem 2.

tree of Schnyder's realizer [5], one can obtain alternate proofs for Theorems 1 and 2.

4 NP-completeness

In this section we prove that it is NP-complete to decide whether a graph with maximum degree 7 has an acyclic 4-coloring or not. We denote the problem by ACYCLIC 4-COLOR MAX-DEG 7. The equivalent decision problem is given below.

Instance: A graph G with maximum degree 7.
Question: Can the vertices of G be acyclically colored with 4 colors?

Theorem 3. ACYCLIC 4-COLOR MAX-DEG 7 *is NP-complete.*

Proof. The problem is in NP. If a valid 4-coloring of the vertices of G is given, we can check in polynomial time whether that is an acyclic coloring or not. We consider each pair of colors and the subgraph induced by the vertices of those two colors. We check whether that subgraph contains a cycle. If none of the $\binom{4}{2}$ subgraphs contains any cycles, the 4-coloring is an acyclic coloring.

We will prove the NP-hardness by reducing the problem of deciding acyclic 3-colorability of plane graphs with maximum degree 4 to our problem. The problem of acyclic 3-colorability, which was proved to be NP-complete by Angelini and Frati [2], is given below.

Instance: A plane graph H with maximum degree 4.
Question: Can the vertices of H be acyclically colored with 3 colors?

Let H be an instance of the problem of deciding acyclic 3-colorability of plane graphs with maximum degree 4, as in Figure 7. Let p be the number of vertices in H. Then we construct a plane 3-tree G_{4p} of $4p$ vertices as in Figure 7 as follows. We first take a triangle with vertices v_1, v_2 and v_3. Next we take a vertex v_4 in the inner face of the triangle and connect v_4 to v_1, v_2 and v_3 to get

G_4. In any valid coloring of G_4, v_1, v_2, v_3 and v_4 must be assigned four different colors and hence the coloring is acyclic. Let the colors assigned to v_1, v_2, v_3 and v_4 be c_1, c_2, c_3 and c_4, respectively. Now we place a new vertex v_5 inside the face bounded by the triangle v_2, v_3, v_4 and connect v_5 with the three vertices on the face to get G_5. It is obvious that G_5 is 4–colorable and v_5 must be assigned the same color as v_1 in a 4-coloring of G_5. In this recursive way, we construct the graph G_{4p} with $4p$ vertices, where each inner vertex of G_{4p} has degree exactly six. In any valid 4-coloring of G_{4p}, each of the four colors is assigned to exactly p vertices.

We now prove that any valid 4-coloring of a plane 3-tree G_n with n vertices is an acyclic coloring. The proof is by induction on n. When $n \leq 4$, any 4-coloring of G_3 is an acyclic coloring. We thus assume that $n > 4$ and that any valid 4-coloring of a plane 3-tree with less than n vertices is an acyclic coloring. By definition of plane 3-tree, G_n has a vertex v of degree three. We remove v from G_n to get another plane 3-tree G_{n-1} with $n-1$ vertices. By the induction hypothesis, any 4-coloring of G_{n-1} is an acyclic coloring. We now add v to G_{n-1} to get G_n. By construction of plane 3-trees, v must be placed in a face of G_{n-1} and must be connected to the three vertices on the face. Let the colors assigned to three neighbors of v be c_1, c_2 and c_3. Then v is assigned color c_4. Now, any cycle that goes through v must also go through at least two of the neighbors of v. Hence any cycle containing v contains vertices of at least three colors. Therefore, G_n has no bichromatic cycle.

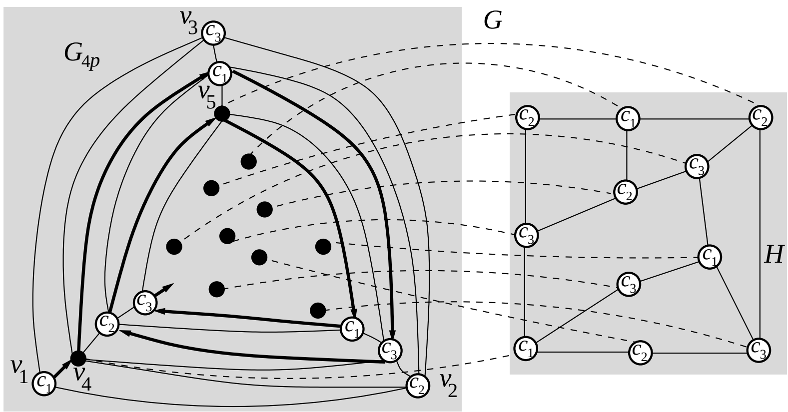

Fig. 7. Illustration for the proof of Theorem 3.

Let S be the set of the vertices of G_{4p} that are assigned color c_4 in a 4-coloring when the outer vertices use the colors c_1, c_2, c_3. We connect each vertex of H to exactly one vertex of S as illustrated in Figure 7, so that the edges connecting vertices of H and G_{4p} form a matching. Let the resulting graph be G. It is easy to see that the degree of each vertex of G is at most seven. We argue that G has an acyclic 4-coloring if and only if H has an acyclic 3-coloring.

First we assume that G admits an acyclic 4-coloring. Let the colors assigned to the vertices of G be c_1, c_2, c_3 and c_4. Let the colors assigned to the outer vertices of G_{4p} be c_1, c_2 and c_3. Then each vertex in S has color c_4 and hence no vertex in H receives color c_4. Therefore, the vertices of H are acyclically colored with three colors c_1, c_2 and c_3.

We now assume that H has an acyclic 3-coloring where the colors assigned to the vertices of H are c_1, c_2 and c_3. We assign the three colors to the three outer vertices v_1, v_2 and v_3 of G_{4p}. Clearly the common neighbor v_4 of the three outer vertices must be assigned a fourth color c_4. In the same way, all the vertices of S get the color c_4. Suppose for a contradiction that G contains a bichromatic cycle C. C cannot be a cycle of H. Since any 4-coloring of G_{4p} is acyclic, C cannot be a cycle of G_{4p}. Therefore, C must contain vertices from both G_{4p} and H. Since the edges connecting G_{4p} and H form a matching, no two vertices of G_{4p} have the same neighbor in H. Therefore, C must contain at least one edge e of H. The end vertices of e have two of the three colors c_1, c_2, c_3. Since C must contain a vertex in G_{4p} with color c_4, C contains vertices of at least three colors and hence cannot be a bichromatic cycle. Therefore, the 4-coloring of G described above is acyclic. □

5 Open Problems

Acyclic colorings of plane graph subdivisions with fewer division vertices will be an interesting direction to explore. We ask the following question:

What the minimum positive constant c such that every triangulated planar graph with n vertices has an acyclic k-coloring, $k \in \{3, 4\}$, with at most cn division vertices?

Every cubic graph is acyclically 4-colorable [14]. On the other hand, we have proved that testing acyclic 4-colorability is NP-complete for graphs with the maximum degree 7. The problem of obtaining acyclic 4-colorings for graphs with maximum degree greater than three and less than seven remains open, as does using our results to improve volume bounds on 3-dimensional polyline grid drawings.

References

1. M. O. Albertson and D. M. Berman. Every planar graph has an acyclic 7-coloring. *Israel Journal of Mathematics*, 28(1–2):169–174, 1977.
2. P. Angelini and F. Frati. Acyclically 3-colorable planar graphs. In *Workshop on Algorithms and Computation (WALCOM '10)*, volume 5942 of *Lecture Notes in Computer Science*, pages 113–124. Springer, 2010.
3. O. V. Borodin. On acyclic colorings of planar graphs. *Discrete Mathematics*, 306(10–11):953–972, 2006.

4. T. F. Coleman and J. Cai. The cyclic coloring problem and estimation of sparse hessian matrices. *SIAM Journal on Algebraic and Discrete Methods*, 7(2):221–235, 1986.
5. R. Dhandapani. Greedy drawings of triangulations. *Discrete & Computational Geometry*, 43(2):375–392, 2010.
6. V. Dujmović, P. Morin, and D. R. Wood. Layout of graphs with bounded tree-width. *SIAM Journal of Computing*, 34:553–579, March 2005.
7. A. H. Gebremedhin, A. Tarafdar, A. Pothen, and A. Walther. Efficient computation of sparse hessians using coloring and automatic differentiation. *INFORMS Journal on Computing*, 21:209–223, April 2009.
8. B. Grünbaum. Acyclic colorings of planar graphs. *Israel Journal of Mathematics*, 14(4):390–408, 1973.
9. A. V. Kostochka. Acyclic 6-coloring of planar graphs. *Diskretn. Anal.*, 28:40–56, 1976.
10. J. Mitchem. Every planar graph has an acyclic 8-coloring. *Duke Mathematical Journal*, 41(1):177–181, 1974.
11. K. Miura, M. Azuma, and T. Nishizeki. Canonical decomposition, realizer, schnyder labeling and orderly spanning trees of plane graphs. *International Journal of Foundations of Computer Science*, 16(1):117–141, 2005.
12. T. Nishizeki and M. S. Rahman. *Planar Graph Drawing.* World Scientific, 2004.
13. P. Ochem. Negative results on acyclic improper colorings. In *European Conference on Combinatorics (EuroComb '05)*, pages 357–362, 2005.
14. S. Skulrattanakulchai. Acyclic colorings of subcubic graphs. *Information Processing Letters*, 92(4):161–167, 2004.
15. D. R. Wood. Acyclic, star and oriented colourings of graph subdivisions. *Discrete Mathematics & Theoretical Computer Science*, 7(1):37–50, 2005.