Acyclic Coloring with Few Division Vertices

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Abstract. An acyclic k-coloring of a graph G is a mapping ϕ from the set of vertices of G to a set of k distinct colors such that no two adjacent vertices receive the same color and ϕ does not contain any bichromatic cycle. In this paper we prove that every triangulated plane graph with n vertices has a 1-subdivision that is acyclically 3-colorable (respectively, 4-colorable), where the number of division vertices is at most $2n-5$ (respectively, $1.5n-3.5$). Our results imply $O(n \log^{16} n)$ and $O(n \log^{24} n)$ upper bounds on volume of 3D polyline drawings of planar graphs, where each edge has at most one bend and the total number of bends are $2n-5$ and $1.5n-3.5$, respectively. On the other hand, we prove an $1.28n$ (respectively, $0.3n$) lower bound on the number of division vertices for acyclic 3-colorings (respectively, 4-colorings) of triangulated planar graphs. Furthermore, we establish the NP-completeness of deciding acyclic 4-colorability for graphs with the maximum degree 5 and for planar graphs with the maximum degree 7.

1 Introduction

A *k-coloring* of a graph G is a mapping ϕ from the set of vertices of G to a set of k distinct colors such that no two adjacent vertices receive the same color. We call ϕ an *acyclic k-coloring* if it does not contain any bichromatic cycle. The *acyclic chromatic number* of a graph G is the minimum number of colors required in any acyclic coloring of G.

Grünbaum [1] first introduced the concept of acyclic coloring in 1973 and proved that every planar graph admits an acyclic 9-coloring. Then Mitchem [2], Albertson and Berman [3], Kostochka [4] and finally Borodin [5] improved this upper bound to $8, 7, 6$ and 5, respectively. Since there exist planar graphs requiring 5 colors in any acyclic coloring [1], much research effort has been devoted to characterize planar graphs that are acyclically 3 or 4-colorable [6]. Both the problems of deciding acyclic 3 and 4-colorability are NP-hard for planar graphs with maximum degree 4 and 8, respectively [7, 8].

Grünbaum also considered acyclic colorings of non-planar graphs. He proved that any graph of maximum degree 3 admits an acyclic 4-coloring. Alon et al. [9]

gave an $O(\Delta^{4/3})$ upper bound and an $\Omega(\Delta^{4/3}/(\log \Delta)^{1/3})$ lower bound on acyclic chromatic number for the graphs with maximum degree Δ. The currently best known upper bounds on acyclic chromatic numbers for the graphs with maximum degree $3, 4, 5$ and 6 are $4, 5, 7$ and 11, respectively [10–13]. Little is known about the time complexity of deciding acyclic colorability for bounded degree graphs. Acyclic 3-colorability (respectively, 4-colorability) is NP-complete for bipartite planar graphs with maximum degree 4 (respectively, maximum degree 8) [8]. Recently, Mondal et al. [14] proved that acyclic 4-colorability is NP-complete for the graphs with maximum degree 7.

A *k-subdivision* of a graph G is a graph G' obtained by replacing every edge of G with a path that has at most k internal vertices. We call these internal vertices the *division vertices* of G. Wood [15] observed that every graph has a 2-subdivision that is acyclically 3-colorable. Angelini and Frati [7] proved that every triangulated planar graph with n vertices has a 1-subdivision with $3n-6$ division vertices that is acyclically 3-colorable. This upper bound on the number of division vertices reduces to $2n-6$ in the case of acyclic 4-coloring [14].

Acyclic colorings of graphs and their subdivisions find applications in diverse areas [16–18]. For example, an acyclic coloring of a planar graph has been used to obtain upper bounds on the volume of a 3-dimensional straight-line grid drawing of a planar graph [16]. Consequently, an acyclic coloring of a planar graph subdivision can give upper bounds on the volume of a 3-dimensional polyline grid drawing, where the number of division vertices gives an upper bound on the number of bends sufficient to achieve that volume. The acyclic chromatic number of a graph helps to obtain an upper bound on the size of "feedback vertex set" of a graph, which has wide applications in operating system, database system, genome assembly, and VLSI chip design [17]. Acyclic colorings are also used in efficient computation of Hessian matrix [18].

In this paper we examine acyclic colorings of 1-subdivisions of planar graphs. We also show some improvement over the previous NP-completeness results in terms of maximum degree. Our results are as follows.

– In Section 3 we prove that every triangulated plane graph with n vertices has a 1-subdivision with at most $2n-5$ (respectively, $1.5n-3.5$) division vertices that is acyclically 3-colorable (respectively, 4-colorable), which significantly improves the previously best known upper bounds $3n-6$ and $2n-6$ on the number of division vertices presented in [7, 14].
– In Section 4 we establish a $1.28n$ (respectively, $0.3n$) lower bound on the number of division vertices for acyclic 3 and 4-colorings of triangulated planar graphs, respectively.
– In Section 5 we show that deciding acyclic 4-colorability is NP-complete for graphs with maximum degree 5 and for planar graphs with maximum degree 7. Our results improve the previously known NP-completeness results on acyclic 4-colorability for graphs with maximum degree 7 [14] and for planar graphs with maximum degree 8 [8].

2 Preliminaries

In this section we present some definitions and preliminary results that are used throughout the paper.

Let $G = (V, E)$ be a connected graph with vertex set V and edge set E. By $\deg(v)$ we denote the degree of the vertex v in G. The *maximum degree* Δ of G is the maximum of all $\deg(v), v \in V$. Let $P = u_0, u_1, u_2, \ldots, u_{l+1}$, $l \geq 1$, be a path of G such that $\deg(u_0) \geq 3$, $\deg(u_1) = \deg(u_2) = \ldots = d(u_l) = 2$ and $d(u_{l+1}) \geq 3$. Then we call the subpath $u_1, u_2, \ldots, u_l$ of P a *chain* of G. A *spanning tree* of G is a subgraph of G that is a tree and contains all the vertices in G. G is k*-connected* if the minimum number of vertices required to remove from G to obtain a disconnected graph or a single-vertex graph is k. The following remark is easy to verify. See the Appendix.

Remark 1. *Let G be a graph and let G' be a graph obtained from G by adding a chain $w_1, \ldots, w_j$ between two distinct vertices u and v of G. Assume that G admits an acyclic 3-coloring, which can be extended to a 3-coloring ϕ of G' such that the vertices on the path $u, w_1, \ldots, w_j, v$ receive three different colors. Then ϕ is an acyclic 3-coloring of G'.*

A *plane graph* G is a planar graph with a fixed planar embedding on the plane. G delimits the plane into connected regions called faces. The unbounded face is the outer face of G and all the other faces are the inner faces of G. G is called *triangulated* (respectively, internally triangulated) if every face (respectively, every inner face) of G contains exactly three vertices on its boundary. The vertices on the outer face of G are called the *outer vertices* and all the remaining vertices are called the *inner vertices*. The edges on the outer face are called the *outer edges* of G.

Let $G = (V, E)$ be a triangulated plane graph with the outer vertices x, y and z in anticlockwise order on the outer face. Let $\pi = (v_1(= x), v_2(= y), ..., v_n(= z))$ be an ordering of all vertices of G. By G_k, $3 \leq k \leq n$, we denote the subgraph of G induced by $v_1 \cup v_2 \cup ... \cup v_k$ and by C_k the outer cycle (i.e., the boundary of the outer face) of G_k. We call π a *canonical ordering* of G with respect to the outer edge (x, y) if for each index k, $3 \leq k \leq n$, the following conditions are satisfied [19].

(a) G_k is 2-connected and internally triangulated.
(b) If $k + 1 \leq n$, then v_{k+1} is an outer vertex of G_{k+1} and the neighbors of v_{k+1} in G_k appears consecutively on C_k.

Assume that for some $k \geq 3$, the outer cycle C_k is $w_1(= x), \ldots, w_p, w_q(= v_k), w_r \ldots, w_t(= y)$, where the vertices appear in clockwise order on C_k. Then we call the edges (w_p, v_k) and (v_k, w_r) the *left-edge* and the *right-edge* of v_k, respectively. By $L(v_k)$ and $R(v_k)$ we denote the vertices w_p and w_r, respectively. Let E^* be the set of edges that does not belong to any C_k, $3 \leq k \leq n$. Assume that $V^* = V - \{v_1, v_2\}$. Then the graph $T_\pi = (V^*, E^*)$ is a tree. The graph induced by the right-edges (respectively, left-edges) of the vertices $v_k, 3 \leq k \leq$

$n-1$, is also a tree, which we denote by T^r_π (respectively, T^l_π). In fact, T_π, T^l_π and T^r_π form a "Schnyder's realizer" of G [20]. By G_π we denote the graph obtained from G by removing all the edges of T_π. Figure 1(a) illustrates π, T_π and G_π.

The existence of Schnyder's realizer implies that there exists another canonical ordering $\pi' = (u_1(=z), u_2(=x), ..., u_n(=y))$ of G with respect to the outer edge (z, x) such that the following properties hold [20].

(i) For each index $k, 3 \le k \le n-1$, the right-edge e of vertex u_k in π' coincides with the left-edge of that vertex in π, and hence both $G_{\pi'}$ and G_π contains e. On the other hand, the left-edge of vertex u_k belongs to $G_{\pi'}$, but does not belong to G_π.

(ii) $G_{\pi'}$ contains all the edges of G except the edges of T^r_π.

Figure 1(b) illustrates π' and T^r_π. The vertex v_6 in Figure 1(a) and the vertex u_3 in Figure 1(b) are the same, where the left-edge (v_6, v_1) of v_6 coincides with the right-edge (u_3, u_2) of u_3.

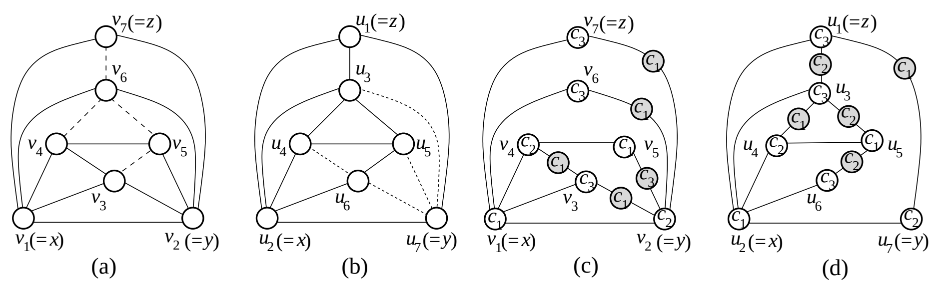

Fig. 1. (a) A graph G and its canonical ordering π. The edges of T_π are shown in dashed lines. All the remaining edges belong to G_π. (b) The canonical decomposition π' of G. The edges of T^r_π are shown in dotted lines. All the remaining edges belong to G'_π. (c) An acyclic 3-coloring of S. (d) An acyclic 3-coloring of S'. The division vertices are shown in gray color.

3 Acyclic Colorings of Planar Graph Subdivisions

In this section we prove that every triangulated plane graph with n vertices has a 1-subdivision with at most $2n-5$ (respectively, $1.5n-3.5$) division vertices that is acyclically 3-colorable (respectively, 4-colorable). To achieve our results we exploit the properties of canonical orderings of triangulated plane graphs.

Theorem 1. *Every triangulated plane graph G with $n \ge 3$ vertices has a 1-subdivision G' with at most $2n-5$ division vertices that is acyclically 3-colorable.*

Proof. Let x, y, z be the outer vertices of G in anticlockwise order on the outer face. Let $\pi = (v_1(=x), v_2(=y), \ldots, v_n(=z))$ and $\pi' = (u_1(=z), u_2(=x), \ldots, u_n(=y))$ be the canonical orderings of G as defined in Section 2. We use G_π and $G_{\pi'}$ to construct the required 1-subdivision G' and an acyclic 3-coloring of G'.

We first construct a 1-subdivision S of G_π and compute an acyclic 3-coloring ϕ of S with the colors c_1, c_2, c_3. We assign colors c_1 and c_2 to the vertices v_1 and v_2, respectively. For $3 \le k \le n$, we assign v_k a color other than the colors of $L(v_k)$ and $R(v_k)$. We then subdivide the right-edge $(v_k, R(v_k))$ with one division vertex and assign the division vertex the color different from its neighbors. The resulting 1-subdivision of G_π is the required subdivision S. It is now straightforward to prove inductively using Remark 1 that the resulting coloring ϕ of S is an acyclic 3-coloring[1]. Observe that every edge e of T^r_π contains a division vertex z in S and z along with the two end vertices of e receive three different colors in ϕ. This property also holds for the edge (y, z). Figure 1(c) illustrates S.

We then construct a 1-subdivision of $G_{\pi'}$ and color that subdivision acyclically with colors c_1, c_2, c_3 without changing the colors assigned to the original vertices by ϕ. For $3 \le k \le n-1$, we subdivide the left-edge of u_k with one division vertex z'. If $col(u_k) = col(L(u_k))$, then we color z' with the color other than the colors of u_k and $R(u_k)$. Otherwise, we assign z' the color different from its neighbors. Finally, we subdivide the left-edge of u_n with a division vertex z'' and color it with the color of the division vertex on (y, z) in S. By Property (i) of $G_{\pi'}$ along with the computation of ϕ observe that $col(u_k) \ne col(R(u_k))$. Consequently, the coloring of z' ensures that the path $L(u_k), z', u_k, R(u_k)$ contains vertices of three different colors. This property holds when $k = n$, i.e., the path $L(u_n), z'', u_n, R(u_n)$ contains vertices of three different colors. It is now straightforward to prove inductively using Remark 1 that the coloring of the resulting 1-subdivision S' of $G_{\pi'}$ is an acyclic 3-coloring, which we denote by ϕ'. Figure 1(d) illustrates S'.

We now use S and S' to construct G'. For each edge e in G, we subdivide e if the corresponding edge in S or S' contains a division vertex. The resulting 1-subdivision is the required G'. Since ϕ and ϕ' do not contain any conflict, we can color the vertices of G' unambiguously. Suppose for a contradiction that the coloring we compute for G' contains a bichromatic cycle C. Since S' is colored acyclically, C must contain an edge e of G' that does not correspond to any edge in S'. By Property (ii) of $G_{\pi'}$, e must be an edge of T^r_π. Recall that every edge e of T^r_π contains a division vertex z in S and z along with the two end vertices of e receive three different colors in ϕ. Therefore, C cannot be a bichromatic cycle, a contradiction.

Observe that each of the subdivisions S and S' contains exactly $n-2$ division vertices, where only the division vertex on (y, z) is common to both subdivisions. Therefore, the number of division vertices in G' is $2n - 5$. $\square$

Theorem 2. *Every triangulated plane graph G with $n \ge 3$ vertices has a 1-subdivision G' with at most $\lceil 3(n-3)/2 \rceil \approx 1.5n - 3.5$ division vertices that is acyclically 4-colorable.*

[1] Note that the graph induced by v_1, v_2 in G_π is colored acyclically. Assume inductively that this holds for the graph H, which is a subgraph of G_π induced by $v_1, v_2, \ldots, v_z$, where $2 < z < n$. The graph induced by $v_1, v_2, \ldots, v_z, v_{z+1}$ in G_π is obtained by adding a chain to H and the colors assigned to the vertices on the chain satisfy the condition of Remark 1.

Proof. Zhang and He [20] proved that G has a canonical ordering π such that T_π contains $\lceil (n-3)/2 \rceil$ leaves. (The corresponding Schnyder's realizer is known as minimum Schnyder's realizer.) We use π to compute G'.

We first construct a 1-subdivision H of G_π and compute an acyclic 3-coloring of H as follows. We assign colors c_1 and c_2 to the vertices v_1 and v_2, respectively. For $3 \le k \le n$, we assign v_k a color other than the colors of $L(v_k)$ and $R(v_k)$. If $col(L(v_k)) = col(R(v_k))$, then we subdivide the edge $(v_k, R(v_k))$ with one division vertex and assign the division vertex the color different from its neighbors. The resulting 1-subdivision is the required subdivision H. It is now straightforward to prove inductively using Remark 1 that the resulting coloring of H is an acyclic 3-coloring.

We now count the number of division vertices in H. Observe that for each $v_k, 3 \le k \le n$, if v_k is a leaf in T_π or $k = n$, then the edge $(L(v_k), R(v_k))$ exists. Consequently, $col(L(v_k)) \ne col(R(v_k))$ and we do not add any division vertex in this situation. Since there are $\lceil (n-3)/2 \rceil$ leaves in T_π, the number of division vertices in H is at most $n - 2 - \lceil (n-3)/2 \rceil - 1 = \lceil (n-3)/2 \rceil$.

To construct G', we add the edges of T_π to H by subdividing each edge of T_π with one division vertex. We color all the new division vertices with color c_4. The resulting subdivision is the required subdivision G' of G. Suppose for a contradiction that the coloring we compute for G' contains a bichromatic cycle C. Since H is colored acyclically and T_π is a tree, C must contain at least one edge e from T_π and at least one edge e' from G_π. Since the division vertex on e is colored with c_4, and the end vertices of e' along with the division vertex on e' (if any) contribute two different colors to C other than c_4, C cannot be a bichromatic cycle, a contradiction.

Finally, the number of division vertices in G' is at most $n - 3 + \lceil (n-3)/2 \rceil = \lceil 3(n-3)/2 \rceil \le 1.5n - 3.5$. □

Since the canonical orderings of plane graphs used in Theorems 1 and 2 can be computed in linear time [20], the proofs of these theorems lead us to linear-time acyclic coloring algorithms.

Every graph G with acyclic chromatic number c and 'queue number' q has 'track number' $tn(G) \le c(2q)^{c-1}$ [16]. The upper bound on the queue number of planar graphs is $O(\log^4 n)$ [21]. Since every planar graph G has a subdivision G' with $2.86n - 6$ division vertices that is acyclically 3-colorable, the track number of G' is $tn(G') = O(\log^8 n)$. Since G' has an $O(tn(G')) \times O(tn(G')) \times O(n)$ volume 3D straight-line drawing [16], Theorems 1 and 2 imply the following.

Remark 2. *Every planar graph G admits an $O(n \log^{16} n)$ volume (respectively, an $O(n \log^{24} n)$ volume) 3D polyline drawing with at most one bend per edge and at most $2n - 5$ bends (respectively, $1.5n - 3.5$ bends) in total.*

4 Lower Bounds on the Number of Division Vertices

In this section we present a triangulated planar graph G with n vertices such that any of its 1-subdivisions that is acyclically 3-colorable (respectively, 4-colorable), contains at least $1.28n$ (respectively, $0.3n$) division vertices.

In Figures 2(a) and (b) we exhibit two planar graphs $\mathcal{M}$ and $\mathcal{N}$ such that the following lemma holds.

Lemma 1. *Any* 1*-subdivision of* $\mathcal{M}$ *(respectively,* $\mathcal{N}$*) that is acyclically acyclically* 3*-colorable (respectively,* 4*-colorable), contains at least* 9 *(respectively,* 3*) division vertices.*

Lemma 1 can be verified by case study or by computer programs.

We use $\mathcal{M}$ and $\mathcal{N}$ along with a recursive graph structure $G_k, k \in \mathbb{Z}^+$, to construct the triangulated planar graphs that give rise to the lower bound. G_1 is the graph shown in Figure 2(c). A set of four edge disjoint empty triangles of G_1 are shown in gray, which we call *cells*. $G_k, k > 1$, is constructed by inserting a copy of G_1 into each cell of G_{k-1} and then identifying the outer cycle of each copy of G_1 with the boundary of the corresponding cell. Figure 2(d) shows G_2. The number of cells and the number of vertices in G_k is 4^k and $4^k + 2$, respectively.

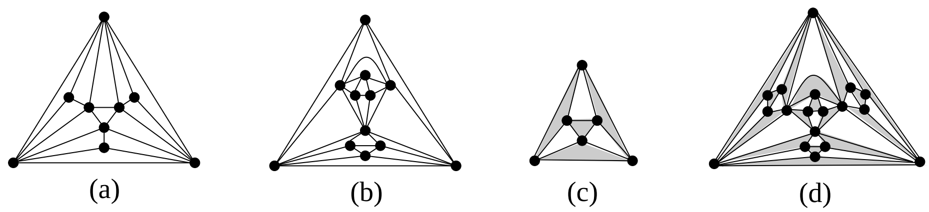

Fig. 2. Illustration for (a) $\mathcal{M}$, (b) $\mathcal{N}$, (c) G_1, and (d) G_2.

Let $\mathcal{M}_k$ be the graph obtained by inserting a copy of $\mathcal{M}$ into each cell of G_{k-1} and then identifying the outer cycle of each copy of $\mathcal{M}$ with the boundary of the corresponding cell. Then the number of vertices in $\mathcal{M}_k$ is $4^k + 2 + 6 \cdot 4^k = 7 \cdot 4^k + 2$. The copies of $\mathcal{M}$ are edge disjoint in $\mathcal{M}_k$. Therefore by Lemma 1, any 1-subdivision of $\mathcal{M}_k$ that is acyclically acyclically 3-colorable contains at least $9 \cdot 4^k = (9t - 18)/7$ division vertices, where $t = 7 \cdot 4^k + 2$.

Similarly, for any $k \in \mathbb{Z}^+$, we use $\mathcal{N}$ to construct a triangulated planar graph with $t' = 4^k + 2 + 9 \cdot 4^k$ vertices such that any of its 1-subdivisions that is acyclically 4-colorable contains at least $3 \cdot 4^k = (3t' - 6)/10$ division vertices.

Theorem 3. *For every* $k \in \mathbb{Z}^+$*, there exists a triangulated planar graph with* $t = 7 \cdot 4^k + 2$ *vertices (respectively,* $t' = 10 \cdot 4^k + 2$ *vertices) such that any of its* 1*-subdivisions, which is acyclically* 3*-colorable (respectively,* 4*-colorable), contains at least* $(9t - 18)/7$ *(respectively,* $(3t' - 6)/10$*) division vertices.*

5 NP-Completeness

In this section we first prove that acyclic 4-colorability is NP-hard for the graphs with maximum degree 5. We then prove that the problem remains NP-hard for planar graphs with maximum degree 7.

5.1 Acyclic 4-Colorability of Graphs with $\Delta = 5$

To prove the NP-hardness of acyclic 4-colorability for maximum degree 5, we use the same technique as used in [14] to show the NP-hardness of 4-colorability for graphs with maximum degree 7. The crucial step is to construct a graph with low maximum degree such that in any acyclic 4-coloring of G a set of vertices of G receives the same color.

We use the graph shown in Figure 3(a) for this purpose. We call the graph of Figure 3(a) a *bead* and the vertices x, y the *poles*. A bead contains exactly one vertex s of degree 4, which we call the *center* of the bead.

Remark 3. *In any acyclic* 4*-coloring of a bead, the poles get different colors.*

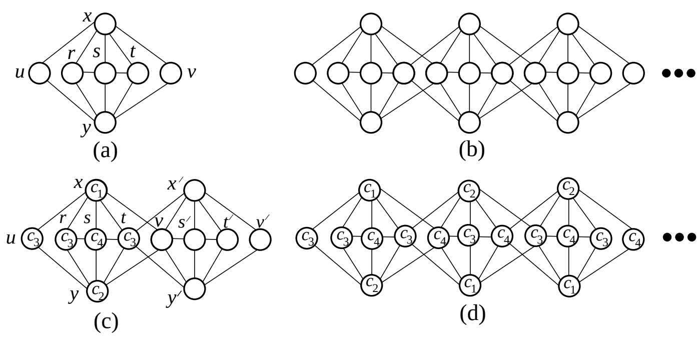

Fig. 3. Illustration for (a) a bead, (b) G_p, (c) a partial acyclic 4-coloring of G_2, and (d) an acyclic 4-coloring of G_p.

For any $i \in \mathbb{Z}^+$, we now define a graph G_i with maximum degree 5 as follows.

(a) G_1 is a bead.
(b) $G_i, i > 1$, is constructed with an ordered sequence $B_1, B_2, \ldots, B_i$ of i beads by merging a vertex of degree two of B_j with a vertex of degree three of B_{j+1} and a vertex of degree three of B_j with a vertex of degree two of B_{j+1}, where $0 < j < i$. A construction for G_i is shown in Figure 3(b).

Observe that every bead in G_i contains exactly one vertex of degree 4. The following lemma gives some properties of acyclic 4-colorings of G_i, whose proof is omitted.

Lemma 2. *For any* $p \in \mathbb{Z}^+$, G_p *contains an independent set* $I(G_p)$ *of size* $\lfloor (p+1)/2 \rfloor$ *such that every vertex of* $I(G_p)$ *is a vertex of degree* 4 *and in any acyclic* 4*-coloring of* G_p*, the vertices of* $I(G_p)$ *receive the same color.*

We now prove the NP-completeness of acyclic 4-colorability for graphs with maximum degree 5. Observe that given a valid 4-coloring of the vertices of the input graph, one can check in polynomial time whether the vertices of each pair of color classes induces a forest. Therefore, the problem is in NP.

To prove the NP-hardness we reduce the NP-complete problem of deciding acyclic 3-colorability of maximum degree 4 graphs [7] to the problem of deciding acyclic 4-colorability of maximum degree 5 graphs. Let G be an instance of acyclic 3-colorability problem, where G has n vertices and the maximum degree of G is 4. Take a copy of G_{2n-1} and connect each vertex of G with a distinct vertex of $I(G_{2n-1})$ by an edge. Let the resulting graph with maximum degree 5 be G', which is straightforward to construct in polynomial time. Using the proof technique of Theorem 3 of [14] we can show that G admits an acyclic 3-coloring if and only if G' admits an acyclic 4-coloring. We thus have the following theorem. A stand-alone proof of the theorem is included in the Appendix.

Theorem 4. *It is NP-complete to decide whether a graph with maximum degree* 5 *admits an acyclic* 4*-coloring.*

5.2 Acyclic 4-Colorability of Planar Graphs with $\Delta = 7$

In this section we prove the NP-completeness of acyclic 4-colorability of planar graphs with maximum degree 7. Ochem [8] proved the NP-hardness of acyclic 4-colorability of bipartite Planar Graphs with maximum degree 8. It does not seem straightforward to adapt his proof to show the NP-hardness of acyclic 4-colorability of planar graphs with maximum degree 7, even if we relax the 'bipartite' condition and try to replace his vertex gadget with another vertex gadget of low maximum degree that have the same functionality.

To define our vertex and edge gadgets, we first define some special graphs. A *jewel* is a graph obtained from a bead by connecting the vertices of degree 2 with distinct vertices of degree 3, as shown in Figure 4(a). By the *connectors of a jewel* J we denote the vertices of degree three in J. For any $i \in \mathbb{Z}^+$, a *necklace* N_i is a graph with maximum degree 6, which is constructed with an ordered sequence $J_1, J_2, \ldots, J_i$ of i jewels by merging a connector of J_q with a connector of J_{q+1}, where $0 < q < i$. We use the necklace N_{15} as the vertex gadget, as shown in Figure 4(c) inside the dashed rectangle.

We call the graph of Figure 4(b) a *link* L, where the vertex w is the *connector of* L and each of the vertices l, m, n is a *tail of* L. In a similar technique as we used in Lemma 2, we can prove the following lemma. See the Appendix.

Lemma 3. *The following claims hold:*

(a) Let ϕ be an acyclic 4-coloring of a necklace $N_i, i \in \mathbb{Z}^+$. Then all the connectors receive the same color c in ϕ. Let $c', c' \neq c$, be any color among the 4 colors used in ϕ. Then for any two connectors in N_i, there is a bichromatic path with colors c' and c in ϕ.

(b) In any acyclic 4-coloring of the link L, if $col(l) = col(m) = col(n) = c$ and there is no bichromatic path between any pair of the tails l, m, n, then $col(w) = c$. Furthermore, there exists an acyclic coloring ϕ of L such that there is no bichromatic path between any pair of the vertices w, l, m, n.

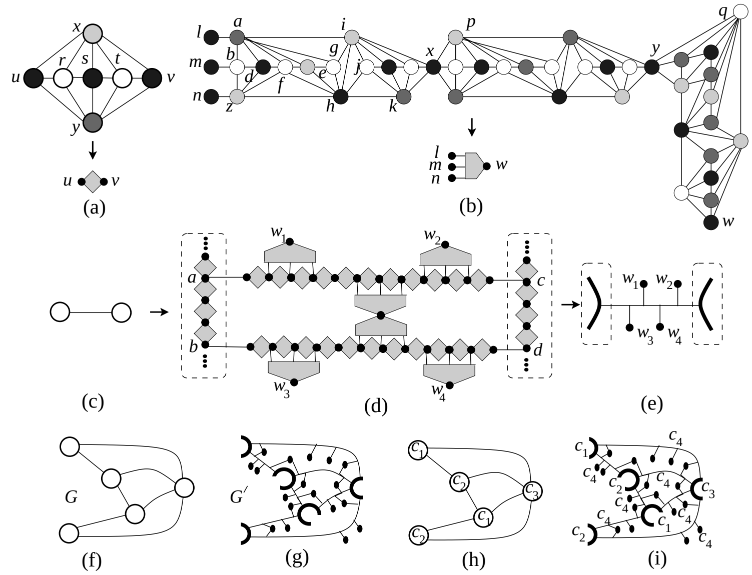

Fig. 4. (a) A jewel. (b) A link. (c) An edge e. (d) A vertex and edge gadgets replacing e. (e) A hypothetical representation of the gadgets. (f) A graph G. (g) G', which is obtained from G by first replacing the vertices and edges with appropriate gadgets and then merging the free connectors as necessary. (h) A 3-coloring ϕ of G. (i) An acyclic 4-coloring of G' that corresponds to ϕ, where a color associated with a vertex or edge gadget denotes the color of the connectors in that gadget.

We use two copies of N_{11} along with six copies of the link to construct the edge gadget. Figures 4(c)–(e) illustrate the edge gadget and its hypothetical representation. We call the vertices w_1, w_2, w_3 and w_4 the *free connectors* of the edge gadget. We now have the following theorem.

Theorem 5. *It is NP-complete to decide whether a planar graph with maximum degree* 7 *admits an acyclic* 4*-coloring.*

Proof. In a similar way as in Theorem 4, we can observe that the problem is in NP. To prove the NP-hardness we reduce the NP-complete problem of deciding 3-colorability of planar graphs with maximum degree 4 [22] to the problem of deciding acyclic 4-colorability of planar graphs with maximum degree 7.

Let G be an instance of 3-colorability problem, where G has n vertices and the maximum degree of G is 4. We now construct a graph G' by replacing the vertices and edges with appropriate gadgets, as illustrated in Figures 4(c)–(e). For every vertex gadget $\mathcal{X}$, we connect the edge gadgets incident to $\mathcal{X}$ by merging some of the free connectors such that the resulting graph remains planar and the maximum degree does not exceed 7. As a consequence, all the edge gadgets become connected, i.e., removal of all the vertex gadgets leaves a connected component. See Figures 4(f)-(g). Let the resulting planar graph be G', which is straightforward to construct in polynomial time. We now show that G admits a 3-coloring if and only if G' admits an acyclic 4-coloring.

We first assume that G admits a 3-coloring with the colors c_1, c_2, c_3 and then construct an acyclic 4-coloring of G'. For every vertex v in G, we color the connectors of the corresponding vertex gadget in G' with $col(v)$. We then color all the remaining connectors with color c_4. See Figures 4(h)-(i). Finally, we color the remaining vertices of G' according to the Figures 4(a)–(b). Suppose for a contradiction that the resulting coloring contains a bichromatic path C. It is straightforward to verify that every vertex gadget and edge gadget is colored acyclically. Moreover, we have colored every link L in such a way that there is no bichromatic path between the connector and any of the tails of L (See Lemma 3 and Figure 4(b)). Therefore, the cycle C must pass through at least one edge gadget $\mathcal{Y}$ and its two incident vertex gadgets. Since the connectors of $\mathcal{Y}$ are colored with c_4 and the connectors of its two incident vertex gadgets are colored with two different colors other than c_4, the cycle C cannot be bichromatic, a contradiction. We now assume that G' admits an acyclic 4-coloring ϕ' and then construct a 3-coloring of G. By Lemma 3, all the connectors in each vertex gadget receive the same color in ϕ'. We assign the color associated to the connectors of a vertex gadget in G' to its corresponding vertex in G. Suppose for a contradiction that the resulting coloring ϕ of G is either a 4-coloring or contains two vertices with the same color that are adjacent in G.

By construction of G', all the edge gadgets are connected through the connectors. Therefore, the color of the connectors in all the edge gadgets must be the same. Without loss of generality let that color be c_4. Since every vertex gadget has a connector that is adjacent to some connector in some edge gadget in G', no connector of the vertex gadgets can receive color c_4. Therefore, ϕ contains only three different colors. We are now left with the case when ϕ contains two vertices with the same color z that are adjacent in G.

Let $\mathcal{Y}$ be the corresponding edge gadget and let $\mathcal{X}_1, \mathcal{X}_2$ be its incident vertex gadgets. Figure 4(d) illustrates an example, where $\mathcal{Y}$ meets $\mathcal{X}_1$ at the connectors a, b and $\mathcal{X}_2$ at the connectors c, d. If both the connectors of $\mathcal{X}_1$ and $\mathcal{X}_2$ are colored with color z, then by Lemma 3 we can construct a bichromatic cycle through a, b, c, d that is contained in the necklaces of $\mathcal{X}_1, \mathcal{X}_2$ and $\mathcal{Y}$. Therefore, any two connectors that lie in two different vertex gadgets must receive two different colors if those vertex gadgets are adjacent in G'. Hence, no two adjacent vertices of G can receive the same color in ϕ, a contradiction. □

6 Conclusion

The question "What is the minimum positive constant c such that every triangulated planar graph with n vertices has an acyclic k-coloring, $k \in \{3, 4\}$, with at most cn division vertices?" was posed in the 22nd International Workshop on Combinatorial Algorithms (IWOCA 2011) [23]. Although we proved that $1.28 \leq c \leq 2$ and $0.3 \leq c \leq 1.5$ for $k = 3$ and $k = 4$, respectively, there is a gap between the upper bound and the lower bound leaving a scope for improvement.

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Appendix

Proof of Remark 1.

Proof. Suppose for a contradiction that there is a bichromatic cycle C in G'. C cannot be a cycle in G because the coloring in G is bichromatic. Then C must contain the path $u, w_1, \ldots, w_j, v$. Since the vertices on the path $u, w_1, \ldots, w_j, v$ receives three different colors in ϕ, C cannot be a bichromatic cycle. Therefore, ϕ is an acyclic coloring of G'. □

Proof of Remark 3.

Proof. Suppose for a contradiction that there is an acyclic coloring of a bead where the poles x, y receive the same color c_1. The vertex s must receive a different color than r, t and x, y. Let the color of vertex r be c_2. Then, s must be assigned a different color c_3. If the color of t is c_2 then we have a bichromatic cycle x, r, y, t, x. Therefore we assume that the color of t is c_4. Then u, v must receive a color $c \in \{c_2, c_3, c_4\}$ and we will have a bichromatic cycle. □

Proof of Lemma 2.

Proof. Let $B_1, B_2, \ldots, B_p$ be the ordered sequence of beads in G_p. It suffices to prove that in any acyclic 4-coloring of $G_p, p \geq 3$, the center of bead B_j and the center of bead B_{j+2} receive the same color, where $1 \leq j \leq p-2$. To prove this claim we show that an acyclic 4-coloring of a single bead enforce the subsequent beads to follow some color pattern.

Figure 3(a) depicts a drawing of a single bead B. By Remark 3, in any acyclic 4-coloring ϕ of B the poles x, y receive different colors. Let the color of the poles be c_1 and c_2. Then all the vertices of B other than the poles are colored with c_3 or c_4. Without loss of generality assume that $col(x) = c_1, col(y) = c_2, col(u) = col(r) = col(t) = c_3$ and $col(s) = c_4$, as shown in Figure 3(c). Then the color of vertex v in ϕ can be c_3 or c_4.

Add another bead B' to B to form a G_2, as shown in Figure 3(c). Let the poles of B' be x' and y'. If $col(v) = c_3$, then both x' and y' must be colored with c_4 to avoid any bichromatic cycle. But by Remark 3, this partial coloring cannot be extended to an acyclic 4-coloring of G_2. Consequently, we have $col(v) = c_4$, which leaves us with the choice $\{col(x'), col(y')\} \subseteq \{c_1, c_2\}, col(t') = c_4$ and $col(v') = c_3$. It is now straightforward to verify that the resulting coloring is an acyclic 4-coloring of G_2.

Observe that each pole vertex in G_2 receives a color from $\{c_1, c_2\}$ and the colors of the center vertices alternate between c_3 and c_4. Since G_p is obtained from a repeated addition of beads, the center vertices of the beads B_j and B_{j+2} receive the same color, where $1 \leq j \leq p-2$. Figure 3(d) illustrates an acyclic 4-coloring of G_p. □

Proof of Lemma 3.

Proof. (a) We use an induction on i. The claim is easy to verify for N_1. Assume inductively that the claim holds for $N_z, z < i$. Now consider the case for N_{z+1}.

N_{z+1} is obtained by merging a connector of N_z with a connector of N_1. Let the connector common to N_z and N_1 be w'. Since the connectors of N_z receive the same color and the connectors of N_1 receive the same color, all the connectors in N_{z+1} must obtain the same color. Let that color be c. We now prove that for any two connectors u', v' and for any color $c' \neq c$, there exists a bichromatic path between x, y with colors c and c'. If u' and v' both lie in N_z or N_1, then our claim holds by induction hypothesis. Hence, without loss of generality assume that u' is in N_z and v' is in N_1. Then there is a bichromatic path between u' and w' with colors c and c' by the induction hypothesis. Similarly, there is a bichromatic path between w' and v' with colors c and c'.

(b) First we prove that in any acyclic coloring of L, where there are no bichromatic path between any pair of tails, $col(x) = col(l) = col(m) = col(n)$ and there is no bichromatic path from x to l. If a, z receive the same color, then b, d, f must receive the three other distinct colors to avoid bichromatic cycle in the graph induced by a, b, z, d, f. But this will force a bichromatic path between l and n through L. Therefore, a and z must receive different colors. Let the colors of a, z, b be c_1, c_2, c_3, respectively. Then d and f must receive colors c and c_3, respectively. Since h is adjacent to z, f it can only have color c or c_1. If it receives color c_1, at least one vertex among e, g, i must receive color c_3 which will create a bichromatic cycle. Therefore, the color of h must be c and e, g, i receive colors c_2, c_3, c_2, respectively. Since i and k are the poles of a jewel and k is adjacent to h, k cannot have the colors c_2 and c. In any case based on the assignment of color to k and j, The vertex x must have the color c. Note that there are no bichromatic path of colors c, c_1 between x and l.

Let the subgraph of L with shaded background be G'. We make two more copies of G' and connect it to x and y as shown in Figure 4(b). We claim that if we assign three different colors other than c to the vertices a, p and q in an acyclic coloring of L then there would be no bichromatic path between any of the tails l, m, n and w. Let the colors assigned to a, p, q be c_1, c_2, c_3, respectively. Then there is no bichromatic path of colors c, c_3 from w to y and hence from w to any tail. There are no bichromatic path of colors c, c_2 between x and y and hence from w to any tail. In a similar way we can show that there are no bichromatic paths of the colors c, c_1 between w and any of the tails. □

Proof of Theorem 4.

Proof. Given a valid 4-coloring of the vertices of the input graph, we can check in polynomial time whether the vertices of each pair of color classes induces a forest. Therefore, the problem of deciding 4-colorability is in NP. To prove the NP-hardness we reduce the NP-complete problem of deciding acyclic 3-colorability of maximum degree 4 graphs [7] to the problem of deciding acyclic 4-colorability of maximum degree 5 graphs.

Let G be an instance of acyclic 3-colorability problem, where G has n vertices and the maximum degree of G is 4. Take a copy of G_{2n-1} and connect each vertex of G with a distinct vertex of $I(G_{2n-1})$ by an edge. Let the resulting graph with maximum degree 5 be G', which is straightforward to construct in polynomial time. By the *linkers* of G' we denote these edges that connect the vertices of $I(G_{2n-1})$ with the vertices in G. We now show that G admits an acyclic 3-coloring if and only if G' admits an acyclic 4-coloring.

We first assume that G admits an acyclic 3-coloring with the colors c_1, c_2, c_3 and then construct an acyclic 4-coloring of G'. For each vertex v in G, color the corresponding vertex in G' with $col(v)$. We color G_{2n-1} acyclically with the colors c_1, c_2, c_3 and c_4 such that the vertices of $I(G_{2n-1})$ receive color c_4. Which can be done in polynomial time by Lemma 2. If the resulting coloring of G' is not acyclic, then there is a bichromatic cycle C. Since G and G_{2n-1} are colored acyclically, C must contain a linker. Therefore, some vertex on C must be colored with color c_4. Since no two linkers have a common end vertex, C must contain an edge e of G. The end vertices of e must have two of the three colors c_1, c_2, c_3. Consequently, C cannot be a bichromatic cycle, a contradiction.

We now assume that G' admits an acyclic 4-coloring and then construct an acyclic 3-coloring of G. By Lemma 2, the vertices in $I(G_{2n-1})$ are colored with the same color. Since each vertex in G is adjacent to some vertex in $I(G_{2n-1})$, the vertices of G are colored with three colors. Since G' is colored acyclically, the coloring of G is acyclic. $\square$