On the Hardness of Point-Set Embeddability*
(Extended Abstract)

Stephane Durocher and Debajyoti Mondal

Department of Computer Science,
University of Manitoba.
durocher@cs.umanitoba.ca, jyoti@cs.umanitoba.ca

Abstract. A point-set embedding of a plane graph G with n vertices on a set S of n points is a straight-line drawing of G, where the vertices of G are mapped to distinct points of S. The problem of deciding whether a plane graph admits a point-set embedding on a given set of points is NP-complete for 2-connected planar graphs, but polynomial-time solvable for outerplanar graphs and plane 3-trees. In this paper we prove that the problem remains NP-complete for 3-connected planar graphs, which settles an open question posed by Cabello (Journal of Graph Algorithms and Applications, 10(2), 2000). We then show that the constraint of convexity makes the problem easier for klee graphs, which is a subclass of 3-connected planar graphs. We give a polynomial-time algorithm to decide whether a klee graph with exactly three outer vertices admits a convex point-set embedding on a given set of points and compute such an embedding if one exists.

1 Introduction

A *planar graph* admits a planar embedding on the Euclidean plane. A *plane graph* is a fixed planar embedding of a planar graph. A *straight-line drawing* of a plane graph is a planar drawing, where the vertices are drawn as points and edges are drawn as straight line segments. In a *straight-line grid drawing*, the vertices of the corresponding graph are placed at integer grid points. The *width*, *height* and *area* of a straight-line grid drawing are respectively the width, height and area of the smallest axis-parallel rectangle that encloses the drawing. In 1990, de Fraysseix et al. [9] and Schnyder [17] gave two algorithms to find straight-line drawings of any plane graph with n vertices on $(2n-4)\times(n-2)$ and $(n-2)\times(n-2)$ grids, respectively. Since the lower bound on area for straight-line grid drawings is $\Omega(n^2)$ [15], an interesting open problem in this direction is to characterize nontrivial classes of planar graphs that admit straight-line grid drawings on $o(n^2)$ area. In a *minimum-area straight-line grid drawing* of a plane graph G, the area of the drawing is the minimum among all possible straight-line grid drawings of G. Determining minimum-area straight-line grid drawings of planar graphs is NP-hard [14].

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To meet the requirements of different practical applications researchers have explored various related problems, where the the goal is to find an embedding of a given plane graph placing the vertices on a set of pre-specified locations satisfying different aesthetic requirements [4, 13]. Let G be a plane graph with n vertices and let S be a set of n pre-specified points on the Euclidean plane. A *point-set embedding of G on S* is a straight-line embedding of G on S, where each vertex of G is mapped to a distinct point of S. Although the point-set embeddability problem is polynomial-time solvable for outerplanar graphs [4] and plane 3-trees [16], Cabello proved that it is NP-complete in general to decide whether a given 2-connected plane graph admits a point-set embedding on a given set of points [5]. As mentioned by Cabello, previous techniques used to show NP-hardness for point-set embeddability problems do not seem to apply to the case of 3-connected planar graphs. The problem deserves investigation since many graph drawing problems (e.g., upward planarity testing [11], bend minimization in planar orthogonal drawings [18]) that are NP-hard for 2-connected planar graphs, become polynomial-time solvable for 3-connected planar graphs. The problem seems more interesting with the following two observations. Firstly, a variant of the point-set embeddability problem, where we allow one bend per edge, is NP-complete for 3-connected planar graphs [13]. Secondly, the point-set embeddability problem is polynomial-time solvable for plane 3-trees [16], which is a subclass of 3-connected planar graphs.

In this paper we prove that the point-set embeddability problem remains NP-complete for 3-connected plane graphs, which settles the open problem posed by Cabello [5]. We then show that the constraint of convexity makes both the point-set embeddability problem and the area minimization problem polynomial-time solvable for graphs having a plane 3-tree as weak dual.

A *convex drawing* of a plane graph is a straight-line drawing of the graph, where each face is a convex polygon. A *convex point-set embedding* of a graph G on a point set S is a point-set embedding of G on S, where each face in the embedding is a convex polygon. See Figure 1. A *minimum-area convex grid drawing* of a graph G is a convex grid drawing of G, where the area of the drawing is minimized. Much research effort has been devoted to computing convex grid drawings of 3-connected plane graphs on small integer grids [2, 3]. Recently, both the problems of computing minimum-area convex grid drawings and convex point-set embeddings have been solved for plane 3-trees in polynomial time [15, 16]. Since every plane 3-tree is a maximal planar graph, all of its straight-line drawings are convex. Therefore, a natural question is whether the problems of computing convex point-set embeddings and minimum-area convex grid drawings are polynomial-time solvable for graphs having a plane 3-tree as dual. This leads us to examine klee graphs.

A *klee graph* G is a plane cubic graph with $n{\geq}4$ vertices, which is defined as follows: (a) If $n{=}4$, then G is a K_4. (b) Otherwise, G has a face f of length three such that contraction of the three edges on the boundary of f gives another klee graph with $n{-}2$ vertices. In other words, every klee graph can be constructed from K_4 by repeatedly replacing a vertex with a face of length three. The weak

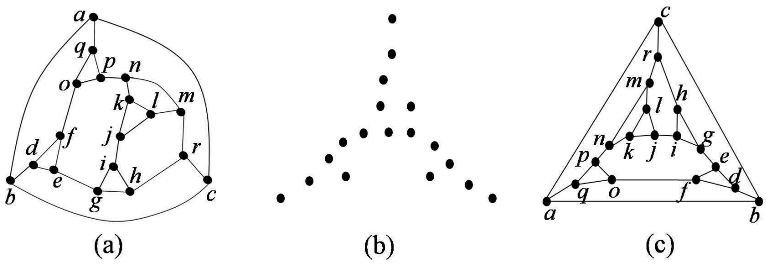

Fig. 1. (a) A plane graph G with eighteen vertices. (b) A set S of eighteen points. (c) A convex point-set embedding of G on S.

dual of every klee graph with exactly three outer vertices is a plane 3-tree. Klee graphs arise naturally in many theoretical and applied fields [1, 7].

We give a polynomial-time algorithm to decide whether a klee graph with exactly three outer vertices admits a convex point-set embedding on a given set of points and compute such an embedding if one exists. We then extend the idea to give a polynomial-time algorithm for finding a minimum-area convex grid drawing of a klee graph with exactly three outer vertices.

Our algorithms are motivated by the recursive structure of plane 3-trees, i.e., every plane 3-tree with $n \geq 4$ vertices can be decomposed into three smaller plane 3-trees [15]. An outline of our algorithm for computing a convex point-set embedding of a klee graph is as follows. Let G be a klee graph with exactly three outer vertices. We use dynamic programming to test whether G admits a convex point-set embedding on a set of given points. We show that this decision problem can be recursively divided into three subproblems. We solve those subproblems recursively and combine their results to obtain the result of the original decision problem. Figure 2(a) depicts a klee graph G, where the weak dual G^* is shown in dotted lines. The decomposition of G^* into three smaller plane 3-trees determines a decomposition of G into three subgraphs, as shown in Figure 2(b) in red, blue and green. Figure 2(c) depicts a point set S. If G admits a point-set embedding on S, where the outer vertices a, b, c of G are mapped respectively to the points s_1, s_2, s_3 of S, then those three subgraphs must be mapped to the point sets shown in the corresponding colors.

2 Preliminaries

In this section we introduce some definitions and present preliminary results.

Let G be a graph. By $n(G)$ we denote the number of vertices of G. *Contraction of an edge* (u, v) of G is an operation that removes edge (u, v) from G by merging u and v, and replaces any resulting multiedge by a single edge.

A plane graph G delimits the plane into connected regions called *faces*. The unbounded face is the *outer face* of G and all the other faces are the *inner faces* of G. The *length of a face* is the number of vertices on the boundary of the face.

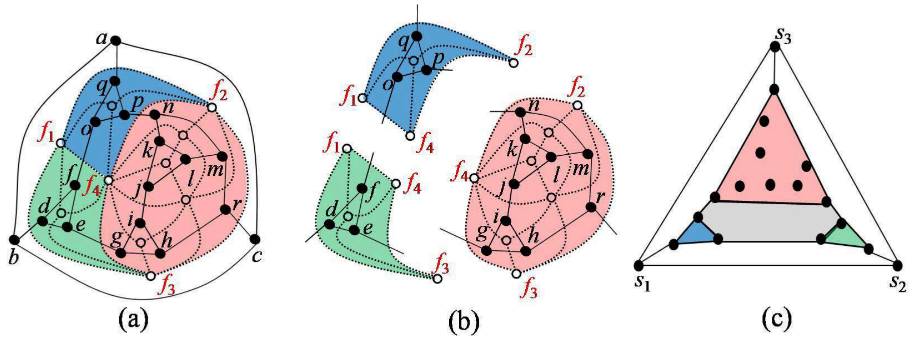

Fig. 2. Illustration of the algorithm for convex point-set embedding.

The vertices on the outer face are the *outer vertices* and all the other vertices are the *inner vertices*. Let u and v be two outer vertices of G. Then the *outer chain between* u *and* v is the anticlockwise path between u and v on the outer face of G. By C_{abc} we denote a cycle C that has exactly three vertices a, b, c on its boundary. By $G(C_{abc})$ we denote the plane subgraph of G induced by the vertices on C_{abc} and the vertices interior to C_{abc}. A *plane* 3*-tree* G with $n \geq 3$ vertices is a plane graph for which the following hold: (a) G is a triangulated plane graph; (b) if $n > 3$, then G has a vertex whose deletion gives a plane 3-tree with $n-1$ vertices. Every plane 3-tree G contains a vertex which is the common neighbor of all the three outer vertices of G. We call this vertex the *representative vertex* of G.

Let S be a set of points on the Euclidean plane. For any three points $\{x, y, z\} \subseteq S$, $S(xyz)$ is the set that consists of the points of S that are on or inside the triangle xyz. $S'(xyz)$ is the set that consists of the points of S that are in the proper interior of triangle xyz. $|S|$ denotes the number of points in S. Let $s_1, s_2, \ldots, s_n$ be an ordered sequence of $n > 1$ points. By $\langle s_1, s_2, \ldots, s_n\rangle$ we denote a drawing of a path obtained by drawing a straight line between s_i and s_{i+1}, $1 \leq i < n$. We call $\langle s_1, s_2, \ldots, s_n\rangle$ a *convex path*, if joining s_1 and s_n with a straight line segment results in a convex polygon.

3 Point-Set Embeddings of 3-Connected Planar Graphs

In this section we prove that point-set embeddability problem is NP-complete for 3-connected planar graphs.

A formal definition of the problem is as follows:

Problem: Point-Set Embeddings of 3-Connected Planar Graphs (PSE)

Instance: A 3-connected planar graph $\mathcal{G}$ with n vertices and a set $\mathcal{S}$ of n points not necessarily in general position.

Question: Does $\mathcal{G}$ admit a point-set embedding on $\mathcal{S}$?

We prove the NP-hardness of PSE by reducing the problem of deciding whether a given 3-connected cubic planar graph is Hamiltonian or not, which has been proved to be NP-complete in [10].

Problem: HAMILTONIAN CYCLE IN 3-CONNECTED CUBIC GRAPHS (HC)

Instance: A 3-connected cubic planar graph M.

Question: Does M contain a Hamiltonian cycle?

Here is an outline of our proof for NP-hardness. For a given instance M of HC with n vertices, we construct a 3-connected planar graph $\mathcal{G}$ with $9n^2 + 7n$ vertices and a set $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k\}$ of k point sets, where $k = 3n-1$ and each point set $\mathcal{S}_i$, $1 \leq i \leq k$, contains $9n^2+7n$ points. We prove that $\mathcal{G}$ admits a point-set embedding on some $\mathcal{S}_i$ if and only if M contains a Hamiltonian cycle. We first assume that M contains a Hamiltonian cycle, and then show a construction of a point-set embedding of $\mathcal{G}$ on some $\mathcal{S}_i$. We then assume that $\mathcal{G}$ admits a point-set embedding Γ on some $\mathcal{S}_i$ and then prove that M contains a Hamiltonian cycle. To prove this, we show that the subgraphs of $\mathcal{G}$, each of which corresponds to a distinct vertex of M, must be embedded in some ordered fashion in Γ. This will in turn correspond to an order of the vertices of M determining a Hamiltonian cycle.

We now describe the formal reduction. Let M be an instance of HC with n vertices. We now construct a plane graph $\mathcal{G}$ with $9n^2 + 7n$ vertices as follows:

(a) Take any plane embedding of M. For each face f in M, place a cycle R of l vertices inside f, where l is the length of f. We call each cycle R a *ring*. Let the vertices on the boundaries of f and R be $f_1, f_2, \ldots, f_l$ and $r_1, r_2, \ldots, r_l$, respectively. Then add the edges $(f_i, r_i), 1 \leq i \leq l$. Let the resulting graph be M'. Then remove the edges of M' that originally belonged to M. See Figures 3(a) and (b).

(b) Let N be a plane 3-tree with the three outer vertices a', b', c' in anticlockwise order and let the representative vertex of N be p. Assume that each of the three subgraphs of N induced by the vertices interior to $C_{a'b'p}, C_{b'c'p}$ and $C_{c'a'p}$ is a path of $3n$ vertices. We call such a plane 3-tree a *supernode*. Now for each vertex u of M' that originally belonged to M, replace u with a copy of N as follows. Let r, s, t be the three neighbors of u in anticlockwise order. Delete u and the edges adjacent to u and then add the edges $(a', r), (b', s), (c', t)$. An example is illustrated in Figure 3(c). The resulting plane graph determines the required planar graph $\mathcal{G}$.

Since M is a 3-connected cubic planar graph, the number of vertices in all the rings of $\mathcal{G}$ is twice the number of edges of M, i.e., $3n$. Moreover, there are n supernodes in $\mathcal{G}$. Therefore, the number of vertices in $\mathcal{G}$ is $3n + n(9n + 4) = 9n^2 + 7n$ vertices. We now have the following lemma.

Lemma 1. *$\mathcal{G}$ is a 3-connected planar graph.*

We now construct a set $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k\}$ of k point sets, where $k = 3n-1$ and each $\mathcal{S}_j, 1 \leq j \leq k$, contains the following points:

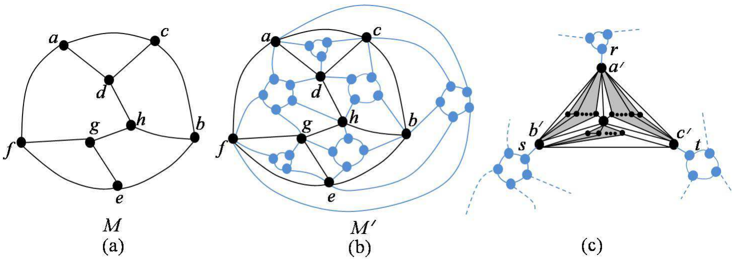

Fig. 3. (a)A 3-connected cubic planar graph M. (b) M', where the vertices and the edges not in M are shown in blue. (c) Replacement of the vertex d with a supernode.

(a) The set P_t of n points on line $y = 1$, where $P_t = \{(3i, 1) \mid 0 \leq i \leq n-1\}$. The set P_m of n points on line $y = 0$, where $P_m = \{(3i + 1, 0) \mid 0 \leq i \leq n - 1\}$. The set P_b of n points on line $y = -1$, where $P_b = \{(3i, -1) \mid 0 \leq i \leq n - 1\}$. Observe that each triple of points $\{(3i, 1), (3i + 1, 0), (3i, -1)\}$ form an isosceles triangle. We denote such a triangle by T_i, where we call the points $(3i, 1), (3i + 1, 0)$ and $(3i, -1)$ the *top*, *middle* and *bottom* points of T_i, respectively. See Figure 4(a).

(b) The set $A = \{(9n - x, 10n + x^2) \mid 0 \leq x \leq j - 1\}$ of j points on a parabola. The set $B = \{(9n - x, -10n - x^2) \mid 0 \leq x \leq 3n - j - 1\}$ of $3n - j$ points on a parabola. See Figure 4(a). Observe that for each T_i the following holds:

  (i) Every straight line joining the bottom of T_i with a point in A, properly intersects the edge between the top point and the middle point of T_i. We denote the triangle determined by the points $(9n, 10n), (9n - j + 1, 10n+(j-1)^2)$ and the bottom of T_i as A_i. Observe that every straight line joining the bottom point of T_i with a point in A is contained in A_i. See Figure 4(b).

  (ii) Every line joining the top of T_i with a point in B, properly intersects the edge between the bottom point and the middle point of T_i. We denote the triangle determined by the points $(9n, -10n), (9n-3n+j+1, -10n-(3n-j-1)^2)$ and the top of T_i as B_i.

  (iii) For every i and every $\epsilon \in (0, 1)$, line $y = \epsilon$ intersects both A_i and the edge e_i between the top point and the middle point of T_i. Let r, s and t denote the respective points of intersection between $y = \epsilon$ and A_i, and between $y = \epsilon$ and e_i. Assume that s is closer to t than r. Then there exists a small constant $\epsilon > 0$ such that for each T_i, s lies interior to T_i and $s \neq t$. See Figure 4(c). We will show how to choose ϵ later in Lemma 2. Let l_i be the open line segment from s to t.

(c) The sets $W_i, 0 \leq i \leq n - 1$, where each W_i contains $6n + 1$ points on the line $y = \epsilon$ interior to T_i. The points in each W_i are divided into two subsets X_i and Y_i. X_i contains $3n$ points that are on segment l_i. Y_i contains the remaining $3n + 1$ points that are interior to T_i, but not on l_i. The $3n + 1$ points in Y_i includes the intersection points of the lines joining the points of

A and B with the bottom point and the top point of T_i, respectively. See Figure 4(c).

(d) The sets $Z_i, 0 \le i \le n-1$, each containing $3n$ points, which are the perpendicular projections of the points of X_i on the line $y = 0$. See Figure 4(c).

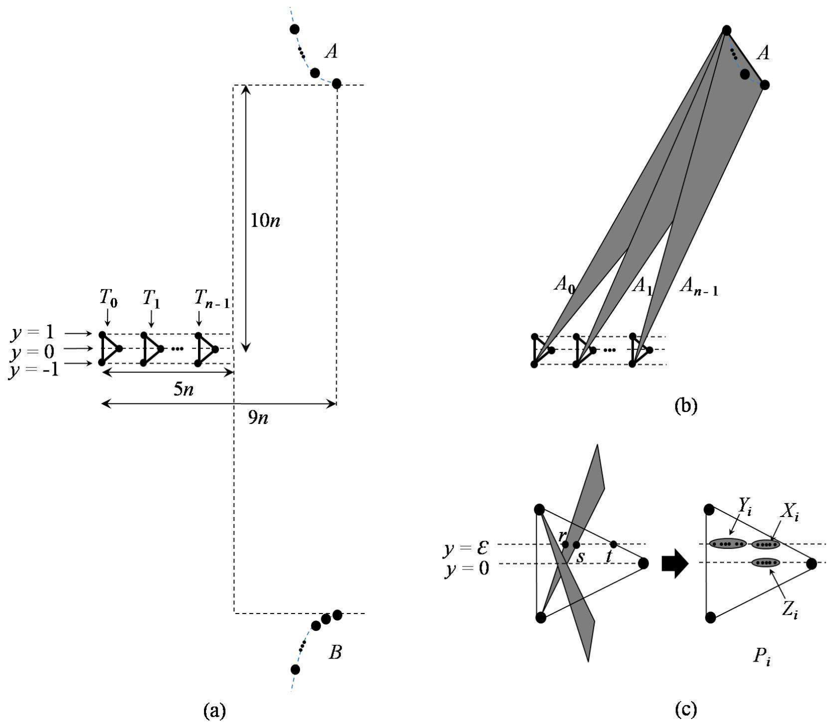

Fig. 4. (a) Construction of $T_i, 0 \le i \le n-1, A$ and B. (b) Illustration for A_i. (c) Construction of X_i, Y_i and Z_i.

Observe that the total number of points in $\mathcal{S}_j$ is $|P_t|+|P_m|+|P_b|+|A|+|B|+n(|W_i|+|Z_i|) = 6n+n(9n+1) = 9n^2+7n$. Let $P_i = W_i \cup Z_i \cup T_i, 0 \le i \le n-1$, and let $C = \mathcal{S}_j \setminus (A \cup B)$. We now have the following lemma.

Lemma 2. *$\mathcal{S}_j$ has the following properties:*

(a) The coordinates of the points in $\mathcal{S}_j$ are expressible using a number of bits that is polynomial in n.

(b) If $\mathcal{G}$ admits a point-set embedding on $\mathcal{S}_j$, then the supernodes of $\mathcal{G}$ must be mapped into the points in C, where each supernode is mapped to a distinct point set P_i.

(c) For any given mapping of the outer vertices of a supernode N to the three points on the convex hull of P_i, N admits a point-set embedding on P_i.

We now use the reduction described above to prove the following theorem.

Theorem 1. PSE *is NP-complete.*

Proof. Given a mapping of the vertices of the input 3-connected planar graph G to the input point set S, it is straightforward to check if the drawing determined by this mapping is a planar straight-line drawing of G in polynomial time. Therefore, the problem is in NP.

We now prove NP-hardness. Let M be a given instance of HC, where M has n vertices. We construct the corresponding graph $\mathcal{G}$ with $9n^2+7n$ vertices and a set $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{3n-1}\}$ of $3n-1$ point sets, where each $\mathcal{S}_j, 1 \le j \le 3n-1$, contains $9n^2+7n$ points. The number of vertices of $\mathcal{G}$ and the number of points in $\mathcal{S}$ are polynomials in n. Moreover, by Property (a) of Lemma 2, the coordinates of the points are bounded by polynomials in n. Consequently, we can construct $\mathcal{G}$ and $\mathcal{S}$ in polynomial time. We now prove that $\mathcal{G}$ admits a point-set embedding on some $\mathcal{S}_j$ if and only if M contains a Hamiltonian cycle.

We first assume that M contains a Hamiltonian cycle H and then give a construction of a point-set embedding of $\mathcal{G}$ on some $\mathcal{S}_j$ as follows. Choose a plane embedding $\mathit{\Gamma}$ of M, where $\mathit{\Gamma}$ contains an edge of H on the outer face. Let $H = (u_0, u_1, \ldots, u_{n-1}, u_0)$ and let (u_0, u_{n-1}) be the edge of H that lies on the outer face in $\mathit{\Gamma}$. H partitions the faces of M into two sets F_1 and F_2, where F_1 contains the faces outside of H and F_2 contains the faces interior to H. In any plane embedding $\mathit{\Gamma}'$ of M, where the outer face is similar to the outer face of $\mathit{\Gamma}$, if the path $u_0, u_1, \ldots, u_{n-1}$ lies along a horizontal line L, then the faces interior to H will be on one side (above or below) of L and all the other faces of M will be on the other side of L. Recall that each face of M correspond to a ring of $\mathcal{G}$. Let l_1 and l_2 be the number of vertices in all the rings that correspond to the faces in F_1 and F_2, respectively. Then $l_1 + l_2$ is twice the number of edges of M, i.e., $l_1 + l_2 = 3n$. In the following we show a construction of a point-set embedding of $\mathcal{G}$ on $\mathcal{S}_{l_1}$.

Recall that $\mathcal{S}_{l_1}$ consists of the point sets A, B and $P_i, 0 \le i \le n-1$, where $|A| = l_1, |B| = l_2$ and $|P_i| = 9n + 4$. First map the rings that correspond to the faces in F_1 and F_2 on the point set A and B, respectively, such that the order of the rings corresponds to the order of the faces in $\mathit{\Gamma}'$. We now only need to map the supernodes of $\mathcal{G}$ into P_i, where we can draw the edges between the rings and the supernodes avoiding edge crossings. Recall that each supernode corresponds to a node u_i of M. Since H passes through two adjacent edges at each $u_i, 0 \le i \le n-1$, the three faces incident to u_i cannot be all in F_1 or F_2. Hence, each ring must be connected with a supernode with at most two edges. Using this observation along with Property (c) of Lemma 2, it is now straightforward to map the supernode corresponding to u_i into P_i avoiding any edge crossings.

We now assume that $\mathcal{G}$ admits a point-set embedding $\mathit{\Gamma}$ on some $\mathcal{S}_j$ and then prove that M contains a Hamiltonian cycle as follows. By Property (b) of Lemma 2, each supernode of $\mathcal{G}$ is mapped to a distinct P_i. Let u_i be the vertex of M that corresponds to the supernode mapped to P_i. We then claim that $u_0, u_1, \ldots, u_{n-1}, u_0$ is a Hamiltonian cycle in M. Otherwise, assume that $(u_i, u_{(i+1) \bmod n})$ is not an edge of M. From the construction of $\mathcal{G}$, observe that if we consider each ring and each supernode of $\mathcal{G}$ as a giant node, then each face of $\mathcal{G}$ is a quadrilateral that contains two rings and two supernodes on its boundary. Therefore, if $(u_i, u_{(i+1) \bmod n})$ is not an edge of M, then two rings will be inside the same face of M, deriving a contradiction to the fact that each ring of $\mathcal{G}$ corresponds to a distinct face of M. □

It may initially appear that we have proved NP-hardness for a version of the point-set embeddability problem in which the number of points is greater than the number of vertices. Note that we have reduced an instance of HC to a polynomial number of instances of PSE. If there exists a polynomial-time algorithm for PSE, then we can simulate that algorithm for those instances of PSE to obtain the answer to the instance of HC in polynomial time, which is impossible if P $\neq$ NP.

The point set we construct in our reduction contains many collinear points. It would be interesting to study whether a similar reduction is possible when no three points in the point set are collinear. Although our NP-hardness proof is inspired by the NP-hardness proofs for some other problems presented in [8, 12], our reduction technique is significantly different from the techniques used in [8, 12] in several aspects.

4 Convex Point-Set Embeddings of Klee Graphs

In this section we examine a specific subclass of 3-connected planar graphs, namely klee graphs. We present an $O(n^8)$-time algorithm for computing a convex point-set embedding of a klee graph with n vertices and exactly three outer vertices on a set of n points in general position. Before describing the algorithm we focus on some properties of a klee graph and its convex point-set embedding.

Let G be a plane graph. The *dual graph* G^* *of* G is a plane graph which has a vertex for each face of G, and two vertices in G^* are adjacent if the corresponding faces in G share an edge. The *weak dual* of G is the subgraph of G^* whose vertices correspond to the inner faces of G. We now have the following lemma, whose proof is omitted due to page limit.

Lemma 3. *Let G be a plane graph with exactly three outer vertices. Then G is a klee graph if and only if the weak dual G^* of G is a plane 3-tree.*

A *near klee graph* K is a plane graph that is obtained by deleting the outer vertices of some klee graph having exactly three outer vertices. Observe that every near klee graph with $n \geq 3$ vertices contains exactly three outer vertices of degree two. We call these vertices the *poles* of K. If K is obtained from a klee graph G, then the three outer vertices of G are the *legs* of K. The legs and the poles of the near klee graph obtained from the klee graph of Figure 2(a) are a, b, c and q, d, r, respectively. The following lemma proves that every near klee graph can be decomposed into three smaller near klee graphs.

Lemma 4. *Let K be a near klee graph with $n \geq 3$ vertices, which is obtained from a klee graph G with exactly three outer vertices. Let G^* be a weak dual of G, where the outer vertices of G^* are f_1, f_2, f_3 and the representative vertex of G^* is f_4. Then the weak duals of $G^*(C_{f_1 f_2 f_4})$, $G^*(C_{f_2 f_3 f_4})$ and $G^*(C_{f_3 f_1 f_4})$ are three vertex-disjoint near klee graphs, which are subgraphs of K.*

By a *klee partition* of K we denote the three near klee graphs obtained from the decomposition of K according to Lemma 4. See Figures 2(a) and (b). We now have the following lemma.

Lemma 5. *Let G be a klee graph with exactly three outer vertices. Let $n \geq 6$ be the number of vertices of G and let S be a set of n points. If G admits a convex point-set embedding on S, then the following conditions hold.*
(a) The convex hull of S has exactly three points s_1, s_2 and s_3 on its boundary.
(b) The convex hull $\mathcal{C}$ of $S \setminus \{s_1, s_2, s_3\}$ has exactly three points on its boundary.
(c) Let K be the near klee graph obtained from G. Then for a fixed mapping of the legs of K to $\{s_1, s_2, s_3\}$, the mapping of the poles of K to the three vertices on $\mathcal{C}$ is uniquely defined.

Let K be a near klee graph with n vertices and let S be a set of n points. Let a, b, c be the legs of K and let p, q, r be the poles of K. Then a *near convex point-set embedding* of K on S is a point-set embedding of K on S with the following properties.

(a) The legs a, b, c of K are mapped respectively to the points s_1, s_2, s_3 of S, where s_1, s_2, s_3 determine the convex hull $\mathcal{C}$ of S.
(b) The poles p, q, r of K are respectively mapped to the points s_4, s_5, s_6 of S, where s_4, s_5, s_6 determine the convex hull of $S \setminus \{s_1, s_2, s_3\}$.
(c) All the interior faces of K are drawn as convex polygons.
(d) Let the outer chains of K between a, b; b, c and c, a be $a, p, \ldots, q, b$; $b, q, \ldots, r, c$ and $c, r, \ldots, p, a$. Then $\langle a, p, \ldots, q, b\rangle$, $\langle b, q, \ldots, r, c\rangle$ and $\langle c, r, \ldots, p, a\rangle$ are convex paths.

Observe that if G is a klee graph with exactly three outer vertices and K is a near klee graph obtained from G, then G admits a convex point-set embedding on S if and only if K admits a near convex point-set embedding on S. Since the outer vertices of G can be mapped to the three points of the convex hull of S in six different ways, we check whether K admits a near convex point-set embedding on S for each of those six mappings. In the rest of this section we give a dynamic programming algorithm to test whether K admits a near convex point-set embedding on S respecting a given mapping of its legs. A formal definition of the problem is as follows.

Problem: NEAR CONVEX POINT-SET EMBEDDING

Instance: A near klee graph K with $n \geq 3$ vertices, a set S of n points in general position whose convex hull contains exactly three points s_i, s_j and s_k, and a mapping of the legs a, b, c of K to the points s_i, s_j, s_k.

Question: Does K admit a near convex point-set embedding on S respecting the given mapping of its legs?

Let v_p be the mapping of vertex v on point p and let $P(K, a_{s_i}, b_{s_j}, c_{s_k})$ be the problem of finding a near convex point-set embedding of K respecting $a_{s_i}, b_{s_j}, c_{s_k}$. The following theorem gives a recursive solution for $P(K, a_{s_i}, b_{s_j}, c_{s_k})$.

Theorem 2. *Let K be a near klee graph with n vertices and let S be a set of n points whose convex hull $\mathcal{C}$ has exactly three points s_i, s_j, s_k. Let a, b, c be the legs of K and let $a_{s_i}, b_{s_j}, c_{s_k}$ be the mapping of a, b, c. Let the klee partition of K be $\{H_1, H_2, H_3\}$, where the legs of H_1, H_2, H_3 are $\{a, a', a''\}$, $\{b, b', b''\}$ and $\{c, c', c''\}$ in anticlockwise order. Let x_1, y_1, z_1, $x_2, y_2, z_2, x_3, y_3, z_3$ be nine points in $S'(s_i s_j s_k)$ that satisfy the following conditions.*

C_1: *The triangle $x_1x_2x_3$ determines the convex hull of $S \setminus \{s_1, s_2, s_3\}$.*
C_2: *$|S(x_1y_1z_1)| = n(H_1)$, $|S(x_2y_2z_2)| = n(H_2)$ and $|S(x_3y_3z_3)| = n(H_3)$. If $n(H_u) = 1$, for some $u \in \{1, 2, 3\}$, then x_u, y_u and z_u are the same point. Otherwise, x_u, y_u and z_u are distinct.*
C_3: *$\langle s_i, x_1, y_1, z_2, x_2, s_j\rangle$, $\langle s_j, x_2, y_2, z_3, x_3, s_k\rangle$ and $\langle s_k, x_3, y_3, z_1, x_1, s_i\rangle$ are convex paths inside $\mathcal{C}$.*

Let $\mathcal{X}$ be a quantifier that denotes "All possible choices for x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3 that satisfy (C_1)–(C_3)". Then $P(K, a_{s_i}, b_{s_j}, c_{s_k})$ can be defined recursively as follows.

$$P(K, a_{s_i}, b_{s_j}, c_{s_k}) = \begin{cases} True & \text{if } n(K)=1 \text{ and } |S'(s_i s_j s_k)|=1;\\ False & \text{if } \mathcal{X} \text{ is empty};\\ \bigvee_{\mathcal{X}} (P(H_1, a_{s_i}, a'_{z_2}, a''_{y_3}) \wedge & \\ \quad P(H_2, b_{s_j}, b'_{z_3}, b''_{y_1}) \wedge & \\ \quad P(H_3, c_{s_k}, c'_{z_1}, c''_{y_2})) & \text{Otherwise.} \end{cases}$$

Theorem 2 leads us to an $O(n^8)$-time dynamic programming algorithm for computing a near convex point-set embedding of K on S respecting the given mapping of the legs of K. In a similar technique we can devise a polynomial-time dynamic programming algorithm for computing a minimum-area convex grid drawing of a klee graph with exactly three outer vertices.

5 Open Problems

Cabello [5] proved the NP-hardness of the point-set embeddability problem for 2-connected planar graphs. In this paper we have shown the problem remains NP-hard for 3-connected planar graphs. Our NP-hardness proof relies on the NP-hardness of finding a Hamiltonian cycle in a 3-connected planar graph. Since every 4-connected planar graph is Hamiltonian and, furthermore, a Hamiltonian cycle can be found in linear time [6], our NP-hardness proof does not hold for 4-connected planar graphs.

Problem 1: What is the time complexity of deciding whether a 4-connected planar graph with n vertices admits a point-set embedding on a set of n points?

We have given an $O(n^8)$-time dynamic programming algorithm for testing convex point-set embeddability of klee graphs with exactly three outer vertices. However, the time complexity of deciding convex point-set embeddability of 3-connected planar graphs is still unknown.

Problem 2: What is the time complexity of deciding whether a 3-connected planar graph with n vertices admits a convex point-set embedding on a set of n points? Can we achieve faster algorithm for computing convex point-set embeddings of klee graphs?

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