Minimum-Segment Convex Drawings of 3-Connected Cubic Plane Graphs*

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Abstract. A convex drawing of a plane graph G is a plane drawing of G, where each vertex is drawn as a point, each edge is drawn as a straight line segment and each face is drawn as a convex polygon. A maximal segment is a drawing of a maximal set of edges that form a straight line segment. A minimum-segment convex drawing of G is a convex drawing of G where the number of maximal segments is the minimum among all possible convex drawings of G. In this paper, we present a linear-time algorithm to obtain a minimum-segment convex drawing Γ of a 3-connected cubic plane graph G of n vertices, where the drawing is not a grid drawing. We also give a linear-time algorithm to obtain a convex grid drawing of G on an $(\frac{n}{2}+1) \times (\frac{n}{2}+1)$ grid with at most $s_n + 1$ maximal segments, where $s_n = \frac{n}{2} + 3$ is the lower bound on the number of maximal segments in a convex drawing of G.
Keywords. Graph drawing, Convex drawing, Minimum-segment, Grid drawing, Cubic graph.

1 Introduction

From the advent of the field of graph drawing, various graph drawing styles have been studied and "convex drawing" is one of the most widely studied drawing styles. A *convex drawing* Γ of G is a "straight-line drawing" of G such that all the faces of G are drawn as convex polygons in Γ. A *straight-line drawing* Γ of a plane graph G is a plane drawing of G where each vertex of G is drawn as a point and each edge of G is drawn as a straight line segment. Any two clockwise

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consecutive edges incident to a vertex v in Γ form an *angle* θ_v at the vertex v. For $\theta_v = 180°$, we call θ_v a *straight angle*. We call a set of edges S a *maximal segment* in Γ if S is a maximal set of edges that form a straight line segment in Γ. A *minimum-segment convex drawing* of G is a convex drawing of G, where the number of maximal segments is the minimum among all possible convex drawings of G. Figures 1(a), (b) and (c) depict a plane graph G, a convex drawing of G and a minimum-segment convex drawing of G, respectively. We call a straight line drawing of G a *straight-line grid drawing* where the vertices are drawn on integer grid points. Figure 2 (g) depicts a straight-line grid drawing of a plane graph on 8×8 grid.

Convex drawings of plane graphs is one of the classical and widely studied

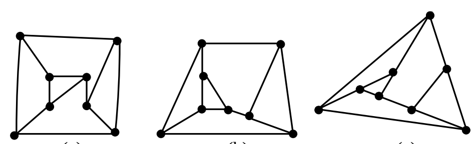

Figure 1. (a) A plane graph G, (b) a convex drawing of G and (c) a minimum-segment convex drawing of G.

drawing styles for plane graphs. Although not every plane graph has a convex drawing, every 3-connected plane graph has such a drawing [13]. Several algorithms are known for finding convex drawings of plane graphs which improve various aesthetic qualities of the drawings [2,3,4]. Dujmović *et al.* first addressed the problem of obtaining a drawing of a planar graph with few segments [6]. They have also shown that any 3-connected cubic plane graph (i.e., a plane graph where every vertex of the graph has degree three) admits a drawing with at most $n + 2$ maximal segments. Recently, Samee *et al.* have given a linear-time algorithm for computing a minimum-segment drawing of a "series-parallel graph" with the maximum degree three [12]. These recent works motivated us to study the problem of finding minimum-segment convex drawings of plane graphs.

In this paper we give a linear-time algorithm to obtain a minimum-segment convex drawing Γ of a 3-connected cubic plane graph G with n vertices. We also give a linear-time algorithm to obtain a convex grid drawing of G on an $(\frac{n}{2} + 1) \times (\frac{n}{2} + 1)$ grid with at most $s_n + 1$ maximal segments, where $s_n = \frac{n}{2} + 3$ is the lower bound on the number of maximal segments in a convex drawing of G. Although several drawing styles are known for 3-connected cubic plane graphs [8,9,10], to the best of our knowledge this is the first work on near-optimal minimum-segment convex drawing where the drawing is a grid drawing.

We now present an outline of our algorithm for grid drawing. Let G be a 3-connected cubic plane graph with n vertices. We partition the input graph G into several vertex disjoint subsets by a "canonical decomposition" described in [11]. We add each subset one after another to construct a drawing Γ of G incrementally. At each addition we ensure that each vertex of degree three in the resulting drawing gets a straight angle except two of the vertices at the initial subset. We add the last subset in such a way that, at the end of the construction, there are at least $n-4$ straight angles which are associated with $n-4$ different vertices of G. Using this property of Γ we prove that Γ has at most s_n+1 maximal segments. Figure 2 depicts a "canonical decomposition" of a 3-connected cubic plane graph G . Figures 2(a)–(g) illustrate the incremental construction of a convex grid drawing Γ of G with at most s_n+1 maximal segments. Note that G has 14 vertices and each of the vertices of G has one straight angle in Γ except the vertices 1, 5, 6 and 14. We derive a relation between the number of straight angles and the number of maximal segments in Γ, and use the relation to obtain the number of maximal segments in Γ.

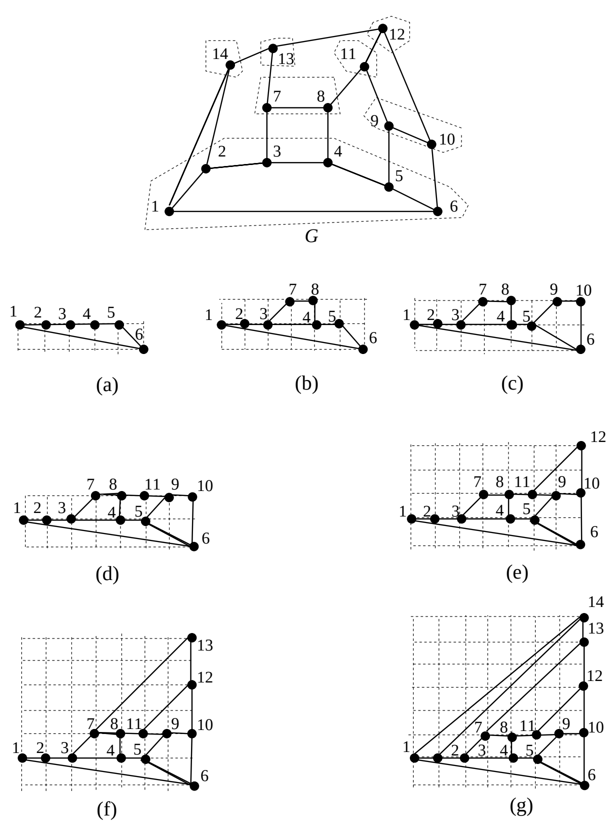

Figure 2. Illustration of the algorithm for convex grid drawing.

The rest of this paper is organized as follows. Section 2 presents some definitions and preliminary results. Section 3 gives a linear-time algorithm for obtaining a convex grid drawing Γ of a 3-connected cubic plane graph G with at most $s_n + 1$ maximal segments. Section 4 gives a linear-time algorithm to obtain a minimum-segment convex drawing Γ of G where Γ is not a grid drawing. Section 5 concludes the paper suggesting some future works. An early version of this paper has been presented at [1].

2 Preliminaries

In this section we give some definitions that will be used throughout the paper and present some preliminary results.

Let $G = (V, E)$ be a connected simple graph with vertex set V and edge set E. We denote by *degree*(v) the degree of a vertex v of G. G is a *cubic graph* if degree(v) is equal to three for every vertex v of G. An edge with the end vertices v_1 and v_2 is denoted by (v_1, v_2). A subgraph of a graph $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. If G' contains all the edges of G that join vertices in V', then G' is called the *subgraph induced* by V'. A graph G is *connected* if there is a path between any two distinct vertices u and v in G. A graph which is not connected is called a *disconnected* graph. The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph. We say that G is *k-connected* if $\kappa(G) \geq k$.

We call a graph *planar* if it can be embedded in the plane without any edge crossing. A *plane graph* is a planar graph with a fixed planar embedding. In a plane graph G, the plane is divided into some connected regions called the *faces*. The unbounded region is called the *outer face* and all the other faces are called the *inner faces*. A vertex on the outer face is called an *outer vertex* and an edge on the outer face is called an *outer edge*. All the vertices other than the outer vertices are called the *inner vertices*.

Let Γ be a straight-line grid drawing of G. The *area* of Γ is measured by the size of the smallest rectangle with sides parallel to the axes which encloses Γ. The *width* W of Γ is the number of vertical grid-lines and the *height* H of Γ is the number of horizontal grid-lines in such a rectangle. The grid size of Γ is usually described as $W \times H$. Note that, if the grid size of Γ is $W \times H$, then the area of Γ is $(W - 1) \times (H - 1)$.

Let Γ be a straight-line drawing of G. Any two clockwise consecutive edges incident to a vertex v in Γ form an *angle* θ_v at the vertex v. We call θ_v a *convex angle* if $0° < \theta_v < 180°$. For $\theta_v = 180°$ we call θ_v a *straight angle*. A *convex polygon* is a simple polygon such that every internal angle of the polygon is either a convex angle or a straight angle. An *internally convex* drawing of G is a straight-line drawing of G where all the inner faces are drawn as convex polygons. We now present some properties of a convex drawing of a 3-connected cubic plane graph.

Lemma 1. *Let G be a 3-connected cubic plane graph with n vertices and Γ be a convex drawing of G. Then the number of straight angles in Γ is at most $n - 3$.*

Proof. Since Γ is a convex drawing of a 3-connected cubic plane graph, the outer face of G must be a convex polygon. A convex polygon with the minimum number of angles is a triangle. Since a triangle has three convex angles, none of the angles at these vertices is a straight angle. Furthermore, at most one angle at a vertex in Γ can be a straight angle since G is cubic. Therefore, the number of straight angles in Γ is at most $n-3$. □

A path $v_1, v_2, \ldots, v_{k+1}$ in a graph G is an alternating sequence of distinct vertices of G, beginning and ending with v_1 and v_{k+1}, in which there is an edge (v_i, v_{i+1}) for $1 \leq i \leq k$. We denote by $< v_1, v_2, \ldots, v_{k+1} >$ the maximal segment which corresponds to the drawing of the path $v_1, v_2, \ldots, v_{k+1}$.

We now explore a relation between the number of maximal segments and the number of straight angles in a convex drawing of a 3-connected cubic plane graph.

Lemma 2. *Let G be a 3-connected cubic plane graph with n vertices and Γ be a convex drawing of G with x maximal segments. Then the number of straight angles in Γ is $\frac{3n}{2} - x$.*

Proof. Let $S =< v_1, v_2, ..., v_{k+1} >$ be a maximal segment in Γ and $|S|$ be the number of edges in S. Since S is maximal, v_1 and v_{k+1} cannot have any other edge incident to them that extends S. Moreover, each v_i, $2 \leq i \leq k$ must have exactly one straight angle since G is cubic. Hence, there must be $k-1 = |S|-1$ straight angles in S as illustrated in Figure 3. Let l and x be the number of

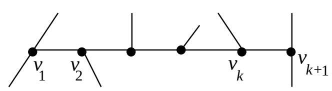

Figure 3. A maximal segment S in Γ.

straight angles and the number of maximal segments in Γ, respectively. We denote those maximal segments by S_1,S_2,...,S_x. Thus the number of straight angles for all of the x maximal segments is $l = |S_1| + |S_2| + ... + |S_x| - x$. Since no two maximal segments share edges, the number of edges in G is $|S_1|+|S_2|+...+|S_x|$. By degree sum formula, $|S_1| + |S_2| + ... + |S_x| = \frac{3n}{2}$. Therefore, $l = \frac{3n}{2} - x$. □

We now compute the lower bound on the number of maximal segments in a convex drawing of a 3-connected cubic plane graph.

Lemma 3. *Let G be a 3-connected cubic plane graph with n vertices and Γ be a convex drawing of G. Then the number of maximal segments in Γ is at least $\frac{n}{2} + 3$.*

Proof. Let x be the number of maximal segments in Γ. Then, by Lemma 2, the number of straight angles in Γ is $\frac{3n}{2} - x$. On the other hand, the number of straight angles in a convex drawing of G is at most $n-3$ by Lemma 1. Therefore, $\frac{3n}{2} - x \le n - 3$ and hence $x \ge \frac{n}{2} + 3$. □

Lemma 3 implies that $\frac{n}{2} + 3$ is the lower bound on the number of maximal segments in a convex drawing of G. We denote the lower bound by S_n.

3 Convex Grid Drawings

In this section we give an algorithm to obtain a convex grid drawing of a 3-connected cubic plane graph with $S_n + 1$ maximal segments. To present the algorithm we need some preparation.

Let G be a 3-connected plane graph. An *internally convex* drawing of G is a straight-line drawing of G where all the inner faces are drawn as convex polygons. Let (v_1, v_2) be an edge on the outer face of G and let $\pi = (V_1, V_2, ..., V_m)$ be an ordered partition of V, that is $V_1 \cup V_2 \cup ... \cup V_m = V$ and $(V_i \cap V_j) = \phi$ for $i \neq j$. We denote by G_k, $1 \le k \le m$, the subgraph of G induced by $V_1 \cup V_2 \cup ... \cup V_k$ and by C_k the outer cycle of G_k. Let $\{v_1, v_2, \ldots, v_p\}$, $p \ge 3$, be a set of outer vertices consecutive on C_k such that degree$(v_1) \ge 3$, degree(v_2) =degree$(v_3) = \ldots =$ degree$(v_{p-1}) = 2$, and degree$(v_p) \ge 3$ in G_k. Then we call the set $\{v_2, \ldots, v_{p-1}\}$ an *outer chain* of G. We now describe the properties of a canonical decomposition π of G with the outer edge (v_1, v_2) in the following (a)-(b) [11].

(a) V_1 is the set of all vertices on the inner face containing the edge (v_1, v_2). V_m is a singleton set containing an outer vertex v such that v is a neighbor of v_1 and $v \notin (v_1, v_2)$.

(b) For each index k, $2 \le k \le m-1$, all vertices in V_k are outer vertices of G_k and the following conditions hold:
  (1) if $|V_k| = 1$, then the vertex in V_k has two or more neighbors in G_{k-1} and has at least one neighbor in $G - G_k$; and
  (2) If $|V_k| > 1$, then V_k is an outer chain $\{z_1, z_2, ..., z_l\}$ of G_k.

Throughout the paper we assume that the vertices of an outer chain $\{z_1, z_2, ..., z_l\}$ are ordered clockwise. The vertex v_1 has two neighbors in G_1, one neighbor is v_2 and we denote the other neighbor by v_4. The vertex v_2 has two neighbors in G_1, one neighbor is v_1 and we denote the other neighbor by v_3. The following result on canonical decomposition is known [9].

Lemma 4. *Every 3-connected plane graph G has a canonical decomposition π, and π can be found in linear time.*

Let z be a vertex of G and $P(z)$, $x(z)$, $y(z)$ be the (x, y)-coordinates of z, the x-coordinate of z and the y-coordinate of z, respectively. We will later associate a set with each vertex z of G, which we denote by $L(z)$. By $\Gamma(G)$ we denote a drawing of G. We are now ready to describe Algorithm **Cubic Drawing** for finding a convex grid drawing of G.

Let $V_1 = (z_1 = v_1, z_2 = v_4, ..., z_{l-1} = v_3, z_l = v_2)$. We draw G_1 by a triangle as follows. Set $P(z_i) = (i-1, 0)$ for $1 \le i < l$, $P(z_l) = (l-1, -1)$ and $L(z_i) = \{z_i\}$

for $1 \leq i \leq l$. Figure 4 illustrates the drawing of G_1. Let $\Gamma(G)$ be a straight-line drawing of G. We now install $V_2, V_3, ..., V_m$ one after another to construct $\Gamma(G_2), \Gamma(G_3), \ldots, \Gamma(G_m) = \Gamma(G)$, respectively. V_k is either a singleton set or an outer chain of G_k, but in the algorithm we will treat both cases uniformly. We now explain how to install V_k to $\Gamma(G_{k-1})$. We denote by $C_{k-1} = (w_1 = v_1, w_2, ..., w_t = v_2)$ the outer cycle of G_{k-1}. Let w_p and w_q be the leftmost and rightmost neighbors of V_k on C_{k-1} where $1 < k \leq m$. For each $V_k = \{z_1, z_2, \ldots, z_l\}$, $1 < k < m$, we set $L(z_1) = \{z_1\} \cup (\bigcup_{x=p}^{t} L(w_x))$ and $L(z_i) = (\bigcup_{x=i}^{l} \{z_x\}) \cup (\bigcup_{x=q}^{t} L(w_x))$, where $2 \leq i \leq l$.

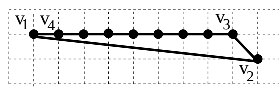

Figure 4. Drawing of G_1.

At each step we prove that the resulting drawing $\Gamma(G_k)$, $1 < k < m$, is internally convex. We also prove that the slopes of the maximal segments in $\Gamma(G_k)$ are in $\{0, 1, \infty, \lambda_1, \lambda_2\}$ where λ_1 and λ_2 are the slopes of (v_1, v_2) and (v_2, v_3), respectively. Moreover, each vertex $z \notin \{v_1, v_2, v_3\}$ of degree three in $\Gamma(G_k)$ has exactly one straight angle and no two vertices of degree two have the same x-coordinate in $\Gamma(G_k)$. For each V_k, $2 \leq k \leq m$, let $D_x = |x(w_q) - x(w_p)|$ and $D_y = |y(w_q) - y(w_p)|$.

We now have the following lemma.

Lemma 5. *Let $G_k = V_1 \cup V_2 \cup ... \cup V_k$, $1 \leq k \leq m-1$. Then G_k admits a straight-line drawing $\Gamma(G_k)$ which is internally convex and the slopes of the maximal segments in $\Gamma(G_k)$ are in $\{0, 1, \infty, \lambda_1, \lambda_2\}$ where λ_1 and λ_2 are the slopes of (v_1, v_2) and (v_2, v_3), respectively. Moreover, each vertex $v \notin \{v_1, v_2, v_3\}$ of degree three of G_k has a straight angle in $\Gamma(G_k)$ and no two vertices of degree two have the same x-coordinate in $\Gamma(G_k)$.*

Proof. We will prove the claim by induction on k.

The case for $k = 1$ is trivial since G_1 is drawn as a triangle, where v_1, v_2 and v_3 are the corner vertices of the triangle. The slopes of the maximal segments in $\Gamma(G_1)$ are in $\{0, \lambda_1, \lambda_2\}$. We may thus assume that k is greater than one and the claim holds for all $\Gamma(G_{k-i})$, $0 < i < k$. Let $C_{k-1} = (w_1 = v_1, w_2, ..., w_t = v_2)$ be the outer cycle of G_{k-1} and we denote by w_p and w_q the leftmost and rightmost neighbors of V_k on C_{k-1}, $1 < k < m$. Let f_k be the inner face of G_k containing v_k. We are now going to add $V_k = (z_1, z_2, ..., z_l)$ to $\Gamma(G_{k-1})$ to obtain $\Gamma(G_k)$.

We first assume that $w_p, w_q \notin \{v_1, v_2, v_3\}$. (We will consider the case where w_p or w_q is in $\{v_1, v_2, v_3\}$ at the end of this proof.) We now have the following four cases to consider.

Case 1: Both of w_p and w_q have straight angles in $\Gamma(G_{k-1})$.

Since each of the vertices $w_{p+1}, \ldots, w_{q-1}$ must be of degree three, each of those vertices must have a straight angle by induction hypothesis. Therefore,

$y(w_p) = y(w_q)$. Hence if $|V_k| = 1$, then we set $P(z_1)$=$(x(w_q), y(w_p) + D_x)$, as illustrated in Figure 5(a). Otherwise we shift $\bigcup_{i=q}^{t} L(w_i)$ by $|V_k| - D_x$ unit to the right when $|V_k| - D_x > 0$. Then we set $P(z_i) = (x(w_p) + i, y(w_p) + 1)$, $1 \leq i < l$, and $P(z_l) = (x(w_q), y(w_p) + 1)$ as illustrated in Figure 5(b). We thus obtain a drawing of $\Gamma(G_k)$.

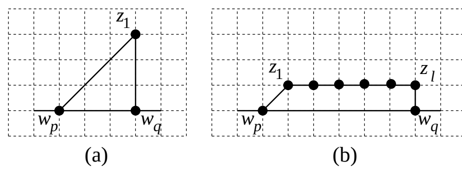

Figure 5. Both of w_p and w_q have straight angles. (a) $|V_k| = 1$ and (b) $|V_k| > 1$.

We now show that $\Gamma(G_k)$ satisfies the claim in Lemma 5.

We first prove that $\Gamma(G_k)$ is internally convex. By induction hypothesis, $\Gamma(G_{k-1})$ is internally convex. By Lemma 6, shifting of $\bigcup_{i=q}^{t} L(w_i)$ to the right keeps $\Gamma(G_{k-1})$ internally convex. Moreover, the vertices of f_k which are in $G_{k-1} - \{w_p, w_q\}$ are the vertices of degree three in $\Gamma(G_{k-1})$ and each of these vertices has a straight angle by the induction hypothesis. These straight angles remain the same after the shift by Lemma 6. According to the installation of V_k, each of w_p and w_q obtains a straight angle in $\Gamma(G_k)$ and the vertices of V_k do not obtain any concave angle inside f_k. Therefore, f_k is a convex polygon and $\Gamma(G_k)$ is internally convex.

We next prove that the slopes of the maximal segments in $\Gamma(G_k)$ are in $\{0, 1, \infty, \lambda_1, \lambda_2\}$. By induction hypothesis, the slopes of the maximal segments in $\Gamma(G_{k-1})$ are in $\{0, 1, \infty, \lambda_1, \lambda_2\}$. By Lemma 6, shifting of $\bigcup_{i=q}^{t} L(w_i)$ to the right keeps the slopes of the maximal segments in $\Gamma(G_{k-1})$ in $\{0, 1, \infty, \lambda_1, \lambda_2\}$. In this case, the slopes of (w_p, z_1) and (w_q, z_l) are in $\{0, 1, \infty, \lambda_1, \lambda_2\}$ and (z_1, z_2), (z_2, z_3),..., (z_{l-1}, z_l) form one maximal segment of slope 0 when $|V_k| > 1$. Therefore, the slopes of the maximal segments in G_k are in $\{0, 1, \infty, \lambda_1, \lambda_2\}$.

We then prove that each vertex $v \notin \{v_1, v_2, v_3\}$ of degree three of G_k has a straight angle in $\Gamma(G_k)$. By induction hypothesis, each vertex $v \notin \{v_1, v_2, v_3\}$ of degree three in G_{k-1} has a straight angle in $\Gamma(G_{k-1})$. Moreover, w_p and w_q are the two new vertices which become vertices of degree three in G_k. Each of w_p and w_q obtains a straight angle according to our drawing method. Therefore, each vertex $v \notin \{v_1, v_2, v_3\}$ of degree three of G_k has a straight angle in $\Gamma(G_k)$.

We finally prove that no two vertices of degree two have the same x-coordinate in $\Gamma(G_k)$. According to the installation of V_k, the x-coordinates of the vertices $z_1, \ldots, z_l$ are different. One can observe that any vertex $v \notin \{z_1, \ldots, z_l\}$ of degree two in G_k is contained in either on the path $w_1, w_2, \ldots, w_p$ or on the path $w_{q+1}, w_q, \ldots, w_t$ where $x(v) < x(z_1)$ or $x(v) > x(z_l)$, respectively. These vertices of degree two are also in $\Gamma(G_{k-1})$ and have different x-coordinates by induction hypothesis. Therefore, no two vertices of degree two in G_k have the same x-coordinates in $\Gamma(G_k)$.

Case 2: Only w_q has a straight angle in $\Gamma(G_{k-1})$.

In this case, the slope of (w_{p-1}, w_p) is either 1 or 0.

Consider first the case where the slope of (w_{p-1}, w_p) is +1. Then $y(w_p) \geq y(w_q)$ since each vertex $v \in \{w_{p+1}, \ldots, w_{q-1}\}$ has degree three in G_{k-1} and has a straight corner in $\Gamma(G_{k-1})$ when $v \neq v_3$ by induction hypothesis. Therefore, if $|V_k| = 1$, we set $P(z_1) = (x(w_q), y(w_p) + D_x)$ as illustrated in Figure 6(a). Otherwise we shift $\bigcup_{i=q}^{t} L(w_i)$ to the right by $|V_k| - D_x$ units when $|V_k| - D_x > 0$. Then we set $P(z_i)$=$(x(w_p) + i, y(w_p) + 1)$ for $1 \leq i < l$ and $P(z_l) = (x(w_q), y(w_p) + 1)$ as illustrated in Figure 6(b).

The slopes of (w_p, z_1) and (w_q, z_l) are 1 and ∞, respectively and (z_1, z_2), (z_2, z_3),..., (z_{l-1}, z_l) form one maximal segment of slope 0 when $|V_k| > 1$. Therefore, the slopes of the maximal segments in $\Gamma(G_k)$ are in $\{0, 1, \infty, \lambda_1, \lambda_2\}$. In a way similar to the proof of Case 1 we can prove that $\Gamma(G_k)$ is internally convex, each vertex $v \notin \{v_1, v_2, v_3\}$ of degree three in G_k has a straight angle and no two vertices of degree two have the same x-coordinate in $\Gamma(G_k)$.

We now Consider the case where the slope of (w_{p-1}, w_p) is 0. By a similar way as shown above it can be shown that $y(w_p) > y(w_q)$. We shift $\bigcup_{i=q}^{t} L(w_i)$ to the right by $|V_k| - D_x$ units when $|V_k| - D_x > 0$. For $|V_k| = 1$, we set $P(z_1) = (x(w_q), y(w_p))$. Otherwise we set $P(z_i) = (x(w_p) + i, y(w_p))$, $1 \leq i < l$, and $P(z_l) = (x(w_q), y(w_p))$ as illustrated in Figure 6(c).

The slopes of (w_p, z_1) and (w_q, z_l) are 0 and ∞, respectively and (z_1, z_2), (z_2, z_3),..., (z_{l-1}, z_l) form one maximal segment of slope 0 when $|V_k| > 1$. Therefore, the slopes of the maximal segments in $\Gamma(G_k)$ are in $\{0, 1, \infty, \lambda_1, \lambda_2\}$. In a way similar to the proof of Case 1 we can prove that $\Gamma(G_k)$ is internally convex, each vertex $v \notin \{v_1, v_2, v_3\}$ of degree three in G_k has a straight angle and no two vertices of degree two have the same x-coordinate in $\Gamma(G_k)$.

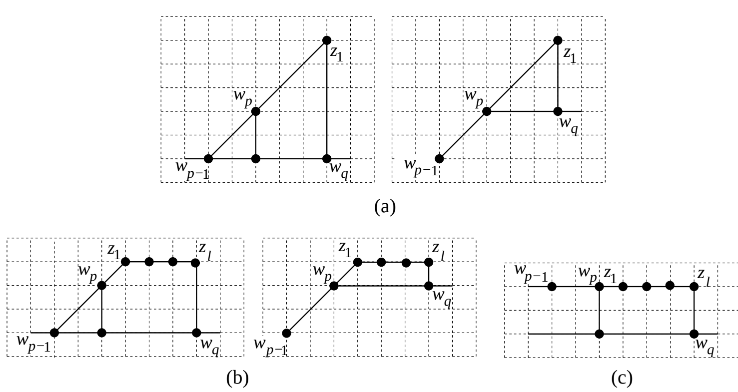

Figure 6. Only w_q has a straight angle. (a) $|V_k| = 1$, (b) $|V_k| > 1$ and the slope of $(w_{p-1}, w_p) = 1$, (c) $|V_k| > 1$ and the slope of $(w_{p-1}, w_p) = 0$.

***Case* 3:** Only w_p has a straight angle in $\Gamma(G_{k-1})$.

In this case, the slope of (w_{q+1}, w_q) is either 0 or ∞.

We first consider the case where the slope of (w_{q+1}, w_q) is 0. Then the slope of (w_{q-1}, w_q) is $+1$ and by a similar way as in Case 2 it can be shown that $y(w_p) < y(w_q)$. If $|V_k| = 1$, we set $P(z_1) = (x(w_p) + D_y, y(w_q))$. Otherwise we shift $\bigcup_{i=q}^{t} L(w_i)$ to the right by $|V_k| + D_y - D_x$ units when $|V_k| + D_y - D_x > 0$. Then we set $P(z_i)$=$(x(w_p) + D_y + i - 1, y(w_q))$ for $1 \le i \le l$ as illustrated in Figure 7(a).

The slopes of (w_p, z_1) and (w_q, z_l) are 1 and 0, respectively and (z_1, z_2), (z_2, z_3),..., (z_{l-1}, z_l) form one maximal segment of slope 0 when $|V_k| > 1$. Therefore, the slopes of the maximal segments in $\Gamma(G_k)$ are in $\{0, 1, \infty, \lambda_1, \lambda_2\}$. In a way similar to the proof of Case 1 we can prove that $\Gamma(G_k)$ is internally convex, each vertex $v \notin \{v_1, v_2, v_3\}$ of degree three in G_k has a straight angle and no two vertices of degree two have the same x-coordinate in $\Gamma(G_k)$.

We now consider the case where the slope of (w_{q+1}, w_q) is ∞. If the slope of (w_{q-1}, w_q) is 0 then in a similar way as in Case 2 one can observe that $y(w_p) = y(w_q)$. Otherwise (w_{q-1}, w_q) belongs to a maximal segment with the slope $+1$ which implies that $y(w_p) < y(w_q)$ and $D_x > D_y$. Therefore, if $|V_k| = 1$, we set $P(z_1) = (x(w_q), y(w_p) + D_x)$ as illustrated in Figure 7(b). Otherwise we shift $\bigcup_{i=q}^{t} L(w_i)$ to the right by $|V_k|+D_y-D_x$ units when $|V_k|+D_y-D_x > 0$ and set $P(z_i)$=$(x(w_p) + D_y + i, y(w_q) + 1)$, $1 \le i < l$, and $P(z_l)$=$(x(w_q), y(w_q) + 1)$, as illustrated in Figure 7(c).

The slopes of (w_p, z_1) and (w_q, z_l) are 1 and ∞, respectively and (z_1, z_2), (z_2, z_3),..., (z_{l-1}, z_l) form one maximal segment of slope 0 when $|V_k| > 1$. Therefore, the slopes of the maximal segments in $\Gamma(G_k)$ are in $\{0, 1, \infty, \lambda_1, \lambda_2\}$. In a way similar to the proof of Case 1 we can prove that $\Gamma(G_k)$ is internally convex, each vertex $v \notin \{v_1, v_2, v_3\}$ of degree three in G_k has a straight angle and no two vertices of degree two have the same x-coordinate in $\Gamma(G_k)$.

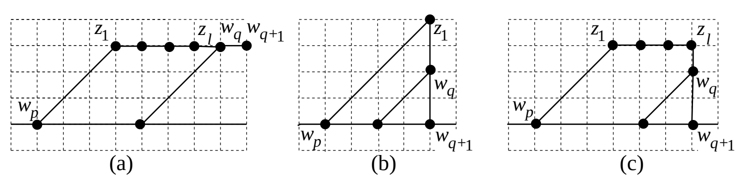

Figure 7. Only w_p has a straight angle. (a) The slope of $(w_{q+1}, w_q) = 0$, (b) $|V_k| = 1$ and the slope of $(w_{q+1}, w_q) = \infty$, (c) $|V_k| > 1$ and the slope of $(w_{q+1}, w_q) = \infty$.

***Case* 4:** None of w_p and w_q has a straight angle in $\Gamma(G_{k-1})$.

In this case one can observe that, the slope of (w_{p-1}, w_p) is either 1 or 0 and the slope of (w_{q+1}, w_q) is either ∞ or 0. Therefore, we have the following four subcases to consider.

Subcase 4a: The slope of (w_{p-1}, w_p) is 1 and the slope of (w_{q+1}, w_q) is ∞ in $\Gamma(G_{k-1})$.

Consider first the case where $|V_k| = 1$. If, $y(w_p) < y(w_q)$ then (w_{q-1}, w_q) belongs to a maximal segment with the slope $+1$ and $D_x > D_y$. Otherwise $y(w_p) \geq y(w_q)$. Therefore we set $P(z_1) = (x(w_q), y(w_p) + D_x)$ as illustrated in Figure 8(a).

One can easily observe that, the slopes of (w_p, z_1) and (w_q, z_l) are $+1$ and ∞, respectively. Therefore, the slopes of the maximal segments in $\Gamma(G_k)$ are in $\{0, 1, \infty, \lambda_1, \lambda_2\}$. In a way similar to the proof of Case 1 we can prove that $\Gamma(G_k)$ is internally convex, each vertex $v \notin \{v_1, v_2, v_3\}$ of degree three in G_k has a straight angle and no two vertices of degree two has the same x-coordinate in $\Gamma(G_k)$.

We next consider the case where $|V_k| > 1$ and $y(w_p) \geq y(w_q)$. We shift $\bigcup_{i=q}^{t} L(w_i)$ to the right by $|V_k| - D_x$ units when $|V_k| - D_x > 0$ and set $P(z_i) = (x(w_p) + i, y(w_p) + 1)$, $1 \leq i < l$ and $P(z_l) = (x(w_q), y(w_p) + 1)$ as illustrated in Figure 8(b).

Here, (z_1, z_2), (z_2, z_3),..., (z_{l-1}, z_l) form one maximal segment of slope 0 and the slopes of (w_p, z_1) and (w_q, z_l) are $+1$ and ∞, respectively. Therefore, the slopes of the maximal segments in $\Gamma(G_k)$ are in $\{0, 1, \infty, \lambda_1, \lambda_2\}$. In a way similar to the proof of Case 1 we can prove that $\Gamma(G_k)$ is internally convex, each vertex $v \notin \{v_1, v_2, v_3\}$ of degree three in G_k has a straight angle and no two vertices of degree two has the same x-coordinate in $\Gamma(G_k)$.

Otherwise $|V_k| > 1$ and $y(w_p) < y(w_q)$ and we shift $\bigcup_{i=q}^{t} L(w_i)$ to the right by $|V_k| + D_y - D_x$ units when $|V_k| + D_y - D_x > 0$. Then we set $P(z_i) = (x(w_p) + D_y + i, y(w_q) + 1)$, $1 \leq i < l$, and $P(z_l) = (x(w_q), y(w_q) + 1)$ as illustrated in Figure 8(c).

The slopes of (w_p, z_1) and (w_q, z_l) are $+1$ and ∞, respectively and (z_1, z_2), (z_2, z_3),..., (z_{l-1}, z_l) form one maximal segment of slope 0. Therefore, the slopes of the maximal segments in $\Gamma(G_k)$ are in $\{0, 1, \infty, \lambda_1, \lambda_2\}$. In a way similar to the proof of Case 1 we can prove that $\Gamma(G_k)$ is internally convex, each vertex $v \notin \{v_1, v_2, v_3\}$ of degree three in G_k has a straight angle and no two vertices of degree two has the same x-coordinate in $\Gamma(G_k)$.

Subcase 4b: The slope of (w_{p-1}, w_p) is 1 and the slope of (w_{q+1}, w_q) is 0 in $\Gamma(G_{k-1})$.

We first consider the case where $y(w_p) \geq y(w_q)$. Then we choose a vertex w_{q-i} for the smallest i, $p < q - i < q$, where w_{q-i} has one edge with slope ∞ or two edges with slope 0. Clearly, there exists such a w_{q-i}. We set $P(w_j) = (x(w_{q-i}), y(w_j))$ where $q-i < j \leq q$. For $|V_k| = 1$, we set $P(z_1) = (x(w_q), y(w_p) + D_x)$ as illustrated in Figure 9(a). After this modification, every w_j, $q-i \leq j < q$, which had a straight angle in $\Gamma(G_{k-1})$ still has a straight angle. Otherwise we shift $\bigcup_{i=q}^{t} L(w_i)$ to the right by $|V_k| - D_x$ units when $|V_k| - D_x > 0$. Then we set $P(z_i) = (x(w_p) + i, y(w_p) + 1)$, for $1 \leq i < l$ and $P(z_l) = (x(w_q), y(w_p) + 1)$ as illustrated in Figure 9(b).

The slopes of (w_p, z_1) and (w_q, z_l) are $+1$ and ∞, respectively and (z_1, z_2), (z_2, z_3),..., (z_{l-1}, z_l) form one maximal segment of slope 0 when $|V_k| > 1$. There-

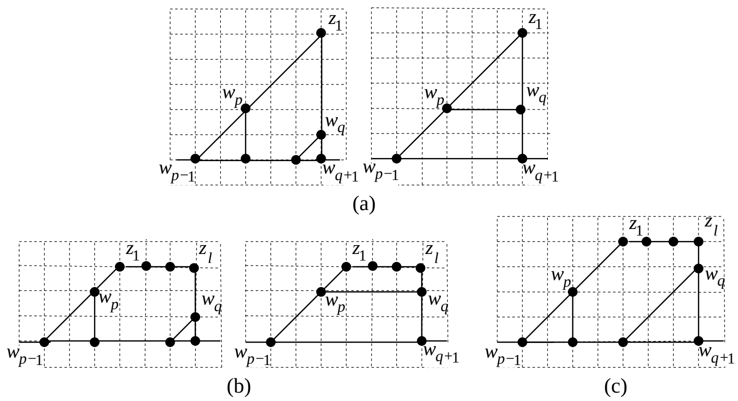

Figure 8. None of w_p, w_q has straight angle, the slope of (w_{p-1}, w_p) is 1, the slope of (w_{q+1}, w_q) is ∞. (a) $|V_k| = 1$, (b) $|V_k| > 1$ and $y(w_p) \geq y(w_q)$, (c) $|V_k| > 1$ and $y(w_p) < y(w_q)$.

fore, the slopes of the maximal segments in $\Gamma(G_k)$ are in $\{0, 1, \infty, \lambda_1, \lambda_2\}$. In a way similar to the proof of Case 1 we can prove that $\Gamma(G_k)$ is internally convex, each vertex $v \notin \{v_1, v_2, v_3\}$ of degree three in G_k has a straight angle and no two vertices of degree two has the same x-coordinate in $\Gamma(G_k)$.

We next consider the case where $y(w_p) < y(w_q)$. Then we shift $\bigcup_{i=q}^{t} L(w_i)$ to the right by $|V_k| + D_y - D_x$ units when $|V_k| + D_y - D_x > 0$ and set $P(z_i) = (x(w_p) + D_y + i - 1, y(w_q))$, $1 \leq i \leq l$, as illustrated in Figure 9(c).

The slopes of (w_p, z_1) and (w_q, z_l) are $+1$ and 0, respectively and (z_1, z_2), (z_2, z_3),..., (z_{l-1}, z_l) form one maximal segment of slope 0 when $|V_k| > 1$. Therefore, the slopes of the maximal segments in $\Gamma(G_k)$ are in $\{0, 1, \infty, \lambda_1, \lambda_2\}$. In a way similar to the proof of Case 1 we can prove that $\Gamma(G_k)$ is internally convex, each vertex $v \notin \{v_1, v_2, v_3\}$ of degree three in G_k has a straight angle and no two vertices of degree two has the same x-coordinate in $\Gamma(G_k)$.

Subcase 4c: The slope of (w_{p-1}, w_p) is 0 and the slope of (w_{q+1}, w_q) is ∞ in $\Gamma(G_{k-1})$.

We consider first the case where $y(w_p) > y(w_q)$. Let $|V_k| = 1$, then we set $P(z_1) = (x(w_q), y(w_p))$. Otherwise $|V_k| > 1$ and we shift $\bigcup_{i=q}^{t} L(w_i)$ to the right by $|V_k| - D_x$ units when $|V_k| - D_x > 0$ and set $P(z_i) = (x(w_p) + i, y(w_p))$, $1 \leq i < l$, and $P(z_l) = (x(w_q), y(w_p))$ as illustrated in Figure 10(a).

The slopes of (w_p, z_1) and (w_q, z_l) are 0 and ∞, respectively and (z_1, z_2), (z_2, z_3),..., (z_{l-1}, z_l) form one maximal segment of slope 0 when $|V_k| > 1$. Therefore, the slopes of the maximal segments in $\Gamma(G_k)$ are in $\{0, 1, \infty, \lambda_1, \lambda_2\}$. In a way similar to the proof of Case 1 we can prove that $\Gamma(G_k)$ is internally convex, each vertex $v \notin \{v_1, v_2, v_3\}$ of degree three in G_k has a straight angle and no two vertices of degree two has the same x-coordinate in $\Gamma(G_k)$.

We next consider the case where $y(w_p) \leq y(w_q)$. Then we choose a vertex w_{p+i} for the smallest i, $p < p + i < q$, where w_{p+i} has one edge with slope 1

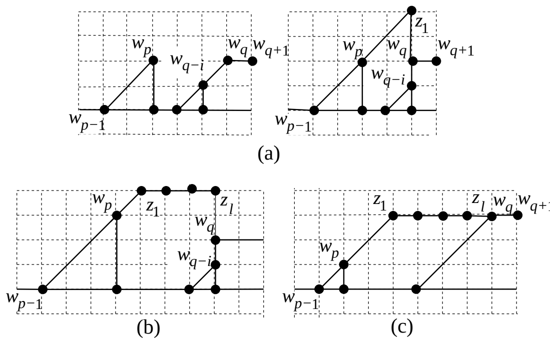

Figure 9. None of w_p, w_q has straight angle, the slope of (w_{p-1}, w_p) is 1, the slope of (w_{q+1}, w_q) is 0. (a) $|V_k| = 1$ and $y(w_p) \geq y(w_q)$, (b) $|V_k| > 1$ and $y(w_p) \geq y(w_q)$, (c) $y(w_p) < y(w_q)$.

or two edges with slope 0. Clearly, there exists such a w_{p+i}. We set $P(w_j) = (x(w_j) + y(w_j) - y(w_{p+i}), y(w_j))$ where $p \leq j < p + i$. After this modification, every w_j, $p < j \leq p + i$, which had a straight angle in $\Gamma(G_{k-1})$ still has a straight angle. We now shift $\bigcup_{i=q}^{t} L(w_i)$ to the right by $|V_k| + D_y - D_x$ units when $|V_k| + D_y - D_x > 0$. Let $|V_k| = 1$, then we set $P(z_1) = (x(w_q), y(w_p) + D_x)$. Otherwise $|V_k| > 1$ and we set $P(z_i) = (x(w_p) + D_y + i, y(w_q) + 1)$, $1 \leq i < l$, and $P(z_l) = (x(w_q), y(w_q) + 1)$ as illustrated in Figure 10(b).

The slopes of (w_p, z_1) and (w_q, z_l) are $+1$ and ∞, respectively and (z_1, z_2), (z_2, z_3),..., (z_{l-1}, z_l) form one maximal segment of slope 0 when $|V_k| > 1$. Therefore, the slopes of the maximal segments in $\Gamma(G_k)$ are in $\{0, 1, \infty, \lambda_1, \lambda_2\}$. In a way similar to the proof of Case 1 we can prove that $\Gamma(G_k)$ is internally convex, each vertex $v \notin \{v_1, v_2, v_3\}$ of degree three in G_k has a straight angle and no two vertices of degree two has the same x-coordinate in $\Gamma(G_k)$.

Subcase 4d: The slope of (w_{p-1}, w_p) is 0 and the slope of (w_{q+1}, w_q) is 0 in $\Gamma(G_{k-1})$.

Consider the case where $y(w_p) > y(w_q)$. Then we choose a vertex w_{q-i} for the smallest i, $p < q - i < q$, where w_{q-i} has one edge with slope ∞ or two edges with slope 0. Clearly, there exists such a w_{q-i}. We set $P(w_j) = (x(w_{q-i}), y(w_j))$ for $q - i < j \leq q$. After this modification, every w_j, $q - i \leq j < q$, which had a straight angle in $\Gamma(G_{k-1})$ still has a straight angle. Let $|V_k| = 1$, then we set $P(z_1) = (x(w_q), y(w_p))$. Otherwise $|V_k| > 1$ and we shift $\bigcup_{i=q}^{t} L(w_i)$ to the right by $|V_k| - D_x$ units when $|V_k| - D_x > 0$ and set $P(z_i) = (x(w_p) + i, y(w_p))$, $1 \leq i < l$ and $P(z_l) = (x(w_q), y(w_p))$ as illustrated in Figure 10(c).

The slopes of (w_p, z_1) and (w_q, z_l) are 0 and ∞, respectively and (z_1, z_2), (z_2, z_3),..., (z_{l-1}, z_l) form one maximal segment of slope 0 when $|V_k| > 1$. Therefore, the slopes of the maximal segments in $\Gamma(G_k)$ are in $\{0, 1, \infty, \lambda_1, \lambda_2\}$. In a way similar to the proof of Case 1 we can prove that $\Gamma(G_k)$ is internally convex, each vertex $v \notin \{v_1, v_2, v_3\}$ of degree three in G_k has a straight angle and no two vertices of degree two has the same x-coordinate in $\Gamma(G_k)$.

Consider next the case where $y(w_p) < y(w_q)$. Then we choose a vertex w_{p+i} for the smallest i, $p < p+i < q$, where w_{p+i} has one edge with slope 1 or two edges with slope 0. Clearly, there exists such a w_{p+i}. We set $P(w_j) = (x(w_j) + y(w_j) - y(w_{p+i}), y(w_j))$ where $p \leq j < p+i$. After this modification, every w_j, $p < j \leq p+i$, which had a straight angle in $\Gamma(G_{k-1})$ still has a straight angle. We now shift $\bigcup_{i=q}^{t} L(w_i)$ to the right by $|V_k| + D_y - D_x$ units when $|V_k| + D_y - D_x > 0$. We set $P(z_i) = (x(w_p) + D_y + i - 1, y(w_q))$, $1 \leq i \leq l$, as illustrated in Figure 10(d).

The slopes of (w_p, z_1) and (w_q, z_l) are $+1$ and 0, respectively and (z_1, z_2), (z_2, z_3),..., (z_{l-1}, z_l) form one maximal segment of slope 0 when $|V_k| > 1$. Therefore, the slopes of the maximal segments in $\Gamma(G_k)$ are in $\{0, 1, \infty, \lambda_1, \lambda_2\}$. In a way similar to the proof of Case 1 we can prove that $\Gamma(G_k)$ is internally convex, each vertex $v \notin \{v_1, v_2, v_3\}$ of degree three in G_k has a straight angle and no two vertices of degree two has the same x-coordinate in $\Gamma(G_k)$.

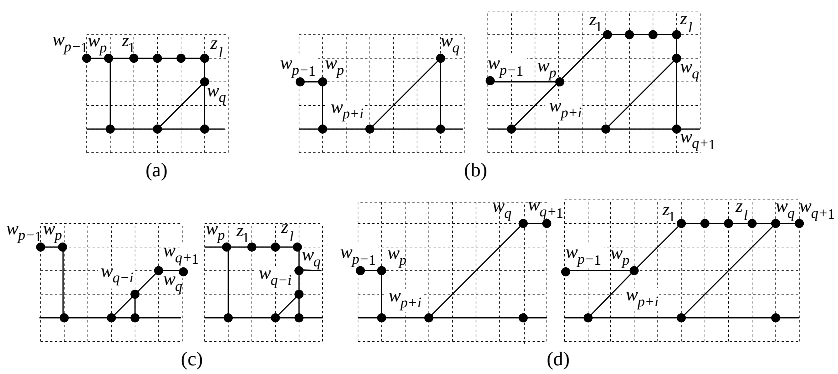

Figure 10. None of w_p, w_q has straight angle, the slope of (w_{p-1}, w_p) is 0. (a)$y(w_p) > y(w_q)$ and the slope of $(w_{q+1}, w_q) = \infty$, (b) $y(w_p) \leq y(w_q)$ and the slope of $(w_{q+1}, w_q) = \infty$, (c) $y(w_p) > y(w_q)$ and the slope of $(w_{q+1}, w_q) = 0$, (d) $y(w_p) < y(w_q)$ and the slope of $(w_{q+1}, w_q) = 0$.

Finally, if $y(w_p) = y(w_q)$ we shift $\bigcup_{i=q}^{t} L(w_i)$ to the right by $|V_k| - D_x + 1$ units when $|V_k| - D_x + 1 > 0$. We set $P(z_i) = (x(w_p) + i, y(w_p))$, $1 \leq i \leq l$.

The slopes of (w_p, z_1) and (w_q, z_l) are both 0 and (z_1, z_2), (z_2, z_3),..., (z_{l-1}, z_l) form one maximal segment of slope 0 when $|V_k| > 1$. Therefore, the slopes of the maximal segments in $\Gamma(G_k)$ are in $\{0, 1, \infty, \lambda_1, \lambda_2\}$. In a way similar to the proof of Case 1 we can prove that $\Gamma(G_k)$ is internally convex, each vertex $v \notin \{v_1, v_2, v_3\}$ of degree three in G_k has a straight angle and no two vertices of degree two has the same x-coordinate in $\Gamma(G_k)$.

It is now remained to show the case where w_p or w_q is in $\{v_1, v_2, v_3\}$. If $w_p, w_q \in \{v_1, v_2, v_3\}$, the proof is similar to the proof in Case 1. If $w_p \notin \{v_1, v_2, v_3\}$, $w_q \in \{v_1, v_2, v_3\}$ and w_p has straight angle, the proof is similar to the proof in Case 1. If $w_p \notin \{v_1, v_2, v_3\}$, $w_q \in \{v_1, v_2, v_3\}$ and w_p does not have straight angle, the proof is similar to the proof in Case 2. If $w_p \in \{v_1, v_2, v_3\}$, $w_q \notin \{v_1, v_2, v_3\}$ and w_q has straight angle, the proof is similar to the proof in Case 1. If $w_p \in \{v_1, v_2, v_3\}$, $w_q \notin \{v_1, v_2, v_3\}$ and w_q does not have straight angle, the proof is similar to the proof in Case 3. Note that, in the proof of Case 1, $y(w_p) = y(w_q)$. In the case when w_p or w_q is in $\{v_1, v_2, v_3\}$, $y(w_p) \geq y(w_q)$ and the reasoning used in the proof of Case 1 also holds for this case. □

The proof of Lemma 5 gives a method of obtaining $\Gamma(G_{m-1})$. Let $w_p(= v_1)$, v_4 and w_q be the three neighbors of V_m where $x(w_p) < x(v_4) < x(w_q)$. We now set $P(V_m) = (x(v_2), y(v_4) + x(v_2) - x(v_4))$ and add V_m to $\Gamma(G_{m-1})$ to complete the drawing $\Gamma(G_m) = \Gamma(G)$, as illustrated in Figure 11. It is obvious that the addition of V_m does not create any edge crossing. Let λ_3 be the slope of (v_1, V_m). Then clearly all the slopes of $\Gamma(G)$ is in $\{0, 1, \infty, \lambda_1, \lambda_2, \lambda_3\}$. Thus we have an algorithm for obtaining a convex grid drawing of a 3-connected cubic graph which we call Algorithm **Cubic Drawing**. We now have Lemma 6 which has been used in the proof of Lemma 5.

Lemma 6. *Let $G_k = V_1 \cup V_2 \cup ... \cup V_k$, $1 \leq k \leq m-1$, where $\Gamma(G_k)$ is a drawing of G_k obtained by Algorithm* ***Cubic Drawing****. Let $C_k = (w_1 = v_1, w_2, \ldots, w_t = v_2)$ be the outer cycle of $\Gamma(G_k)$ and let δ be any integer. Assume that the slope of (w_{i-1}, w_i), $2 \leq i \leq t$, is not ∞ and $\Gamma'(G_k)$ is the drawing obtained from $\Gamma(G_k)$ after shifting $L_k = \bigcup_i^t L(w_i)$ by δ units to the right. Then $\Gamma'(G_k)$ is internally convex, the number of slopes in $\Gamma'(G_k)$ is the same as the number of slopes in $\Gamma(G_k)$ and the slopes of all the maximal segments except (v_1, v_2) and (v_2, v_3) in $\Gamma'(G_k)$ remain the same as the slopes in $\Gamma(G_k)$. Moreover, no two vertices of degree two of G_k have the same x-coordinate in $\Gamma'(G_k)$.*

Proof. We prove the claim by induction on k. For the case when $k = 1$, $\Gamma(G_1)$ is drawn as a triangle and the claim trivially holds. We may thus assume that k is greater than one and the claim holds for all $\Gamma(G_x)$, $x < k$. Note that, we obtain G_k by adding V_k to G_{k-1}. Let $V_k = (z_1, z_2, \ldots, z_l)$ and $C_k = (w_1 = v_1, w_2, \ldots, w_p, w_{p+1} = z_1, w_{p+2} = z_2, \ldots, w_{q-1} = z_l, w_q, \ldots, w_t = v_2)$. Let w_i, $1 \leq i \leq t$, be a vertex on C_k. We now have the following four cases to consider.
Case 1: $w_i \in \{w_2, \ldots, w_p\}$.

In this case $\Gamma'(G_k)$, which is obtained from $\Gamma(G_k)$ by shifting the vertices of L_k by δ units to the right, can also be obtained as follows.

Let $C_{k-1} = (w_1 = v_1, \ldots, w_i, \ldots, w_t = v_2)$ be the outerface of G_{k-1}. One can observe that both C_k and C_{k-1} contain the vertex w_i. Since L_{k-1} includes all the vertices of L_k except $z_1, \ldots, z_l$, we first remove the drawing of V_k from $\Gamma(G_k)$ to obtain $\Gamma(G_{k-1})$. We then shift the vertices of L_{k-1} by δ units to the right. We finally add the drawing of V_k, as in $\Gamma'(G_k)$, to $\Gamma'(G_{k-1})$ to obtain $\Gamma'(G_k)$.

We first prove that $\Gamma'(G_k)$ is internally convex. Let f_k be the inner face in G_k that contains V_k. By induction hypothesis, $\Gamma'(G_{k-1})$ is internally convex. The drawings of the inner faces in $\Gamma'(G_k)$, other than f_k, are internally convex since they are also contained in $\Gamma'(G_{k-1})$. We are only left with the drawing of f_k in $\Gamma'(G_k)$. Since the vertices $w_i, \ldots, w_t$ as well as the vertices in V_k are shifted δ units to the right, the internal angles of f_k in $\Gamma'(G_k)$ remain the same as in $\Gamma(G_k)$. Since f_k is drawn as a convex polygon in $\Gamma(G_k)$, f_k is drawn as a convex polygon in $\Gamma'(G_k)$.

We next prove that the number of slopes in $\Gamma'(G_k)$ is the same as the number of slopes in $\Gamma(G_k)$. Moreover, the slopes of all the maximal segments except (v_1, v_2) and (v_2, v_3) remain the same as the slopes in $\Gamma(G_k)$. By induction hypothesis, the number of slopes in $\Gamma'(G_{k-1})$ is the same as the number of slopes in $\Gamma(G_{k-1})$. Moreover, the slopes of all the maximal segments except (v_1, v_2) and (v_2, v_3) remain the same as the slopes in $\Gamma(G_{k-1})$. Since all the edges of $\Gamma'(G_{k-1})$ are also in $\Gamma'(G_k)$, we are only left with the edges $(w_p, z_1), (z_1, z_2), \ldots, (z_l, w_q)$. Since the vertices $w_i, \ldots, w_t$ as well as the vertices in V_k are shifted δ units to the right, the maximal segments and the slopes of the edges $(w_p, z_1), (z_1, z_2), \ldots, (z_l, w_q)$ in $\Gamma'(G_k)$ remain the same as in $\Gamma(G_k)$.

We finally prove that no two vertices of degree two of G_k have the same x-coordinate in $\Gamma'(G_k)$. By the property of canonical ordering, all the vertices of degree two in G_k are on the outerface. By induction hypothesis no two vertices of degree two of G_{k-1} have the same x-coordinate in $\Gamma'(G_{k-1})$. Since $\Gamma'(G_{k-1})$ is contained in $\Gamma'(G_k)$, we are only left with vertices $z_1, \ldots, z_l$. Since the vertices $w_i, \ldots, w_t$ as well as the vertices in V_k are shifted δ units to the right, the relative distances of the vertices $w_i, \ldots, w_{p+1} = z_1, w_{p+2} = z_2, \ldots, w_t$ in $\Gamma'(G_k)$ remain the same as in $\Gamma(G_k)$. Since no two vertices of degree two of G_k have the same x-coordinate in $\Gamma(G_k)$, no two vertices of degree two of G_k have the same x-coordinate in $\Gamma'(G_k)$.

Case 2: $w_i = z_1$.

In this case, L_k includes w_p by definition of set L. Therefore $\Gamma'(G_k)$, which is obtained from $\Gamma(G_k)$ by shifting the vertices of L_k by δ units to the right, can also be obtained by shifting $\bigcup_p^t L(w_i)$. One can observe that, the shift of $\bigcup_p^t L(w_i)$ is actually the shift of L_k when $w_i = w_p$. Therefore, the proof for this case can be obtained in a similar technique as described in Case 1.

Case 3: $w_i \in \{z_2, \ldots, z_l\}$.

In this case $\Gamma'(G_k)$, which is obtained from $\Gamma(G_k)$ by shifting the vertices of L_k by δ units to the right, can also be obtained as follows.

Let $C_{k-1} = (w_1 = v_1, \ldots, w_q, \ldots, w_t = v_2)$ be the outerface of G_{k-1}. One can observe that both C_k and C_{k-1} contain the vertex w_q. We first remove the drawing of V_k from $\Gamma(G_k)$ to obtain $\Gamma(G_{k-1})$. We then shift the vertices of $\bigcup_q^t L(w_i)$ by δ units to the right. We finally add the drawing of V_k, as in $\Gamma'(G_k)$, to $\Gamma'(G_{k-1})$ to obtain $\Gamma'(G_k)$.

We first prove that $\Gamma'(G_k)$ is internally convex. Let f_k be the inner face in G_k that contains V_k. By induction hypothesis, $\Gamma'(G_{k-1})$ is internally convex. The drawings of the inner faces in $\Gamma'(G_k)$, other than f_k, are internally convex since they are also contained in $\Gamma'(G_{k-1})$. We are only left with the drawing of f_k in $\Gamma'(G_k)$. Since the vertices $w_q, \ldots, w_t$ as well as the vertices $w_i, \ldots, z_l$ are shifted δ units to the right, the horizontal distance between w_{i-1} and w_i increases. But this increase in distance between w_{i-1} and w_i does not change the slope of the segment $(z_1, z_2), \ldots, (z_{l-1}, z_l)$. Therefore, the internal angles of

f_k in $\Gamma'(G_k)$ remain the same as in $\Gamma(G_k)$. Since f_k is drawn as a convex polygon in $\Gamma(G_k)$, f_k is drawn as a convex polygon in $\Gamma'(G_k)$.

We next prove that the number of slopes in $\Gamma'(G_k)$ is the same as the number of slopes in $\Gamma(G_k)$. Moreover, the slopes of all the maximal segments except (v_1, v_2) and (v_2, v_3) remain the same as the slopes in $\Gamma(G_k)$. By induction hypothesis, the number of slopes in $\Gamma'(G_{k-1})$ is the same as the number of slopes in $\Gamma(G_{k-1})$. Moreover, the slopes of all the maximal segments except (v_1, v_2) and (v_2, v_3) remain the same as the slopes in $\Gamma(G_{k-1})$. Since all the edges of $\Gamma'(G_{k-1})$ are also in $\Gamma'(G_k)$, we are only left with the edges $(w_p, z_1), (z_1, z_2), \ldots, (z_l, w_q)$. Since the vertices $w_q, \ldots, w_t$ as well as the vertices $w_i, \ldots, z_l$ are shifted δ units to the right, the horizontal distance between w_{i-1} and w_i increases. But this increase in distance between w_{i-1} and w_i does not change the slope of the segment $(z_1, z_2), \ldots, (z_{l-1}, z_l)$. Therefore, the maximal segments and the slopes of the edges $(w_p, z_1), (z_1, z_2), \ldots, (z_l, w_q)$ in $\Gamma'(G_k)$ remain the same as in $\Gamma(G_k)$.

We finally prove that no two vertices of degree two of G_k have the same x-coordinate in $\Gamma'(G_k)$. By the property of canonical ordering, all the vertices of degree two in G_k are on the outerface. By induction hypothesis no two vertices of degree two of G_{k-1} have the same x-coordinate in $\Gamma'(G_{k-1})$. Since $\Gamma'(G_{k-1})$ is contained in $\Gamma'(G_k)$, we are only left with vertices $z_1, \ldots, z_l$. Since the vertices $w_q, \ldots, w_t$ as well as the vertices $w_i, \ldots, z_l$ are shifted δ units to the right, the relative distances of the vertices $w_i, \ldots, w_t$ in $\Gamma'(G_k)$ remain the same as in $\Gamma(G_k)$. Moreover, the increase in the horizontal distance between w_{i-1} and w_i does not create any overlap among the x-coordinates of the vertices in V_k. Therefore, no two vertices of degree two of G_k have the same x-coordinate in $\Gamma'(G_k)$.

Case 4: $w_i \in \{w_q, \ldots, w_t\}$.

In this case $\Gamma'(G_k)$, which is obtained from $\Gamma(G_k)$ by shifting the vertices of L_k by δ units to the right, can also be obtained as follows.

Let $C_{k-1} = (w_1 = v_1, \ldots, w_i, \ldots, w_t = v_2)$ be the outerface of G_{k-1}. One can observe that both C_k and C_{k-1} contain the vertex w_i. We first remove the drawing of V_k from $\Gamma(G_k)$ to obtain $\Gamma(G_{k-1})$. We then shift the vertices of L_{k-1} by δ units to the right. We finally add the drawing of V_k, as in $\Gamma'(G_k)$, to $\Gamma'(G_{k-1})$ to obtain $\Gamma'(G_k)$.

The claim holds for G_{k-1} by induction hypothesis. Since L_{k-1} includes all the vertices of L_k, the proof follows form the inductive assumption. □

We now have the following lemma on area requirement of the drawing produced by Algorithm **Cubic Drawing**.

Lemma 7. *Let G be a 3-connected cubic plane graph with n vertices. Then Algorithm **Cubic Drawing** produces a convex drawing $\Gamma(G)$ of G on at most $(\frac{n}{2}+1) \times (\frac{n}{2}+1)$ grid.*

Proof. Let $W_{\Gamma(G)}$ and $H_{\Gamma(G)}$ be the width and height of $\Gamma(G)$, respectively. Then one can easily observe that $H_{\Gamma(G)} \leq W_{\Gamma(G)}$. We now calculate $W_{\Gamma(G)}$.

According to the reasoning presented in Cases 1–4 of the proof of Lemma 5, if the shift is δ units to the right then $W_{\Gamma(G_k)} = W_{\Gamma(G_{k-1})} + \delta$. If $\delta = |V_k| + D_y - D_x$

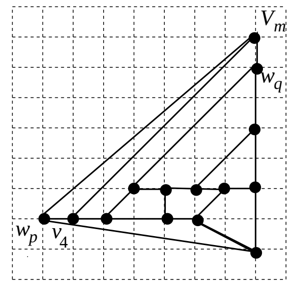

Figure 11. Installation of V_m.

then $\delta < |V_k|$ since $D_x \geq D_y + 1$. If $\delta = |V_k| - D_x$ then $\delta < |V_k|$ since $D_x \geq 1$. If $\delta = |V_k| - D_x + 1$ then $\delta < |V_k|$ since $D_x \geq 2$. Finally, if there is no shift then $W_{\Gamma(G_k)} = W_{\Gamma(G_{k-1})}$. Therefore, the width in each step increases by at most $|V_k| - 1$. The installation of V_m creates two inner faces and the installation of each V_i, $1 \leq i \leq m-1$, creates one inner face. Let the number of inner faces of G be F. Then the number of partitions is $F - 1 = \frac{n}{2}$ by Euler's formula. For the installation of each V_i, $1 < i < m$, the width of the drawing increases by at most $|V_k| - 1$. Moreover, the installation of V_m does not require any shift. Therefore, $W_{\Gamma(G)}$ can be at most $|V_1| + \sum_{i=2}^{\frac{n}{2}}(|V_i| - 1) = n - \sum_{i=2}^{\frac{n}{2}} 1 = n - (\frac{n}{2} - 1) = \frac{n}{2} + 1$. Thus the drawing requires at most $(\frac{n}{2} + 1) \times (\frac{n}{2} + 1)$ grid. □

Theorem 1. *Let G be a 3-connected cubic plane graph. Then Algorithm* ***Cubic Drawing*** *gives a convex drawing of G in $O(n)$ time with at most $s_n + 1$ maximal segments where s_n is the lower bound on the number of maximal segments in a convex drawing of G.*

Proof. The case for $n = 4$ is trivial and hence we may assume that n is greater than four. We construct $\Gamma(G_{m-1})$ by installing V_1, V_2,...,V_{m-1} one after another. Then we install V_m to obtain $\Gamma(G_m) = \Gamma(G)$. Let w_p, v_4 and w_q be the three neighbors of V_m where $x(w_p) < x(w_m) < x(w_q)$. Since all the vertices other than $v_1(= w_p)$, v_4 and w_q are of degree three in G_{m-1}, each of those vertices of degree three except v_2 and v_3 has exactly one straight angle by Lemma 5. Therefore, there are $n - 1$ vertices and at least $n - 6$ straight angles in $\Gamma(G_{m-1})$ when $w_q \neq v_2$. One can easily observe that, Algorithm **Cubic Drawing** installs V_m in such a way that each of v_4 and w_q obtains a straight angle. Thus the number of straight angles in $\Gamma(G)$ is at least $n - 6 + 2 = n - 4$, in total. Similarly, if $w_q = v_2$ then there are $n - 5$ straight angles in $\Gamma(G_{m-1})$ and V_m is installed in such a way that the number of straight angles in $\Gamma(G)$ becomes $n - 4$, in total. Let x be the number of maximal segments in $\Gamma(G)$. Then by Lemma 2, $\Gamma(G)$ has at least $\frac{3n}{2} - x = n - 4$ straight angles and at most $x = \frac{n}{2} + 4$ maximal segments. By Lemma 3 the lower bound on the number of maximal segments s_n in a convex drawing of G is $\frac{n}{2} + 3$. Thus the number of maximal segments in $\Gamma(G)$ is at most $s_n + 1$. To obtain a linear-time implementation of the Algorithm

Cubic Drawing, we use a method similar to the implementation as used in [4]. □

4 Minimum-Segment Drawings

In this section we give an algorithm, which we call **Draw-Min-Segment**, to obtain a minimum-segment convex drawing $\Gamma(G)$ of a 3-connected cubic plane graph G with $n \geq 6$ vertices in linear time, where $\Gamma(G)$ is not a grid drawing.

We now describe Algorithm **Draw-Min-Segment**. We use canonical decomposition to obtain $V_1, \ldots, V_m$ using the same technique as the one in Section 3. We now draw G_1 by a triangle as follows. Set $P(v_i) = (i-1, 0)$ where $1 \leq i < l$, $P(v_l) = (l-1, -1)$. Each V_k, $2 \leq k \leq m$, is either a singleton set or an outer chain of G_k, but in the algorithm we will treat both cases uniformly. We add $V_1, \ldots, V_m$ one after another to obtain $\Gamma(G_1), \ldots, \Gamma(G_m) = \Gamma(G)$. Let w_p and w_q be the leftmost and the rightmost neighbors of V_k on $C(G_{k-1})$ where $1 < k \leq m$. We install $V_k = (z_1, \ldots, z_l)$ in such a way that $(z_1, z_2), \ldots, (z_{l-1}, z_l)$ form a segment of slope 0 and the following Conditions (a) and (b) hold for each index k, $2 \leq k < m$.

(a) If w_p has a straight angle in $\Gamma(G_{k-1})$, then slope of (w_p, z_1) is $+1$. Otherwise w_p has no straight angle in $\Gamma(G_{k-1})$, then slope of (w_p, z_1) is the same as the slope of (w_{p-1}, w_p).
(b) If $x(w_q)$ is the maximum among all the x-coordinates of the vertices of $\Gamma(G_{k-1})$, then slope of (z_l, w_q) is ∞; otherwise if w_q has a straight angle in $\Gamma(G_{k-1})$, then slope of (z_l, w_q) is -1 and if w_q has no straight angle in G_{k-1}, then slope of (z_l, w_q) is the same as the slope of (w_q, w_{q+1}).

We shift the vertices $w_q, w_{q+1}, ..., w_t$ together with some inner vertices to the right or up, to install V_k satisfying Conditions (a) and (b), as illustrated in Figure 12(a)-(f) for the input 3-connected cubic graph in Figure 2. In a similar way as in Section 3, one can maintain a set with each vertex to determine which vertices are to shift.

We now have the following lemma.

Lemma 8. *Let G be a 3-connected cubic plane graph with $n \geq 6$ vertices and $\pi = (V_1, V_2, ..., V_m)$ be a canonical decomposition of the vertices of G with outer edge (v_1, v_2). Let $G_k = V_1 \cup V_2 \cup ... \cup V_k$ and $\Gamma(G_k)$ be a drawing of G_k obtained by Algorithm* ***Draw-Min-Segment****, where $1 \leq k < m$. Then each vertex of degree three in G_k, except the vertices v_1 and v_2, has a straight angle in $\Gamma(G_k)$.*

Proof. The case for $\Gamma(G_1)$ is trivial and we may thus assume that k is greater than one. By the induction hypothesis, each vertex of degree three in G_{k-1}, except the vertices v_1 and v_2, has a straight angle in $\Gamma(G_{k-1})$. Let w_p and w_q be the leftmost and the rightmost neighbors of V_k on C_{k-1}. By a case analysis similar to the one in Lemma 5, one can observe that, addition of V_k with $\Gamma(G_{k-1})$ to obtain $\Gamma(G_k)$ creates two new vertices of degree three, which are w_p and w_q. Each of w_p and w_q has a straight angle by Conditions (a) and (b) when

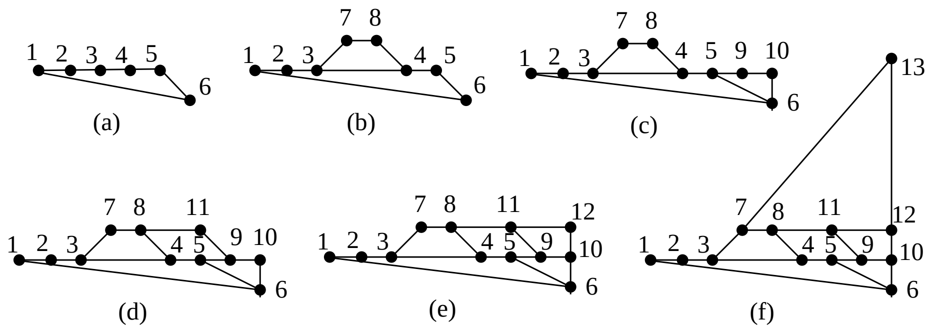

Figure 12. Installation of V_k when $1 \leq k < m$.

$w_p \notin \{v_1, v_2\}$ and $w_q \notin \{v_1, v_2\}$. Therefore, each vertex of degree three in G_k, except v_1 and v_2, has a straight angle in $\Gamma(G_k)$. □

We now describe the installation of V_m. Let w_p, v_4 and w_q be the three neighbors of V_m, where $x(w_p = v_1) < x(v_4) < x(w_q)$. We place V_m in such a way that (v_4, V_m) and (w_q, V_m) obtain the slopes $+1$ and ∞, respectively. Then we simply draw the edge (v_1, V_m).

Theorem 2. *Let G_m be a 3-connected cubic graph with $n \geq 6$ and $\Gamma(G)$ be a drawing of G obtained by Algorithm **Draw-Min-Segment**. Then $\Gamma(G)$ is a minimum-segment convex drawing of G.*

Proof. Let $(V_1, V_2, ..., V_m)$ be an ordered partition of the vertices of G obtained by a canonical decomposition of G. Let $G_k = V_1 \cup V_2 \cup ... \cup V_k$ where $1 \leq k \leq m$. We construct $\Gamma(G_{m-1})$ by installing V_1, V_2,...,V_{m-1} one after another. Then we install V_m to obtain $\Gamma(G_m)$. Let w_p, v_4 and w_q be the three neighbors of V_m where $x(w_p) < x(v_4) < x(w_q)$. Since all the vertices other than $w_p = v_1$, v_4 and w_q are of degree three in G_{m-1}, each of those vertices of degree three except v_1 and v_2 has exactly one straight corner by Lemma 8. Therefore, there are $n-1$ vertices and at least $n-5$ straight corners in $\Gamma(G_{m-1})$. One can easily observe that, Algorithm **Draw-Min-Segment** installs V_m in such a way that each of v_4 and w_q obtains a straight corner. Thus the number of straight corners in $\Gamma(G_m) = \Gamma(G)$ is $n-3$, in total. Let x be the number of maximal segments in $\Gamma(G)$. Then by Lemma 2, G has $\frac{3n}{2} - x = n - 3$ straight corners and therefore, $x = \frac{n}{2} + 3$ segments. Since by Lemma 3 this is the lower bound on the number of maximal segments in a convex drawing of G, $\Gamma(G)$ is a minimum-segment convex drawing of G. □

5 Conclusions

In this paper, we have given a linear time algorithm to obtain a convex grid drawing of a 3-connected cubic plane graph G with $s_n + 1$ maximal segments and on $(\frac{n}{2}+1)\times(\frac{n}{2}+1)$ grid, where s_n is the lower bound on the number of maximal segments in a convex drawing of G. We have also proved that any 3-connected cubic plane graph G with $n \geq 6$ vertices admits a convex drawing with $\frac{n}{2}+3$ maximal segments which is the minimum number of maximal segments required for any convex drawing of G. Keszegh *et al.* showed that every graph with the maximum degree three has a straight-line drawing in the plane, where the edges have at most five different slopes [10]. It is interesting to observe that the drawing produced by our algorithm uses only six different slopes. It is left as a future work to obtain minimum-segment convex drawings of other classes of planar graphs. It seems that the problem of finding minimum-segment convex drawings of general planar graphs is non-trivial and remains as an open problem.

Di Battista *et al.* [5] and Felsner [7] independently proved that any 3-connected plane graph admits a convex grid drawing on $(f-1)\times(f-1)$ area or $f\times f$ grid, where f is the number of faces in the graph. By Euler's formula, the number of faces in a 3-connected cubic plane graph is $\frac{n}{2}+2$. Therefore, the drawings of G produced by their algorithms take $(\frac{n}{2}+2)\times(\frac{n}{2}+2)$ grid, which is close to the grid size obtained by our algorithm, but the number of line segments produced by their algorithm is far from optimal. Since the algorithm in [5,7] deals with convex drawings of 3-connected plane graphs, it will be interesting to investigate whether the method in [5,7] can be applied to find minimum-segment convex drawings of 3-connected plane graphs.

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References

1. S. Biswas, D. Mondal, R. I. Nishat, and M. S. Rahman. Minimum-segment convex drawings of 3-connected cubic plane graphs. In *16th Annual International Computing and Combinatorics Conference (COCOON 2010)*, volume 6196 of *Lecture Notes in Computer Science*, pages 182–191. Springer, 2010.
2. N. Chiba, K. Onoguchi, and T. Nishizeki. Drawing plane graphs nicely. *Acta Informatica*, 22(2):187–201, 1985.
3. N. Chiba, T. Yamanouchi, and T. Nishizeki. Linear algorithms for convex drawings of planar graphs. In *Progress in Graph Theory*, pages 153–173. 1984.
4. M. Chrobak and G. Kant. Convex grid drawings of 3-connected planar graphs. *International Journal of Computational Geometry and Applications*, 7(3):211–223, 1997.
5. G. Di Battista, R. Tamassia, and L. Vismara. Output-sensitive reporting of disjoint paths. *Algorithmica*, 23(4):302–340, 1999.

6. V. Dujmović, D. Eppstein, M. Suderman, and D. R. Wood. Drawings of planar graphs with few slopes and segments. *Computational Geometry: Theory and Applications*, 38(3):194–212, 2007.
7. S. Felsner. Convex drawings of planar graphs and the order dimension of 3-polytopes. *Order*, 18(1):19–37, 2001.
8. G. Kant. Hexagonal grid drawings. In *WG '92*, volume 657 of *Lecture Notes in Computer Science*, pages 263–276. Springer, 1992.
9. G. Kant. Drawing planar graphs using the canonical ordering. *Algorithmica*, 16(1):4–32, 1996.
10. B. Keszegh, J. Pach, D. Pálvölgyi, and G. Tóth. Drawing cubic graphs with at most five slopes. *Computational Geometry*, 40(2):138–147, 2008.
11. T. Nishizeki and M. S. Rahman. *Planar Graph Drawing*. World Scientific, 2004.
12. Md. A. H. Samee, Md. J. Alam, M. A. Adnan, and Md. S. Rahman. Minimum segment drawings of series-parallel graphs with the maximum degree three. In *16th International Symposium on Graph Drawing*, volume 5417 of *Lecture Notes in Computer Science*, pages 408–419. Springer, 2009.
13. W. T. Tutte. Convex representations of graphs. In *London Mathematical Society*, volume 10, pages 304–320, 1960.