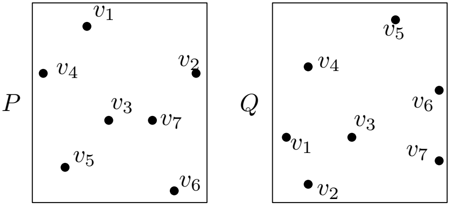
Compatible Paths on Labelled Point Sets

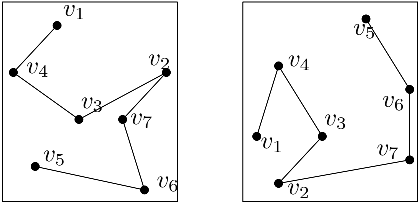
Elena Arseneva, Yeganeh Bahoo, Ahmad Biniaz, Pilar Cano, Farah Chanchary, John Iacono, Kshitij Jain, Anna Lubiw, Debajyoti Mondal, Khadijeh Sheikhan, Csaba D. Tóth

CCCG, 2018

Outline

Definition





Related work

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- $O(n^2)$ lower and upper bound for Steiner points even in the case of polygons with holes. [Babikov, Souvaine, Wenger, 1997 and Pach, Shahrokhi, Szegedy, 1996]

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- NP-hard to find the minimum number of Steiner points in the case of polygons with holes. [Lubiw and Mondal, 2017]

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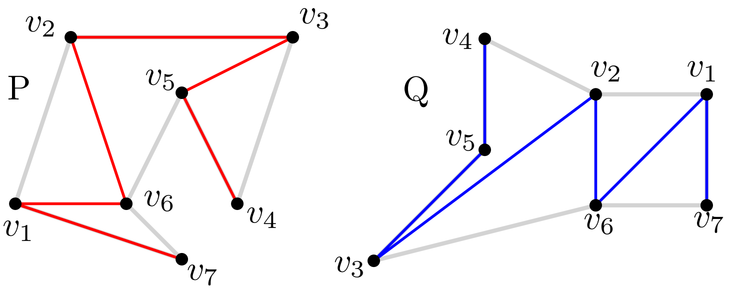
- NP-hard to find the minimum number of Steiner points in the case of polygons with holes. [Lubiw and Mondal, 2017]
- A polynomial-time dynamic programming algorithm to test whether two polygons admit compatible triangulations without Steiner points. [Aronov, Seidel, and Souvaine, 1993]

Outline

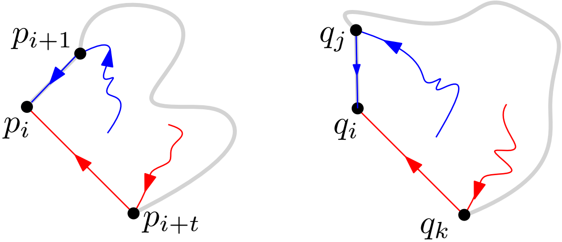
1. Definition
2. Related work
3. Our results
   - Paths in polygons
   - Paths in convex set
   - Monotone paths

Paths in polygons



Paths in polygons

- Calculate visibility graph of the two given polygons.
- Use dynamic programming.
- Paths of length zero (including one vertex) are always compatible.



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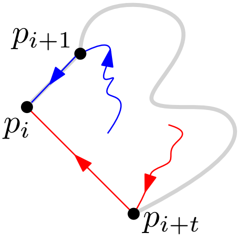

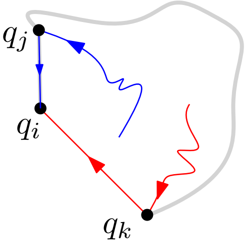

Theorem

Given two n-vertex polygons, each with points labelled from 1 to n in some order, one can find a pair of compatible paths or determine that none exists in $O(n^2)$ time.

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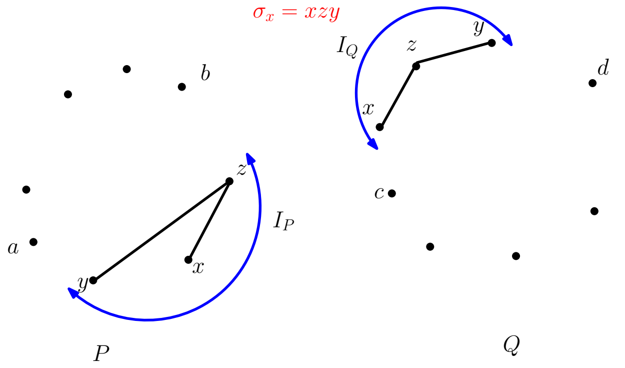
Paths in convex set

- Initialize σ_x with x



Paths in convex set

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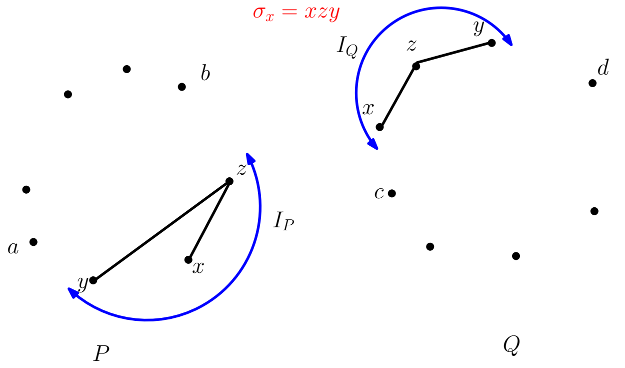

- Three cases:
  - If $\{a, b\} = \{c, d\}$ add both to σ_x.
  - If there is one common vertex add that to σ_x.
  - If there is no common vertex, construction ends.

Paths in convex set

Lemma

P and Q have compatible paths starting at label x if and only if σ_x includes all n labels.

Paths in convex set

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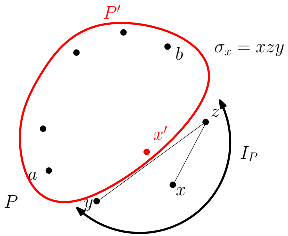
Compatible path can be found in $O(n^2)$.

Paths in convex set

Lemma

If σ_x has length less than n, then no label in σ_x can be the starting label for compatible paths of P and Q.



Lemma

Suppose x is a label appearing in P'. P and Q have compatible paths with initial label x if and only if P' and Q' have compatible paths with initial label x.

Paths in convex set

Theorem

Given two sets of n points in convex position each with points labelled from 1 to n, one can find a pair of compatible paths or determine that none exist in linear time.

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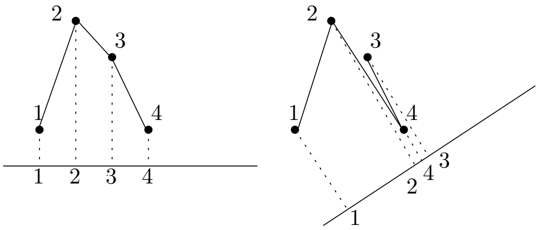
Monotone paths

Monotone Paths

An ordering σ of the point set P is called *monotone* if there exists some line ℓ such that the orthogonal projection of the points on ℓ yields the order σ. A *monotone path* is a path that corresponds to a monotone ordering.



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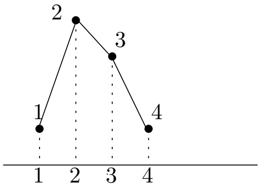
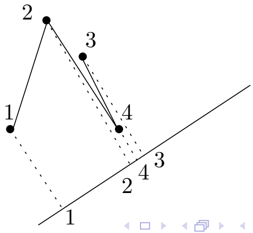
Compatible Monotone Paths

Points sets P and Q have *compatible monotone paths* if there is an ordering of the labels that corresponds to a monotone path in both P and Q.

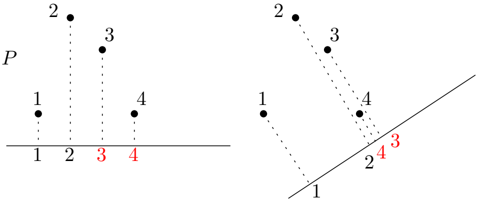




An $O(n^3)$ algorithm

Theorem [Goodman and Pollack, 1980]

Projecting the points onto ℓ as it rotates 360° counter-clockwise about a fixed point gives all the possible monotone orderings of P. There are $n(n-1)$ orderings, and each successive ordering differs from the previous one by a swap of two elements adjacent in the ordering.



L^P

1234
1243
1423
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4132
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⋮

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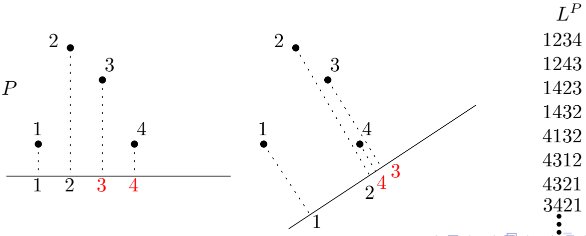
- Generate $O(n^2)$ monotone orderings of P in constant time per ordering.



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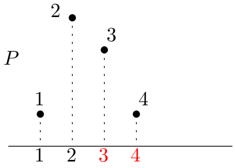
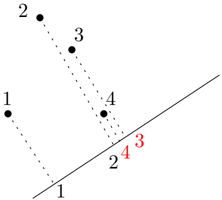
- Generate $O(n^2)$ monotone orderings of P in constant time per ordering.
- Check each ordering for monotonicity in Q in linear time.





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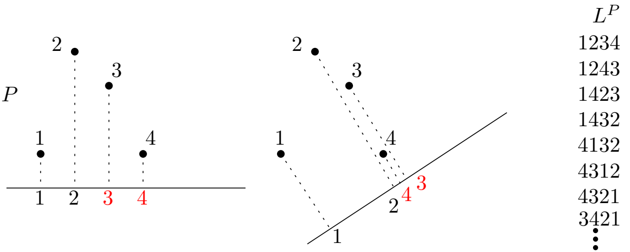
Improved $O(n^2 \log n)$ algorithm

- The sequence of swaps that change each ordering to the next one can be found in $O(n^2 \log n)$ time by sorting the $O(n^2)$ lines by their slopes.



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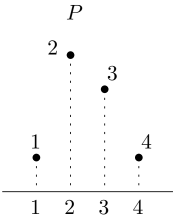

L^P	I_i
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1243	1
1423	2
1432	3
4132	4
4312	5
4321	6
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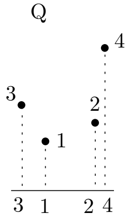
⋮	⋮



L^Q	I_j
3124	2
1324	1
1234	0
2134	1
2143	2
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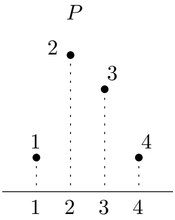
- If the order of the points of P along ℓ_0 is $1, 2, \ldots, n$, inversion numbers of the L_i^P's progress from 0 to $\binom{n}{2}$ and back again.
- If $L_j^Q = L_i^P$, then they have the same inversion number I_j and therefore, sufficient to check if $L_j^Q = L_{I_j}^P$.



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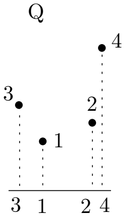

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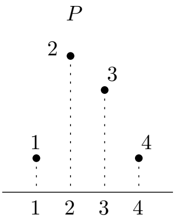
- *Hamming distance* H_j, denotes the number of mismatches between L_j^Q and $L_{I_j}^P$.



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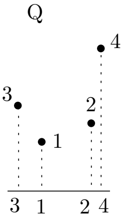

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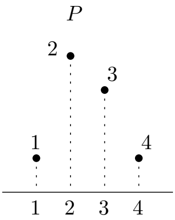
Improved $O(n^2 \log n)$ algorithm

- *Hamming distance* H_j, denotes the number of mismatches between L^Q_j and $L^P_{I_j}$.
- P and Q have compatible monotone paths if and only if $H_j = 0$ for some $0 \leq j < n(n-1)$.



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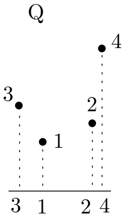

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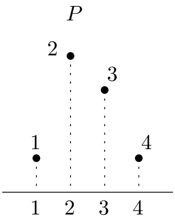
- *Hamming distance* H_j, denotes the number of mismatches between L_j^Q and $L_{I_j}^P$.
- P and Q have compatible monotone paths if and only if $H_j = 0$ for some $0 \leq j < n(n-1)$.
- L_j^Q, I_j, $L_{I_j}^P$, and H_j can be computed in constant time.



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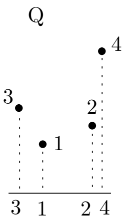

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Compatible Monotone Paths

Theorem

Given two point sets, each containing n points labelled from 1 to n, one can find a pair of compatible monotone paths or determine that none exist in $O(n^2 \log n)$ time.

Negative example

Theorem

For every $n \geq 5$, there exist two sets, P_n and Q_n, each of n labelled points in convex position, such that P_n and Q_n do not admit any compatible tree.

Open Problems

Determine the computational complexity of deciding whether two labelled point sets in general position admit:

- compatible paths or compatible trees?

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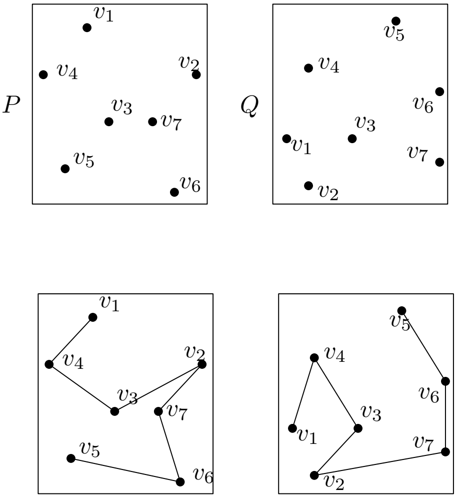
Determine the computational complexity of deciding whether two labelled point sets in general position admit:

- compatible paths or compatible trees?
- admit compatible triangulations (without Steiner points) [CCCG 2017 Open Problem]?

Work done at 2017 Fields Workshop on Discrete and Computational Geometry,
Carleton University, Ottawa.
Missing Ahmad, John, and Csaba.



Questions?