Angle-Monotone Graphs: Construction and Local Routing

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Given a point set, Construct a ‘Good’ Graph
Increasing-Chord Paths

$|bc| \leq |ad|$
Increasing-Chord Paths

$|bc| \leq |ad|$
Increasing-Chord Paths

\[ |bc| \leq |ad| \]

\[ |bc| > |ad| \]
Increasing-Chord Paths

[Icking et al.'95]
Increasing-Chord Paths

[Icking et al.'95]
Construct a small spanning geometric graph that is

- **Monotone** [Mastakas and Symvonis, ‘17]
- **Good spanner - planar** [Chew ‘89; Dobkin et al, ‘90; Bose et al. ‘10, Xia, ‘13]
- **Good spanner - nonplanar** [Narasimhan and Smid (Book) 2007]
- **Increasing-chord** [Alamdari et al.’13; Dehkordi et al.’15; Bonichon et al.’16]
Open Question

[Dehkordi, Frati, and Gudmundsson, 2015]

**Input:** A point set $P$

**Question:** Does there exist an increasing-chord graph on $P$ with $o(n^2)$ edges?
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This presentation

It suffices to construct an **angle-monotone graph of width 90°** with subquadratic number of edges.
A geometric graph in the plane is angle-monotone of width $\gamma$ if every pair of vertices is connected by an angle-monotone path of width $\gamma$. 
If $\gamma = 90^\circ$, then the graph is a $\sqrt{2}$ spanner

Angle-monotone graphs of width $90^\circ$ are increasing chord graphs.
• We can construct an angle-monotone graph of width $90^\circ$ with $O(n^2 \log \log n / \log n)$ edges.

• We can construct an angle-monotone graph of width $(90^\circ + \alpha)$ with $O(n/\alpha)$ edges.

• We also have a **2-local routing algorithm** that can find a path of width $(90^\circ + \alpha)$.
Construction of Angle-Monotone Graphs
width 90°, $O(n^2 \log \log n / \log n)$ edges

Partition the point set into $O(n / \log n)$ convex polygons, each of size $O(\log n)$ [Urabe 1996]
Construction of Angle-Monotone Graphs
width 90°, $O(n^2 \log \log n / \log n)$ edges

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Add all possible edges within and between these convex polygons: $O((n/\log n + n^2/\log^2 n) \log^2 n)$
Construction of Angle-Monotone Graphs
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Does not work.
Construction of Angle-Monotone Graphs
width 90°, \(O(n^2 \log \log n / \log n \) edges

- Partition the point set into \(O(n/\log n)\) convex polygons, each of size \(O(\log n)\) [Urabe 1996]

- For each convex polygon, construct an angle monotone graph: \(O(n \log n)\) edges in total.

- Partition each of these convex polygons into
  - a \((x, y)\)-convex path,
  - a \((x, -y)\)-convex path,
  - a \((x, y)\)-concave path, and
  - a \((x, -y)\)-concave path.
Construction of Angle-Monotone Graphs
width 90°, $O(n^2 \log \log n / \log n)$ edges

➢ For every pair of paths, we now construct an angle-monotone graph spanning these paths.
Construction of Angle-Monotone Graphs
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Case Analysis:

- a (x, y)-convex path
- a (x, y)-convex path
- a (x, -y)-convex path
- a (x, -y)-convex path
- a (x, y)-concave path
Construction of Angle-Monotone Graphs

width 90°, $O(n^2 \log \log n / \log n)$ edges

- An example in more general setting
- Assume that you are given a $(x,y)$-monotone and a $(x,-y)$-monotone path
Construction of Angle-Monotone Graphs
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- Assume that you are given a $(x,y)$-monotone and a $(x,-y)$-monotone path

An angle-monotone graph of linear size
Construction of Angle-Monotone Graphs
width 90°, $O(n^2 \log \log n / \log n)$ edges

- An example in more general setting
- Assume that you are given a (x,y)-monotone and a (x,-y)-monotone path

An angle-monotone graph of linear size

This already covers all combinations except for when both paths are of type (x,y), or both are of type (x,-y).
Construction of Angle-Monotone Graphs
width $90^\circ$, $O(n^2 \log \log n / \log n)$ edges

- For every pair of paths, we now construct an angle-monotone graph spanning these paths.
- In each case we add at most $O(\log n \log \log n)$ edges

Case Analysis:
- $(x,y)$-convex path
- $(x,-y)$-convex path
- $(x,-y)$-convex path
- $(x,y)$-concave path

$O(n^2/\log^2 n) \times O(\log n \log \log n)$
Construction of Angle-Monotone Graphs width $(90^\circ + \alpha)$ with $O(n/\alpha)$ edges
Construction of Angle-Monotone Graphs width $(90°+\alpha)$ with $O(n/\alpha)$ edges

A 3-sweep graph on a point set
Construction of Angle-Monotone Graphs width \((90^\circ + \alpha)\) with \(O(n/\alpha)\) edges

A 3-sweep graph on a point set
Construction of Angle-Monotone Graphs width \((90^\circ + \alpha)\) with \(O(n/\alpha)\) edges

A 3-sweep graph on a point set

- Planar
- If \(\theta_a = \theta_b = \theta_c = 60^\circ\), then the graph coincides with the well-known half-theta-6-graph.
Construction of Angle-Monotone Graphs width $(90^\circ + \alpha)$ with $O(n/\alpha)$ edges

A $k$-layer 3-sweep graph on a point set

$\theta_a = 2\alpha; \ k = \pi/\alpha$
Construction of Angle-Monotone Graphs width \((90° + \alpha)\) with \(O(n/\alpha)\) edges

A \textit{k-layer} 3-sweep graph on a point set

May have high degree but still a union of small number of planar graphs
2-Local Routing on $k$-layer 3-sweep graphs

- Send a message from the source to the destination
- The current vertex knows
  - The locations of its neighbors and their neighbors (2-hop distance).
  - The location of the destination.
- Does not use additional memory (e.g., no routing table)
2-Local Routing on $k$-layer 3-sweep graphs

- Routing Rules
- The path taken is an angle monotone path of width $(90°+\alpha)$
Open Questions

Does every point set admit an angle-monotone graph with linear number of edges?

Find the smallest $\gamma$ such that every point set has a planar angle-monotone graph of width $\gamma$.

Currently it is known that $90^\circ < \gamma \leq 120^\circ$. 
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Thank You