Delays Part 2
Equilibrium Behaviour
Higher-Order Delays

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Our Route Forward:
3 Common Types of Delay-Related Dynamics

- First Order Delays
  - Aging Chains & Higher-Order Delays
  - Competing Risks
  - Delays & Oscillations
First Order Delays in Action: Simple SIT Model

Mean Contacts Per Capita
Per infected contact infection rate
Initial Population

Mean Infectious Contacts Per Susceptible
Per Susceptible Incidence Rate
Prevalence
Recovery Delay
Initial Population

Total Population

New infections
Newly Susceptible
Immunity loss
Delay
Per infected contact
infection rate

Cumulative Illnesses
New Illness

Newly Susceptible

Immunity loss Delay

deptcsc
First Order Delays in Action: Simple SIT Model
Recall: Simple First-Order Decay

Use Initial Value: 1000

Use Formula: People with Virulent Infection/Mean time until Death
First-Order Decay (Variant of Last Time)

Use Initial Value: 1000

Use Formula: People with Virulent Infection * Per Month Likelihood of Death

Recall: How does this relate to the mean time until death?
People in Stock

People with Virulent Infection

Time (Month)

People with Virulent Infection : Baseline
Flow Rate of Deaths

Deaths

Time (Month)

Deaths : Baseline
Cumulative Deaths

Cumulative Deaths vs Time (Month)
Closeup

Why this gap?
50% per Month Risk of Deaths

Cumulative Deaths

Time (Month)

Cumulative Deaths: Baseline pt5
Use Initial Value: 1000

Use Formula: People (x) * Annual Risk of Death (alpha)

Use Value: 0

Use Value: 0.05
Questions

• What is behaviour of stock x?
• What is the mean time until people die?
• Suppose we had a constant inflow – what is the behaviour then?
Answers

• Behaviour Of Stock

• Mean Time Until Death

Recall that if coefficient of first order delay is $\alpha$, then mean time is $1/\alpha$ (Here, $1/0.05 = 20$ years)
Equilibrium Value of a First-Order Delay

• Suppose we have flow of rate $i$ into a stock with a first-order delay out
  – This could be from just a single flow, or many flows
• The value of the stock will approach an equilibrium where inflow=outflow
Equilibrium Value of 1st Order Delay

• Recall: Outflow rate for 1st order delay=$\alpha x$
  – Note that this depends on the value of the stock!
• Inflow rate=i
• At equilibrium, the level of the stock must be such that inflow=outflow
  – For our case, we have
    
    $\alpha x = i$

Thus $x = i/\alpha$

The lower the chance of leaving per time unit (or the longer the delay), the larger the equilibrium value of the stock must be to make outflow=inflow
Scenarios for First Order Delay: Variation in Inflow Rates

• For different immigration (inflows) (what do you expect?)
  – Inflow=10
  – Inflow=20
  – Inflow=50
  – Inflow=100
  – Why do you see this “goal seeking” pattern?
  – What is the “goal” being sought?
Why do we see this behaviour?
Behaviour of *Outflow* for Different Inflows

**Why do we see this behaviour?** Imbalance (gap) causes change to stock (rise or fall) $\Rightarrow$ change to outflow to lower gap until outflow $=$ inflow.
Goal Seeking Behaviour

• The goal seeking behaviour is associated with a negative feedback loop
  – The larger the population in the stock, the more people die per year

• If we have more people coming in than are going out per year, the stock (and, hence, outflow!) rises until the point where inflow=outflows

• If we have fewer people coming in than are going out per year, the stock declines (& outflow) declines until the point where inflow=outflows
What does this tell us about how the system would respond to a sudden change in immigration?
Response to a Change

• Feed in an immigration “step function” that rises suddenly from 0 to 20 at time 50

• Set the Initial Value of Stock to 0

• How does the stock change over time?
Create a Custom Graph & Display it as an Input-Output Object

- Editing
Create Input-Output Object (for Synthesim)
Stock Starting Empty

Flow Rates

Inflow and Outflow

How would this change with alpha?
Stock Starting Empty?
Value of Stock (Alpha=.05)

People (x)

"People (x)" : Step Function 0 to 20 Initial Stock Empty

How would this change with alpha?
For Different Values of $(1/\alpha)$ Alpha Flow Rates (Outflow Rises until = Inflow)

This is for the flows. What do stocks do?
For Different Values of (1/) Alpha

Why do we see this behaviour? A longer time delay (or smaller chance of leaving per unit time) requires $x$ to be larger to make outflow=inflow.
Outflows as Delayed Version of Inputs

Inflow and Outflow

Time (Year)

Immigration: Step Functions 2 yr delay
Deaths: Step Functions 2 yr delay

Immigration: Step Functions 10 yr delay
Deaths: Step Functions 10 yr delay

Immigration: Step Functions 5 yr delay
Deaths: Step Functions 5 yr delay

Inflow and Outflow

Time (Year)
What if stock doesn’t start empty?

Decays at first (no inflow) & then output responds with delayed version of input
Higher Order Delays & Aging Chains
Moving Beyond the “memoryless assumption”

- Recall that first order delays assume that the per-time-unit risk of transitions to the outflow remains equal throughout simulation (i.e. are memoryless).

- Problem: Often we know that transitions are not "memoryless" e.g.
  - It may be the transition reflects some physical delays not endogeneously represented (e.g. Slow-growth of bacterial)
  - Buildup of “damage” of high blood sugars (Glycosylation)
Higher Orders of Delays

- We can capture different levels of delay (with increasing levels of fidelity) using cascaded series of 1\textsuperscript{st} order delays.
- We call the delay resulting from such a series of \( k \) 1\textsuperscript{st} order delays a “\( k^{th} \) order delay”
  - E.g. 2 first order delays in series yield a 2\textsuperscript{nd} order delay.
- The behaviour of a \( k^{th} \) order delay is a reflection of the behaviour of the 1\textsuperscript{st} order delays out of which it is built.
- To understand the behaviour of \( k^{th} \) order delays, we will keep constant the mean time taken to transition across the entire set of all delays.
Recall: Simple 1\textsuperscript{st} Order Decay

Use Formula: \textit{People with Virulent Infection}/Mean time until Death

(Initial Value: 1)
Recall: 1st Order Delay Behaviour

• *Conditional* transition prob: For a 1st Order delay, the per-time-unit likelihood of leaving *given that one has not yet left the stock* remains constant.

• *Unconditional* transition prob: For a 1st Order delay, the unconditional per-time-unit likelihood of leaving declines exponentially.
  – i.e. if were were originally in the stock, our chance of having left in the course of a given time unit (e.g. month) declines exponentially.
    • This reflects the fact that there are fewer people who could still leave during this time unit!
Recall: 1\textsuperscript{st} Order Delay Behaviour

(Likelihood of Still being In System)

(Per-month chance of transitioning out during this month)
2\textsuperscript{nd} Order Delay

Use Formula:
Mean Time to Transition Across All Stages/Stage Count

(Use value of 2)

(Use value of 50)

(Initial Value: 1)

(Initial Value: 0)
2\textsuperscript{nd} Order Delay

Stage 2 Outflow

(Per-month chance of transitioning out during this month)

(Likelihood of Still being In System)
3rd Order Delay

Mean Time to Transition Across All Stages

Mean Time to Transition Across Single Stage

Stage Count

Stage 1
Stage 1 Outflow

Stage 2
Stage 2 Outflow

Stage 3
Stage 3 Outflow

Inflow

Total Likelihood of Still Being in System
3rd Order Delay

Stage 3 Outflow

Per-month chance of transitioning out during this month

Total Likelihood of Still Being in

(Likelihood of Still being In System)
1\textsuperscript{st} through 6\textsuperscript{th} Order Delays

(Order Delays (Likelihood of Still being In System)

(Per-month chance of transitioning out during this month)

(Likelihood of Still being In System)
Mean Times to Depart Final Stage

• Mean time of $k$ stages is just $k$ times mean time of one stage (e.g. if the mean time for leaving 1 stage requires time $\mu$, mean time for $k = k*\mu$

• In our examples, as we added stages, we reduced the mean time per stage so as to keep the total constant!
  – i.e. if we have $k$ stages, the mean time to leave each stage is $1/k$ times what it would be with just 1 stage

• Infinite order delay: As we add more and more stages ($k \rightarrow \infty$), the distribution of time to leave the last stage approaches a normal distribution
  – If we reduce the mean time per stage so as to keep the total time constant, this will approach an impulse function
    • This indicates an exactly fixed time to transition through all stages!
Distribution of Time toDepart Final Stage

- The distributions for the total time taken to transition out of the last of $k$ stages are members of the Erlang distribution family.
  - These are the same as the distribution for the $k$th interarrival time of a Poisson process.

- $k=1$ gives exponential distribution (first order delay).

- As $k \to \infty$, approaches normal distribution (Gaussian pdf).

From Wikipedia, 2009
Notes

- We do not generally define $k^{th}$ order delays simply as a means to the end of capturing a certain distribution
  - Often representing each stage for its own sake is desirable (see examples)
    - Different causal influences
  - Often we represent each such stage as a 1st order delay
- With that proviso, many modeling packages (including Vensim) directly support higher-order delays – use with caution
Delays & Competing Risks
Competing Risks

• Suppose we have another outflow from the stock. How does that change our mean time of proceeding specifically down flow 1 (here, developing diabetes)?
Basic Dynamics

- Diabetic Population
- Population with ESRD
- Deaths of Diabetic Population
- Diabetics Progressing to ESRD
- Mean Time to Develop ESRD
- Annual Risk of Diabetic Mortality
Effect of Doubling Diabetic Mortality Rate
Effect on *Progression* Rates to ESRD

Do the two scenarios have the same or different *mean times to develop ESRD*? If different, which scenario is larger?
Why the Lower Mean Time?

• Why is the mean time to transition different, despite the fact that we didn’t change the transition parameter?

• Mathematical explanation (Following slides): Calculation of mean time varies with mortality rate

• Intuition:
  – Higher death rate $\Rightarrow$ Diabetic population will rapidly decrease & transitions to ESRD will be skewed towards earlier transitions $\Rightarrow$ Earlier mean time to transition
  – Lower death rate $\Rightarrow$ Diabetic population will decrease less rapidly & many will make later transitions to ESRD $\Rightarrow$ Later mean time to transition
Competing Risks Stock Trajectory

Solution Procedure

\[
\frac{dx}{dt} = -\alpha x - \beta x = -(\alpha + \beta) x
\]

• Suppose we start \( x \) at time 0 with initial value \( x(0) \), and we want to find the value of \( x \) at time \( T \)

• This is just like our previous differential equation, except that “\( \alpha \)” has been replaced by “\((\alpha + \beta)\)”
  – The solution must therefore be the same as before, with the appropriate replacement
  – Thus

\[
x(T) = x(0)e^{-(\alpha + \beta)T}
\]
Mean Time to Leave: Competing Risks

- \( p(t)dt \) here is the likelihood of a person leaving via flow 1 (e.g. developing ESRD) exactly between time \( t \) & \( dt+t \)
  - We start the simulation at \( t=0 \), so \( p(t)=0 \) for \( t<0 \)
  - For \( t>0 \), \( P(\text{leaving on flow 1 exactly between time } t \& dt+t) = P(\text{leaving on flow 1 exactly between time } t \& t+dt \mid \text{Still have not left by time } t) P(\text{Still have not left by time } t) \)

For \( T>0 \), \( P(\text{Still have not left by time } T) = e^{-(\alpha+\beta)T} \)

For \( P(\text{leaving exactly between time } t \text{ and } t+dt \mid \text{Still have not left by time } t) \)

Recall: For us, probability of leaving in a time \( dt \) always = \( \alpha dt \)

Thus \( P(\text{leaving exactly between time } t \text{ and } t+dt \mid \text{Still have not left by time } t) = \alpha dt \)

\[ P(t)dt = P(\text{leaving exact b.t. time } t \& dt+t) = \alpha e^{-(\alpha+\beta)T} \]
Mean Time to Transition via Flow 1: Competing Risks

• By the same procedure as before, we have

\[ E[p(t)] = \alpha \int_{t=0}^{t=\infty} te^{-(\alpha+\beta)t} dt \]

• Using the formula we derived for the integral expression, we have

\[ E[p(t)] = \frac{\alpha}{(\alpha + \beta)^2} \]

• Note that this correctly approaches the single-flow case as \( \beta \to 0 \)
“Aging Chains” (including successive 1st Order Delays & Competing Risks) in our Model of Chronic Kidney Disease