Infectious Disease Models 5 – Basic Epidemiological Quantities and Vaccination

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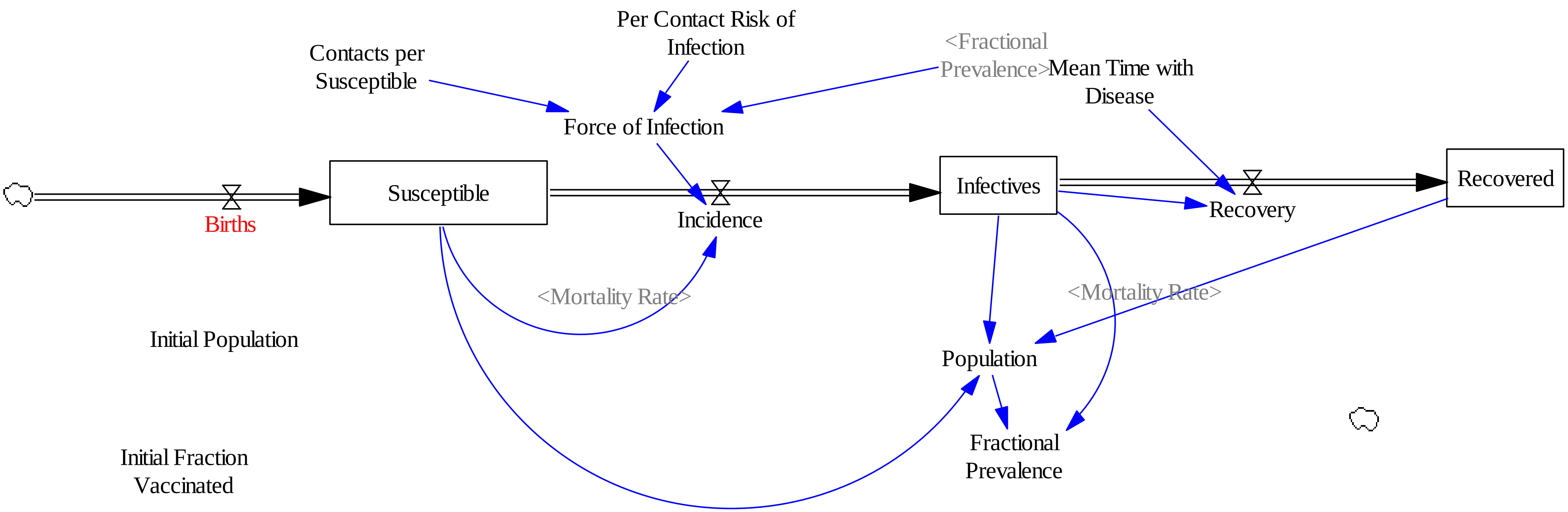
Per Contact Risk of Infection
Contacts per Susceptible
<Fractional Prevalence>
Mean Time with Disease
Force of Infection
Susceptible
Infectives
Recovered
Births
Incidence
Recovery
<Mortality Rate>
<Mortality Rate>
Initial Population
Population
Fractional Prevalence
Initial Fraction Vaccinated

Shorthand for Key Quantities for Infectious Disease Models: Stocks

- *I* (or *Y*): Total number of infectives in population
  - This could be just one stock, or the sum of many stocks in the model (e.g. the sum of separate stocks for asymptomatic infectives and symptomatic infectives)
- N: Total size of population
  - This will typically be the sum of all the stocks of people
- S (or *X*): Number of susceptible individuals

Intuition Behind Common Terms

- I/N: The Fraction of population members (or, by assumption, contacts!) that are infective
  - Important: Simplest models assume that this is also the fraction of a given susceptible's contacts that are infective! Many sophisticated models relax this assumption
- c(I/N): Number of *infectives* that come into contact with a susceptible in a given unit time
- c(I/N)β: "Force of infection": *Likelihood a given susceptible will be infected per unit time*
  - The idea is that if a given susceptible comes into contact with c(I/N) infectives per unit time, and if each such contact gives β likelihood of transmission of infection, then that susceptible has roughly a total likelihood of c(I/N) β of getting infected per unit time (e.g. month)

A Critical Throttle on Infection Spread: Fraction Susceptible (*f*)

- The fraction susceptible (here, S/N) is a key quantity limiting the spread of infection in a population
  - Recognizing its importance, we give this name *f* to the fraction of the population that issusceptible

Key Term: Flow Rate of New Infections

- This is the key form of the equation in many infectious disease models
- Total # of susceptibles infected per unit time

  # of Susceptibles * "Likelihood" a given susceptible will be infected per unit time = S*("*Force of Infection*") $=S(c(I/N)\beta)$

  - Note that this is a term that multiplies both S and I !
    - This is much different than the purely linear terms on which we have previously focused
  - "Likelihood" is actually a likelihood per unit time (e.g. can be >1 – indicating that mean time to infection is <1)

Another Useful View of this Flow

- Recall: Total # of susceptibles infected per unit time = # of Susceptibles * "Likelihood" a given susceptible will be infected per unit time = S*(*"Force of Infection"*) = $S(c(I/N)\beta)$
- The above can also be phrased as the following:$S(c(I/N)\beta)=I(c(S/N)\beta)=I(c*f*\beta)=$
  # of Infectives * Mean # susceptibles infected per unit time by each infective
- This implies that as # of susceptibles falls=># of susceptibles surrounding each infective falls=>the rate of new infections falls ("Less fuel for the fire" leads to a smaller burning rate

Recall: The Importance of Susceptible Fraction

- Recall: Total # of susceptibles infected per unit time = # of Susceptibles * "Likelihood" a given susceptible will be infected per unit time = S*("*Force of Infection*") = $S(c(I/N)\beta)$
- The above can also be phrased as the following: $S(c(I/N)\beta)=I(c(S/N)\beta)$ = # of Infectives * *Average # susceptibles infected per unit time by each infective*
- This implies that as Fraction of susceptibles falls=>Fraction of susceptibles surrounding each infective falls=>the rate of new infections falls ("Less fuel for the fire" leads to a smaller burning rate)

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Critical Notions

- Contact rates & transmission probabilities
- Equilibria
  - Endemic
  - Disease-free
- R_0 , R_*
- Herd Immunity

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Infection

- Recall: For this model, a given infective infects $c(S/N)\beta$ others per time unit
  - This goes up as the number of susceptibles rises
- Questions
  - If the mean time a person is infective is μ, how many people does that infective infect before recovering?
  - With the same assumption, how many people would that infective infect if everyone else is susceptible?
  - Under what conditions would there be more infections after their recovery than before?

Fundamental Quantities

- We have just discovered the values of 2 famous epidemiological quantities for our model
  - Effective Reproductive Number: R_*
  - Basic Reproductive Number: R_0

Effective Reproductive Number: R_*

- Number of individuals infected by an ‘index’ infective in the current epidemological context
- Depends on
  - Contact number
  - Transmission probability
  - Length of time infected
  - # (Fraction) of Susceptibles
- Affects
  - Whether infection spreads
    - If $R_* > 1$, # of cases will rise, If $R_* < 1$, # of cases will fall
      - Alternative formulation: Largest real eigenvalue <> 0
  - Endemic Rate

Basic Reproduction Number: R_0

- Number of individuals infected by an ‘index’ infective *in an otherwise disease-free equilibrium*
  - This is just R_* at disease-free equilibrium all (other) people in the population are susceptible other than the index infective
- Depends on
  - Contact number
  - Transmission probability
  - Length of time infected
- Affects
  - Whether infection spreads
    - If R_0> 1, Epidemic Takes off, If R_0<1, Epidemic dies out
      - Alternative formulation: Largest real eigenvalue <> 0
    - Initial infection rise $\propto$ exp(t*(R0-1)/D)
  - Endemic Rate

Basic Reproductive Number R_0

- If contact patterns & infection duration remain unchanged and if fraction f of the population is **susceptible**, then mean # of individuals infected by an infective over the course of their infection is $f*R_0$
- In endemic equilibrium: Inflow=Outflow $\Rightarrow (S/N)\cdot R_0=1$
  - Every infective infects a "replacement" infective to keep equilibrium
  - Just enough of the population is susceptible to allow this replacement
  - The higher the R_0, the lower the fraction of susceptibles in equilibrium!
    - *Generally some* susceptibles remain: At some point in epidemic, susceptibles will get so low that can't spread

Our model

- Set
  - c=10 (people/month)
  - β=0.04 (4% chance of transmission per S-I contact)
  - μ=10
  - Birth and death rate= 0
  - Initial infectives=1, other 1000 susceptible
- What is R_0?
- What should we expect to see ?

Thresholds

- R_*
  - Too low # susceptibles => $R^* < 1$: # of infectives declining
  - Too high # susceptibles => $R^* > 1$: # of infectives rising
- R_0
  - $R_0>1$: Infection is introduced from outside will cause outbreak
  - $R_0<1$: "Herd immunity": infection is introduced from outside will die out (may spread to small number before disappearing, but in unsustainable way)
    - **This is what we try to achieve by control programs, vaccination, etc.**
- Outflow from susceptibles (infections) is determined by the # of Infectives

Equilibrium Behaviour

- With Births & Deaths, the system can approach an "endemic equilibrium" where the infection stays circulating in the population – but in balance
- The balance is such that (simultaneously)
  - The rate of new infections = The rate of immigration
    - Otherwise # of susceptibles would be changing!
  - The rate of new infections = the rate of recovery
    - Otherwise # of infectives would be changing!

Equilibria

- Disease free
  - No infectives in population
  - Entire population is susceptible
- Endemic
  - Steady-state equilibrium produced by spread of illness
  - Assumption is often that children get exposed when young
- The stability of the these equilibria (whether the system departs from them when perturbed) depends on the parameter values
  - For the disease-free equilibrium on R_0

Vaccination

Equilibrium Behaviour

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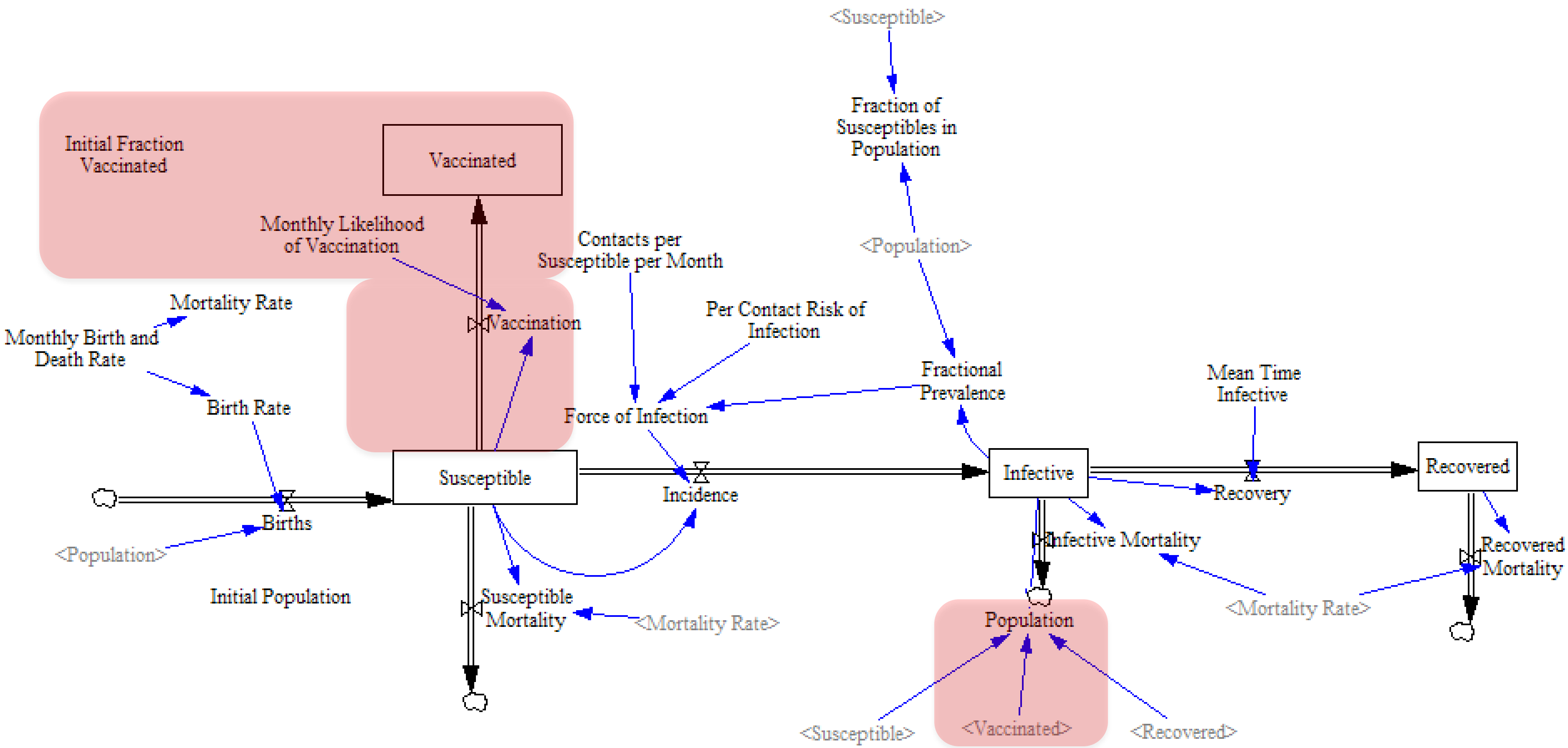
Adding Vaccination Stock

- Add a
  - "Vaccinated" stock
  - A constant called "Monthly Likelihood of Vaccination"
  - "Vaccination" flow between the "Susceptible" and "Vaccinated" stocks
    - The rate is the stock times the constant above
- Set initial population to be divided between 2 stocks
  - Susceptible
  - Vaccinated
- Incorporate "Vaccinated" in population calculation

Additional Settings

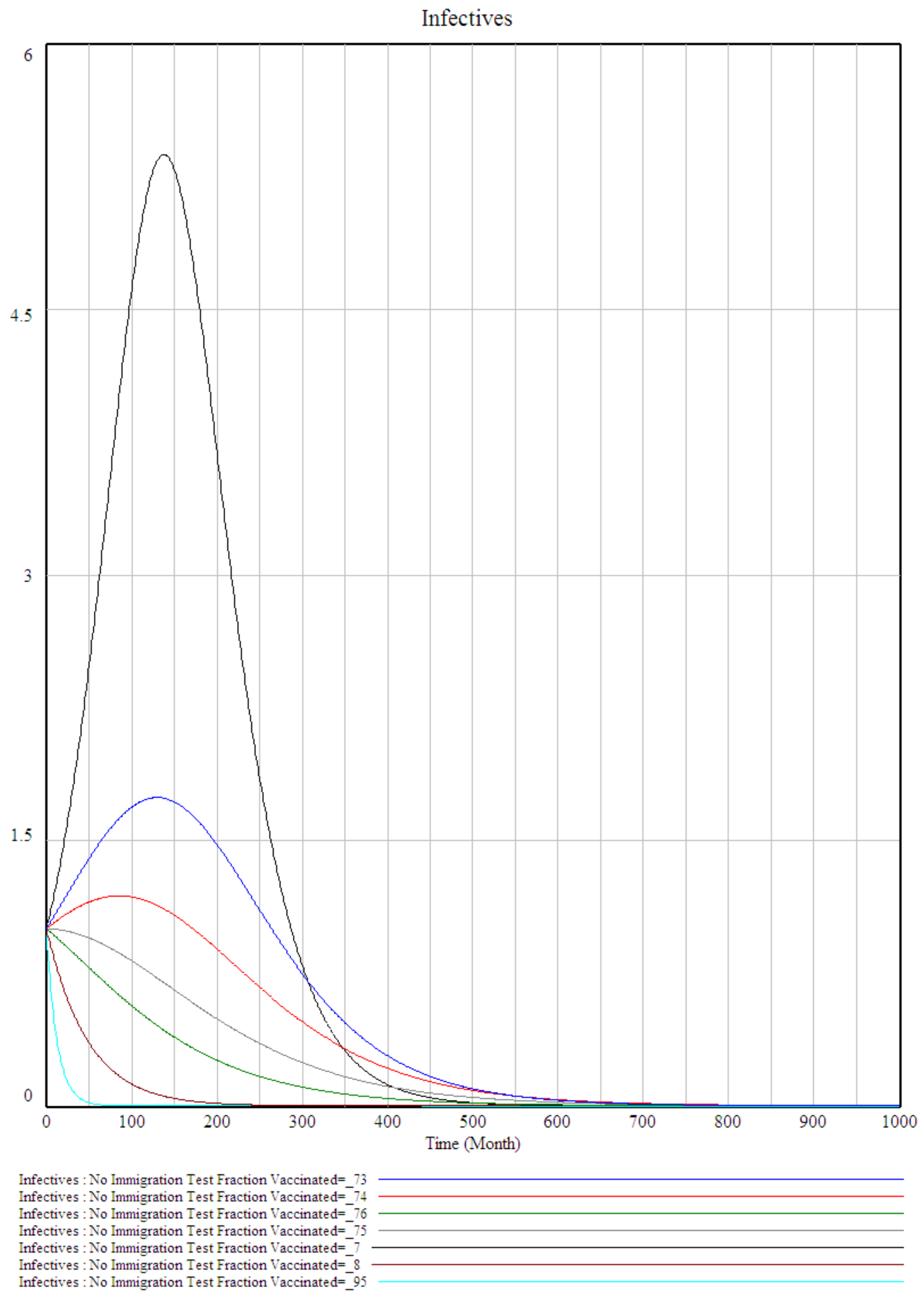
- c= 10
- Beta=.04
- Duration of infection = 10
- Birth & Death Rate=0

Adding Stock



Experiment with Different Initial Vaccinated Fractions

- Fractions = 0.25, 0.50, 0.6, 0.7, 0.8


Infectives
6
4.5
3
1.5
0
0
100
200
300
400
500
600
700
800
900
1000
Time (Month)
Infectives : No Immigration Test Fraction Vaccinated=_73
Infectives : No Immigration Test Fraction Vaccinated=_74
Infectives : No Immigration Test Fraction Vaccinated=_76
Infectives : No Immigration Test Fraction Vaccinated=_75
Infectives : No Immigration Test Fraction Vaccinated=_7
Infectives : No Immigration Test Fraction Vaccinated=_8
Infectives : No Immigration Test Fraction Vaccinated=_95

Recall: Thresholds

- R*
  - Too low # susceptibles => R* < 1: # of infectives declining
  - Too high # susceptibles => R* > 1: # of infectives rising
- Outflow from susceptibles (infections) is determined by the # of Infectives
- Delays:
  - For a while after infectives start declining, they still deplete susceptibles sufficiently for susceptibles to decline
  - For a while after infectives start rising, the # of infections is insufficient for susceptibles to decline

Effective Reproductive Number: R_*

- Number of individuals infected by an ‘index’ infective in the current epidemiological context
- Depends on
  - Contact number
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  - Recognizing its importance, we give this name *f* to the fraction of the population that issusceptible
- If contact patterns & infection duration remain unchanged and, then mean # of individuals infected by an infective over the course of their infection is $f*R_0$

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- The higher the R_0, the lower the fraction of susceptibles in equilibrium!
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Critical Immunization Threshold

- Consider an index infective arriving in a "worst case" scenario when noone else in the population is infective or recovered from the illness
  - In this case, that infective is most "efficient" in spreading
- The goal of vaccination is keep the fraction susceptible low enough that infection cannot establish itself even in this worst case
  - We do this by administering vaccines that makes a person (often temporarily) immune to infection
- We say that a population whose f is low enough that it is resistant to establishment of infection exhibits "herd immunity"

Critical Immunization Threshold

- Vaccination seeks to lower *f* such that $f*R_0<1$
- *Worst case: Suppose we have a population that is divided into immunized (vaccinated) and susceptible*
  - Let q_c be the critical fraction immunized to stop infection
  - *Then* $f=1-q_c$, $f*R_0<1 \Rightarrow (1-q_c)*R_0<1 \Rightarrow q_c>1-(1/R_0)$
- So if $R_0 = 4$ (as in our example), *q_c=0.75(i.e. 75% of population must be immunized – just as we saw!)*