Infectious Disease Models 5 –
Basic Epidemiological Quantities
and Vaccination

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Shorthand for Key Quantities for Infectious Disease Models: Stocks

- $I$ (or $Y$): Total number of infectives in population
  - This could be just one stock, or the sum of many stocks in the model (e.g. the sum of separate stocks for asymptomatic infectives and symptomatic infectives)

- $N$: Total size of population
  - This will typically be the sum of all the stocks of people

- $S$ (or $X$): Number of susceptible individuals
Intuition Behind Common Terms

• I/N: The Fraction of population members (or, by assumption, contacts!) that are infective
  – Important: Simplest models assume that this is also the fraction of a given susceptible’s contacts that are infective! Many sophisticated models relax this assumption

• c(I/N): Number of infectives that come into contact with a susceptible in a given unit time

• c(I/N)\(\beta\): “Force of infection”: Likelihood a given susceptible will be infected per unit time
  – The idea is that if a given susceptible comes into contact with \(c(I/N)\) infectives per unit time, and if each such contact gives \(\beta\) likelihood of transmission of infection, then that susceptible has roughly a total likelihood of \(c(I/N) \beta\) of getting infected per unit time (e.g. month)
A Critical Throttle on Infection Spread: Fraction Susceptible ($f$)

- The fraction susceptible (here, S/N) is a key quantity limiting the spread of infection in a population
  - Recognizing its importance, we give this name $f$ to the fraction of the population that is susceptible
Key Term: Flow Rate of New Infections

• This is the key form of the equation in many infectious disease models

• Total # of susceptibles infected per unit time
  
  # of Susceptibles * “Likelihood” a given susceptible will be infected per unit time = $S^*\left(\text{"Force of Infection"}\right)$
  
  = $S(c(I/N)\beta)$

  – Note that this is a term that multiplies both S and I!
    • This is much different than the purely linear terms on which we have previously focused

  – “Likelihood” is actually a likelihood per unit time (e.g. can be >1 – indicating that mean time to infection is <1)
Another Useful View of this Flow

• Recall: Total # of susceptibles infected per unit time = # of Susceptibles * “Likelihood” a given susceptible will be infected per unit time = $S^*\left(\text{“Force of Infection”}\right) = S(c(I/N)\beta)$

• The above can also be phrased as the following:

  $S(c(I/N)\beta)=I(c(S/N)\beta)=I(c*f*\beta)=#\ of\ Infectives\ *\ Mean\ #\ susceptibles\ infected\ per\ unit\ time\ by\ each\ infective$

• This implies that as # of susceptibles falls=># of susceptibles surrounding each infective falls=>the rate of new infections falls (“Less fuel for the fire” leads to a smaller burning rate...
Recall: The Importance of Susceptible Fraction

• Recall: Total # of susceptibles infected per unit time = # of Susceptibles * “Likelihood” a given susceptible will be infected per unit time = \( S(\text{"Force of Infection"}) = S(c(I/N)\beta) \)

• The above can also be phrased as the following:
  \[ S(c(I/N)\beta) = I(c(S/N)\beta) = \# \text{ of Infectives} \]
  \[ \times \text{Average # susceptibles infected per unit time by each infective} \]

• This implies that as Fraction of susceptibles falls=>Fraction of susceptibles surrounding each infective falls=>the rate of new infections falls
  (“Less fuel for the fire” leads to a smaller burning rate)
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Critical Notions

• Contact rates & transmission probabilities
• Equilibria
  – Endemic
  – Disease-free
• $R_0$, $R^*$
• Herd Immunity
Recall: Flow Rate of New Infections

• This is the key form of the equation in many infectious disease models

• Total # of susceptibles infected per unit time

  \[ \text{# of Susceptibles} \times \text{“Likelihood” a given susceptible will be infected per unit time} = S \times (\text{“Force of Infection”}) = S(c(I/N)\beta) \]

  – Note that this is a term that multiplies both S and I!
    • This is much different than the purely linear terms on which we have previously focused

  – “Likelihood” is actually a likelihood density, i.e.
    “likelihood per unit time” (e.g. can be >1 – indicating that mean time to infection is <1)
Recall: Another Useful View of this Flow

• Recall: Total # of susceptibles infected per unit time = # of Susceptibles * “Likelihood” a given susceptible will be infected per unit time = \( S \times \left( \frac{“Force of Infection”}{N} \right) = S \left( \frac{c(I/N) \beta}{N} \right) \)

• The above can also be phrased as the following: \( S \left( \frac{c(I/N) \beta}{N} \right) = I \left( \frac{c(S/N) \beta}{N} \right) \) = # of Infectives * Average # susceptibles infected per unit time by each infective

• This implies that as # of susceptibles falls => # of susceptibles surrounding each infective falls => the rate of new infections falls (“Less fuel for the fire” leads to a smaller burning rate
Infection

• Recall: For this model, a given infective infects $c(S/N)\beta$ others per time unit
  – This goes up as the number of susceptibles rises

• Questions
  – If the mean time a person is infective is $\mu$, how many people does that infective infect before recovering?
  – With the same assumption, how many people would that infective infect if everyone else is susceptible?
  – Under what conditions would there be more infections after their recovery than before?
Fundamental Quantities

• We have just discovered the values of 2 famous epidemiological quantities for our model
  – Effective Reproductive Number: $R_*$
  – Basic Reproductive Number: $R_0$
Effective Reproductive Number: $R^*$

- Number of individuals infected by an ‘index’ infective in the current epidemiological context
- Depends on
  - Contact number
  - Transmission probability
  - Length of time infected
  - # (Fraction) of Susceptibles
- Affects
  - Whether infection spreads
    - If $R^* > 1$, # of cases will rise, If $R^* < 1$, # of cases will fall
      - Alternative formulation: Largest real eigenvalue $<> 0$
  - Endemic Rate
Basic Reproduction Number: $R_0$

- Number of individuals infected by an ‘index’ infective *in an otherwise disease-free equilibrium*
  - This is just $R_*$ at disease-free equilibrium all (other) people in the population are susceptible other than the index infective

- Depends on
  - Contact number
  - Transmission probability
  - Length of time infected

- Affects
  - Whether infection spreads
    - If $R_0 > 1$, Epidemic Takes off, If $R_0 < 1$, Epidemic dies out
      - Alternative formulation: Largest real eigenvalue $\neq 0$
    - Initial infection rise $\propto \exp(t\times(R_0-1)/D)$
  - Endemic Rate
Basic Reproductive Number $R_0$

- If contact patterns & infection duration remain unchanged and if fraction $f$ of the population is susceptible, then mean # of individuals infected by an infective over the course of their infection is $f \times R_0$

- In endemic equilibrium: Inflow=Outflow $\Rightarrow (S/N) \times R_0 = 1$
  - Every infective infects a “replacement” infective to keep equilibrium
  - Just enough of the population is susceptible to allow this replacement
  - The higher the $R_0$, the lower the fraction of susceptibles in equilibrium!
    - Generally some susceptibles remain: At some point in epidemic, susceptibles will get so low that can’t spread
Our model

• Set
  – \( c = 10 \) (people/month)
  – \( \beta = 0.04 \) (4% chance of transmission per S-I contact)
  – \( \mu = 10 \)
  – Birth and death rate = 0
  – Initial infectives = 1, other 1000 susceptible

• What is \( R_0 \)?

• What should we expect to see?
Thresholds

• $R_*$
  – Too low # susceptibles $\Rightarrow R^* < 1$: # of infectives declining
  – Too high # susceptibles $\Rightarrow R^* > 1$: # of infectives rising

• $R_0$
  – $R_0 > 1$: Infection is introduced from outside will cause outbreak
  – $R_0 < 1$: “Herd immunity”: infection is introduced from outside will die out (may spread to small number before disappearing, but in unsustainable way)
    • This is what we try to achieve by control programs, vaccination, etc.

• Outflow from susceptibles (infections) is determined by the # of Infectives
Equilibrium Behaviour

- With Births & Deaths, the system can approach an “endemic equilibrium” where the infection stays circulating in the population – but in balance

- The balance is such that (simultaneously)
  - The rate of new infections = The rate of immigration
    - Otherwise # of susceptibles would be changing!
  - The rate of new infections = the rate of recovery
    - Otherwise # of infectives would be changing!
Equilibria

• Disease free
  – No infectives in population
  – Entire population is susceptible

• Endemic
  – Steady-state equilibrium produced by spread of illness
  – Assumption is often that children get exposed when young

• The stability of the these equilibria (whether the system departs from them when perturbed) depends on the parameter values
  – For the disease-free equilibrium on $R_0$
Vaccination
Equilibrium Behaviour

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Adding Vaccination Stock

• Add a
  – “Vaccinated” stock
  – A constant called “Monthly Likelihood of Vaccination”
  – “Vaccination” flow between the “Susceptible” and “Vaccinated” stocks
    • The rate is the stock times the constant above

• Set initial population to be divided between 2 stocks
  – Susceptible
  – Vaccinated

• Incorporate “Vaccinated” in population calculation
Additional Settings

- $c = 10$
- Beta = 0.04
- Duration of infection = 10
- Birth & Death Rate = 0
Adding Stock
Experiment with Different Initial Vaccinated Fractions

- Fractions = 0.25, 0.50, 0.6, 0.7, 0.8
Recall: Thresholds

- $R^*$
  - Too low # susceptibles $\Rightarrow R^* < 1$: # of infectives declining
  - Too high # susceptibles $\Rightarrow R^* > 1$: # of infectives rising

- Outflow from susceptibles (infections) is determined by the # of Infectives

- Delays:
  - For a while after infectives start declining, they still deplete susceptibles sufficiently for susceptibles to decline
  - For a while after infectives start rising, the # of infections is insufficient for susceptibles to decline
Effective Reproductive Number: $R^*$

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  - # (Fraction) of Susceptibles

- Affects
  - Whether infection spreads
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Recall: A Critical Throttle on Infection Spread: Fraction Susceptible \((f)\)

• The fraction susceptible (here, \(S/N\)) is a key quantity limiting the spread of infection in a population
  – Recognizing its importance, we give this name \(f\) to the fraction of the population that is susceptible

• If contact patterns & infection duration remain unchanged and, then mean # of individuals infected by an infective over the course of their infection is \(f \times R_0\)
Recall: Endemic Equilibrium

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  - Every infective infects a “replacement” infective to keep equilibrium
  - Just enough of the population is susceptible to allow this replacement
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Critical Immunization Threshold

• Consider an index infective arriving in a “worst case” scenario when no one else in the population is infective or recovered from the illness
  – In this case, that infective is most “efficient” in spreading

• The goal of vaccination is keep the fraction susceptible low enough that infection cannot establish itself even in this worst case
  – We do this by administering vaccines that makes a person (often temporarily) immune to infection

• We say that a population whose $f$ is low enough that it is resistant to establishment of infection exhibits “herd immunity”
Critical Immunization Threshold

• Vaccination seeks to lower $f$ such that $f*R_0<1$
• Worst case: Suppose we have a population that is divided into immunized (vaccinated) and susceptible
  – Let $q_c$ be the critical fraction immunized to stop infection
  – Then $f=1-q_c$, $f*R_0<1 \Rightarrow (1-q_c)*R_0<1 \Rightarrow q_c>1-(1/R_0)$
• So if $R_0 = 4$ (as in our example), $q_c=0.75$ (i.e. 75% of population must be immunized – just as we saw!)