First Order Delays

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CMPT 858
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Simple First-Order Decay
(Create this in Vensim!)

Use Initial Value: 1000

Use Formula: People with Virulent Infection/Mean time until Death
First Order Delays and Transition Processes

• We can think of first order delays as representing a deterministic approximation to a population experiencing a memoryless (Poisson) stochastic transition process.

• The system is “memoryless” because the chance of e.g. a person leaving in the next unit of time is independent of how long they’ve been there!

• The probability distribution of residence time in the stock is exponentially distributed.
Dynamics of Stock?
Dynamics of (Rate of) Death Flow?
• Alpha is per-time-unit likelihood of death
  – Chance of death over small $\Delta t$ is $\alpha \Delta t$
  – If $x$ people are at risk, # dying over $\Delta t$ is $x^*(\text{Likelihood of death over } \Delta t)=x(\alpha \Delta t)= x\alpha \Delta t$
  – When people die, they flow out => cause a negative change in $x$.
  – We denote the change in $x$ over the time $\Delta t$ as $\Delta x$
    Thus $\Delta x= -x\alpha \Delta t$
• As $x$ is depleted (becomes smaller), $\Delta x$ becomes smaller as well (for a fixed $\Delta t$)
Approximate Dynamics

Suppose

\[ x(0) = 1000 \]
\[ \Delta t = 1 \]
\[ \alpha = 0.2 \]

<table>
<thead>
<tr>
<th>Time (t)</th>
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<td>327.68</td>
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Flow Rate Dynamics

• The total change in $x$ over the time $\Delta t$ is $\Delta x$
  
  Thus $\Delta x = -x \alpha \Delta t$

  – This might be 10 people over a timeframe of .1 year (~36.5 days)

• The rate of change of $x$ over given time $\Delta t$ is $\Delta x/\Delta t$
  
  This is just the sum of all of the flows
  
  For system, $\Delta x/\Delta t = (-x \alpha \Delta t)/\Delta t = -x \alpha = -\text{People} \times \text{DeathRate}$

  Because $x$ (People) changes, this flow rate changes over the course of the time we are observing

  Suppose time is measured in years; then for our example above, $\Delta x/\Delta t = 10/.1 = 100$ people per year
# Approximate Dynamics: Net Flow Rate

Reminder: Suppose
Initial $x=1000$
$\Delta t=1$
$\alpha=0.2$

<table>
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Why is This Approximate?

• Our previous graphs used a value of $\Delta t=1$

• In calculating the change ($\Delta x$) from $t$ to $t+\Delta t$ (here, $t+1$), we are assuming that the flow rate (people/year) stays constant in that time
  
  – Recall: In general, this flow rate will be determined by the value of stocks
  
  – So in assuming that the flow rate remains constant, we were basically assuming that the values of the stocks stay constant over time $\Delta t$

• For our system, given that the value of the stock $x$ (People) declines by around 20% per time unit, this is not a very good assumption!
How Can We Reduce the Error?
Try a Smaller $\Delta t$

- Let’s work forward for $\frac{1}{2}$ of a year at a time instead of for a full year
  
  $x(0)=1000$
  
  $\Delta t=.5$
  
  $\alpha=.1$

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<td>-86.1</td>
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Approximate Dynamics: Net Flow Rate

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Vensim has a Step Size!
(Set via Model Menu/Settings Item)
Impact of Step Size on Simulation
• Alpha is per-time-unit likelihood of death
  – Chance of death over small $dt$ is $\alpha dt$
  – If $x$ people are at risk, # dying over $dt$ is $x^\ast$(Likelihood of death over $\Delta t$)=$x(\alpha dt)$= $x\alpha dt$
  – When people die, they flow out => cause a negative change in $x$.
  – We denote the change in $x$ over the time $dt$ as $\Delta x$
    Thus $dx= -x\alpha dt$

• As $x$ is depleted (becomes smaller), $dx$ becomes smaller as well (for a fixed $dt$)
Flow Rate Dynamics: Continuous

• The total change in x over the time dt is dx
  
  Thus \( dx = -x \alpha dt \)
  
  – This might be 10 people over a timeframe of .1 year (~36.5 days)

• The rate of change of x over given time dt is \( \frac{dx}{dt} \)
  
  This is just the sum of all of the flows!

  For system, \( \frac{dx}{dt} = (-x \alpha dt)/dt = - \alpha x = \text{People*DeathRate} \)

  Because x (People) changes, this flow rate changes over the course of the time we are observing

• We will sometimes write \( \frac{dx}{dt} \) as \( \dot{x} \)
  
  \( \frac{dx}{dt} = \dot{x} = -\alpha x \)
The Concept of “Analytic” Solutions

• The model structure describes system behaviour implicitly
  – This indicates how short term changes (flows) depends on the state of the system
  – This does not explicitly state how the system evolves

• Analytic (“closed form”, “exact”) solutions describe system behaviour as an explicit function of time
  – E.g. $a+b*t+c*t^2$, $a +b*t$, $a*sin(t)$, $e^{at}$

• For many systems we will be dealing with (nonlinear systems), an analytic solution is simply not derivable
  – Even when an analytic solution is possible, it is often most convenient to deal with simulations for most needs
An Exact Solution to Our Problem

• The state equation formulation of our system is

\[
\frac{dx}{dt} = \dot{x} = -\alpha x
\]

This is a linear differential equation with constant coefficients – a type of system that can be solved exactly.
Solution Procedure

\[
\frac{dx}{dt} = -\alpha x
\]

- Suppose we start \( x \) at time 0 with initial value \( x(0) \), and we want to find the value of \( x \) at time \( T \).
- Assuming that \( x \) does not start at 0, it will never reach exactly 0, so we can divide the left side by it, and multiply the right side by \( dt \).

\[
\frac{dx}{x} = -\alpha dt
\]

- Integrating both sides

\[
\int_{t=0}^{t=T} \frac{dx}{x} = \int_{t=0}^{t=T} -\alpha dt
\]
Completion of Derivation

\[
\int_{t=0}^{t=T} \frac{dx}{x} = \int_{t=0}^{t=T} -\alpha \, dt = -\alpha \int_{t=0}^{t=T} \, dt
\]

\[\ln x\big|_{t=0}^{t=T} = -\alpha t\big|_{t=0}^{t=T}\]

\[\ln x(T) - \ln x(0) = -\alpha T\]

\[\ln x(T) = \ln x(0) - \alpha T\]

\[x(T) = e^{\ln x(0) - \alpha T} = e^{\ln x(0)} e^{-\alpha T} = x(0)e^{-\alpha T}\]

So the stock \(x\) declines as a negative exponential in time \(T\) i.e. # of people remaining in the stock goes down exponentially w/time
Fraction of Original People Still in Stock or Who have Left

• Assuming no inflows, the fraction of people still in the stock at time $T$ is just

\[
\frac{\text{(\# of people in the stock at time } T\text{)}/\text{(initial \# of people in the stock)}}{x(T)} = x(0)e^{-\alpha T} = e^{-\alpha T}
\]

• Given that people either stay in the stock or leave, the fraction that have left by time $T$= 

\[
1 - \frac{x(T)}{x(0)} = 1 - e^{-\alpha T}
\]
At Time=1

• At time t=1, we have a fraction \( e^{-\alpha \cdot 1} = e^{-\alpha} \) in the stock, and a fraction \( 1 - e^{-\alpha} \) who have left

• Note: By its Taylor Expansion

\[
e^{-\alpha t} = \sum_{i=0}^{\infty} \frac{(-\alpha t)^i}{i!} = 1 + (-\alpha t) + \frac{(-\alpha t)^2}{2!} + \frac{(-\alpha t)^3}{3!} + \ldots
\]

= \( 1 - \alpha t + \frac{(\alpha t)^2}{2} + \ldots \)

• For small \( \alpha t \), the higher order terms are very small, and this will be approximately \( 1 - \alpha t \)

• So by time 1 for small \( \alpha \), approx \( 1 - \alpha \) will remain after, and a fraction of \( \alpha \) will have departed
Mean Time to Transition

• People are leaving via the flow

• Suppose we wish to determine the mean (average) time for a given person in the stock to leave

• Recall: A mean for a continuous probability distribution \( p(t) \) is given by

\[
E[q(a)] = \sum_{a \in \{\text{Possible values of } a\}} aq(a)
\]

• Since \( p(t)dt \) is the probability that will leave between \( t \) and \( t+dt \), this is just the continuous version of
Mean Time to Leave

- \( p(t)dt \) here is the likelihood of a person leaving exactly between time \( t \) & \( dt+t \)
  - We start the simulation at \( t=0 \), so \( p(t)=0 \) for \( t<0 \)
  - For \( t>0 \), \( P(\text{leaving exactly between time } t \text{ and } dt+t)=P(\text{leaving exactly between time } t \text{ and } t+dt|\text{Still have not left by time } t)P(\text{Still have not left by time } t) \)

For \( T>0 \), \( P(\text{Still have not left by time } t)=e^{-\alpha T} \)

For \( P(\text{leaving exactly between time } t \text{ and } t+dt|\text{Still have not left by time } t) \)
  Recall: For us, probability of leaving in a time \( dt \) always=\( \alpha dt \)
  Thus \( P(\text{leaving exactly between time } t \text{ and } t+dt|\text{Still have not left by time } t)= \alpha dt \)

\[ P(t)dt=P(\text{leaving exactly between time } t \text{ and } dt+t)= \left(e^{-\alpha T}\right)(\alpha dt)=\alpha e^{-\alpha T}dt \]
Derivation of Mean

- $P(t) dt = P(\text{leaving exactly between time } t \& dt + t) = \left(e^{-\alpha T}\right)(\alpha dt) = \alpha e^{-\alpha T} dt$

- Now that we have found the function $p(t)$, we must do the integral $\int_{t=-\infty}^{t=\infty} tp(t)dt$ to derive the mean

- Here $E[p(t)] = \int_{t=-\infty}^{t=\infty} tp(t)dt = \int_{t=0}^{t=\infty} tp(t)dt = \int_{t=0}^{t=\infty} t\alpha e^{-\alpha T} dt$
  $= \alpha \int_{t=0}^{t=\infty} te^{-\alpha T} dt$
Recall: Integration by Parts

- We have \[ E[p(t)] = \alpha \int_{t=0}^{t=\infty} te^{-\alpha T} \, dt = \alpha \left( \int_{t=0}^{t=\infty} te^{-\alpha T} \, dt \right) \]
- To solve the term in brackets, we will use integration by parts
- Integration by parts exploits the following:

\[ \frac{d(uv)}{dt} = u \frac{dv}{dt} + v \frac{du}{dt} \]

\[ d(uv) = u \, dv + v \, du \]

\[ \int d(uv) = \int u \, dv + \int v \, du \]

\[ uv = \int u \, dv + \int v \, du \]

and thus

\[ \int u \, dv = uv - \int v \, du \]
Recall: Integration by Parts

- To solve $\int_{t=0}^{t=\infty} te^{-\alpha t} \, dt$ we will use integration by parts
  
  \[ u = t \implies du = \frac{dt}{dt} = \frac{1}{1} \, dt = dt \]

- Here

  \[ dv = e^{-\alpha t} \, dt \implies v = \int e^{-\alpha t} \, dt = \frac{-e^{-\alpha t}}{\alpha} \]

- From the previous page, we know

\[
\begin{align*}
\int_{t=0}^{t=\infty} te^{-\alpha t} \, dt &= \int_{t=0}^{t=\infty} udv = uv - \int_{t=0}^{t=\infty} vdu = \left( t \frac{-e^{-\alpha t}}{\alpha} \right) \bigg|_{t=0}^{t=\infty} - \int_{t=0}^{t=\infty} \left( \frac{-e^{-\alpha t}}{\alpha} \right) \, dt \\
&= \left( \frac{-te^{-\alpha T}}{\alpha} \right) \bigg|_{t=0}^{t=\infty} + \frac{1}{\alpha} \int_{t=0}^{t=\infty} e^{-\alpha t} \, dt = (0 - 0) + \frac{1}{\alpha} \left( \frac{-1}{\alpha} e^{-\alpha t} \right) \bigg|_{t=0}^{t=\infty} = \\
&= \frac{1}{\alpha} \left( 0 - \frac{-1}{\alpha} \right) = \frac{1}{\alpha^2}
\end{align*}
\]
Thus

- The mean time (the \textit{delay associated with a first order delay}) is thus given by

\[
E[p(t)] = \alpha \int_{t=0}^{t=\infty} te^{-\alpha T} dt = \alpha \left( \int_{t=0}^{t=\infty} te^{-\alpha T} dt \right)
\]

\[
= \alpha \left( \frac{1}{\alpha^2} \right) = \frac{1}{\alpha}
\]

- So e.g. if we have an annualized rate of diabetes incident, the mean time to develop diabetes (independent of other risks) is just the reciprocal of that rate (i.e. 1 over that rate)
Computer Exercise: Simulating a First Order Delay

• Create a first order delay
• Feed in a “step function” that rises suddenly at time 10.
• How does the output from the stock change over time?
Competing Risks

- Suppose we have another outflow from the stock. How does that change our mean time of proceeding specifically down flow 1 (here, developing diabetes)?
Competing Risks Stock Trajectory

Solution Procedure

\[
\frac{dx}{dt} = -\alpha x - \beta x = -(\alpha + \beta) x
\]

• Suppose we start \( x \) at time 0 with initial value \( x(0) \), and we want to find the value of \( x \) at time \( T \).

• This is just like our previous differential equation, except that “\( \alpha \)” has been replaced by “\( (\alpha+\beta) \)”
  
  – The solution must therefore be the same as before, with the appropriate replacement

  – Thus

\[
x(T) = x(0)e^{-(\alpha+\beta)T}
\]
Mean Time to Leave: Competing Risks

- $p(t)dt$ here is the likelihood of a person leaving via flow 1 (e.g. developing T2DM) exactly between time $t$ & $dt+t$
  - We start the simulation at $t=0$, so $p(t)=0$ for $t<0$
  - For $t>0$, $P(\text{leaving on flow 1 exactly between time } t \& dt+t)=P(\text{leaving on flow 1 exactly between time } t \& t+dt|\text{Still have not left by time } t)P(\text{Still have not left by time } t)$

For $T>0$, $P(\text{Still have not left by time } T)= e^{-(\alpha+\beta)T}$

For $P(\text{leaving exactly between time } t \text{ and } t+dt|\text{Still have not left by time } t)$

Recall: For us, probability of leaving in a time $dt$ always=$\alpha dt$

Thus $P(\text{leaving exactly between time } t \text{ and } t+dt|\text{Still have not left by time } t)= \alpha dt$

$P(t)dt=P(\text{leaving exact b.t. time } t \& dt+t) = \alpha e^{-(\alpha+\beta)T} dt$
Mean Time to Transition via Flow 1: Competing Risks

- By the same procedure as before, we have
  \[ E[p(t)] = \alpha \int_{t=0}^{t=\infty} te^{-(\alpha+\beta)t} \, dt \]

- Using the formula we derived for the integral expression, we have
  \[ E[p(t)] = \frac{\alpha}{(\alpha + \beta)^2} \]

- Note that this correctly approaches the single-flow case as \( \beta \to 0 \)
Equilibrium Value of a First-Order Delay

• Suppose we have flow of rate $i$ into a stock with a first-order delay out
  – This could be from just a single flow, or many flows
• The value of the stock will approach an equilibrium where inflow=outflow
Equilibrium Value of 1\textsuperscript{st} Order Delay

• Recall: Outflow rate for 1\textsuperscript{st} order delay=$\alpha x$
  – Note that this depends on the value of the stock!
• Inflow rate=$i$
• At equilibrium, the level of the stock must be such that inflow=outflow
  – For our case, we have
    \[ \alpha x = i \]
    Thus \( x = i/\alpha \)

The lower the chance of leaving per time unit (or the longer the delay), the larger the equilibrium value of the stock must be to make outflow=inflow
Computer Exercise: Simulating a First Order Delay

• Create a first order delay
• Feed in a “step function” that rises suddenly from 0 to 20 at time 10
  Use formula if then else(Time > 10, 20, 0)
• Questions to ponder
  – How does the output from the stock change over time?
  – How does the equilibrium value of the stock vary with chance of proceeding (alpha)?
First Order Delays in Action: Simple SIT Model

Mean Contacts Per Capita
Per infected contact infection rate
Initial Population

Mean Infectious Contacts Per Susceptible
Per Susceptible Incidence Rate
Prevalence
Recovery Delay

Total Population
New Recovery

S
New infections
Newly Susceptible

I
New infections

T
Newly Susceptible
Immunity loss Delay

S
Cumulative Illnesses

New Illness
First Order Delays in Action: Simple SIT Model
Recall: Simple First-Order Decay

Use Initial Value: 1000

Use Formula: People with Virulent Infection / Mean time until Death
First-Order Decay (Variant of Last Time)

Use Initial Value: 1000

Use Formula: People with Virulent Infection * Per Month Likelihood of Death

Recall: How does this relate to the mean time until death?
People in Stock

People with Virulent Infection

Time (Month)

People with Virulent Infection : Baseline
Flow Rate of Deaths

Deaths

Time (Month)

Deaths : Baseline
Cumulative Deaths

Cumulative Deaths

Cumulative Deaths : Baseline
Closeup

Why this gap?
50% per Month Risk of Deaths

Cumulative Deaths

Why this gap?
Answer: The “Gap” is Present Because not all 1000 people are at risk for a month!

- The value of the stock is declining over the first month
- The rate of death indicates that 20% of the population will die per month
- While we may have been expecting 200 people (20% of the 1000) to die, this (erroneously) assumes that all 1000 were at risk for the entire month
  - In fact, because the stock was declining, there were considerably fewer people at risk, meaning that we have fewer deaths
- If we had maintained 1000 people in the stock for the 1st month, 1000 people would have died!
Recall: First Order Delay

Use Formula: \( \text{People (x)} \times \text{Annual Risk of Death (alpha)} \)

Use Initial Value: 1000

Use Value: 0

Use Value: 0.05
Questions

• What is behaviour of stock x?
• What is the mean time until people die?
• Suppose we had a constant inflow – what is the behaviour then?
Answers

• Behaviour Of Stock

• Mean Time Until Death
  
  Recall that if coefficient of first order delay is $\alpha$, then mean time is $1/\alpha$ (Here, $1/0.05 = 20$ years)
Equilibrium Value of a First-OrderDelay

• Suppose we have flow of rate $i$ into a stock with a first-order delay out
  – This could be from just a single flow, or many flows

• The value of the stock will approach an equilibrium where inflow=outflow
Equilibrium Value of 1st Order Delay

• Recall: Outflow rate for 1st order delay=$\alpha x$
  – Note that this depends on the value of the stock!

• Inflow rate=$i$

• At equilibrium, the level of the stock must be such that inflow=outflow
  – For our case, we have

$$\alpha x = i$$

Thus $x = i / \alpha$

(equivalently, $x = i * \text{Mean time to Transition}$)

The lower the chance of leaving per time unit (or the longer the delay), the larger the equilibrium value of the stock must be to make outflow=inflow
Scenarios for First Order Delay: Variation in Inflow Rates

- For different immigration (inflows) *(what do you expect?)*
  - Inflow=10
  - Inflow=20
  - Inflow=50
  - Inflow=100
  - Why do you see this “goal seeking” pattern?
  - What is the “goal” being sought?
Behaviour of Stock for Different Inflows

Why do we see this behaviour?
Why do we see this behaviour? Imbalance (gap) causes change to stock (rise or fall) ⇒ change to outflow to lower gap until outflow=inflow
Goal Seeking Behaviour

• The goal seeking behaviour is associated with a negative feedback loop
  – The larger the population in the stock, the more people die per year

• If we have more people coming in than are going out per year, the stock (and, hence, outflow!) rises until the point where inflow=outflows

• If we have fewer people coming in than are going out per year, the stock declines (& outflow) declines until the point where inflow=outflows
As a Causal Loop Diagram

Rate of Outflow

Rate of Inflow

Surplus of Inflow beyond Outflow

Stock

What does this tell us about how the system would respond to a sudden change in immigration?
Response to a Change

- Feed in an immigration “step function” that rises suddenly from 0 to 20 at time 50

- Set the Initial Value of Stock to 0

- How does the stock change over time?
Create a Custom Graph & Display it as an Input-Output Object

- Editing
Create Input-Output Object (for Synthesim)
Stock Starting Empty

Flow Rates

Inflow and Outflow

Immigration: Step Function 0 to 20 Initial Stock Empty
Deaths: Step Function 0 to 20 Initial Stock Empty

How would this change with alpha?
How would this change with alpha?
For Different Values of (1/) Alpha Flow Rates (Outflow Rises until = Inflow)

This is for the flows. What do stocks do?
For Different Values of (1/) Alpha

Value of Stocks

People (x)

Why do we see this behaviour?  A longer time delay (or smaller chance of leaving per unit time) requires $x$ to be larger to make outflow=inflow.
Outflows as Delayed Version of Inputs

- Inflow and Outflow graphs with time (Year) on the x-axis and value on the y-axis.
- Immigration: Step Functions 2 yr delay
- Deaths: Step Functions 2 yr delay
- Immigration: Step Functions 5 yr delay
- Deaths: Step Functions 5 yr delay

- Inflow and Outflow graphs with time (Year) on the x-axis and value on the y-axis.
- Immigration: Step Functions 10 yr delay
- Deaths: Step Functions 10 yr delay
What if stock doesn’t start empty?
Decays at first (no inflow) & then output responds with delayed version of input
Classic Feedbacks

Susceptibles

Contacts of Susceptibles with Infectives

New Infections

Infectives
Dynamics

State variables over time

S : Alternative 30 HC Workers Exogenous Recovery Delay
I : Alternative 30 HC Workers Exogenous Recovery Delay
R : Alternative 30 HC Workers Exogenous Recovery Delay
Broadening the Model Boundaries: Endogenous Recovery Delay

Mean Contacts Per Capita
Mean Infectious Contacts Per Susceptible
Per Susceptible Incidence Rate
Fractions Prevalence
Recovery Delay
Healthcare Workers
Time Until Seek Treatment
Staff Time per Patient
Initial Population
New infections
Newly Susceptible
Immunity loss Delay

S I R

Per infected contact infection rate
Total Population
Per Capita
Mean Infectious Contacts Per Susceptible
Per Capita
Staff Time per Patient
Healthcare Workers
Per Susceptible Incidence Rate
Fractional Prevalence
Recovery Delay
Healthcare Workers
Time Until Seek Treatment
Staff Time per Patient
Initial Population
New infections
Newly Susceptible
Immunity loss Delay
Broadening the Model Boundaries: Endogenous Recovery Delay

- Susceptibles
  - Contacts of Susceptibles with Infectives
    + New Infections
    + Infectives
      + People Presenting for Treatment
        + Waiting Times
          + Health Care Staff

- New Infections
- People Presenting for Treatment
- Waiting Times
- Health Care Staff
- Susceptibles
A Different Behaviour Mode

Prevalence, Infectious

Time (Day)

Prevalence : Baseline 30 HC Workers
I : Baseline 30 HC Workers

Day

Person

200,000
100,000
0

Person

0.5
0
Structure as Shaping Behaviour

• System structure is defined by
  – Stocks
  – Flows
  – Connections between them

• Nonlinearity: The behaviour of the whole is more than the sum of the behaviour of the parts
  – “Emergent” behaviour would not be anticipated from simple behaviour of each piece in turn

• Stock and flow structure (including feedbacks) of a system determines the qualitative behaviour modes that the system can take on