First Order Delays

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CMPT 858

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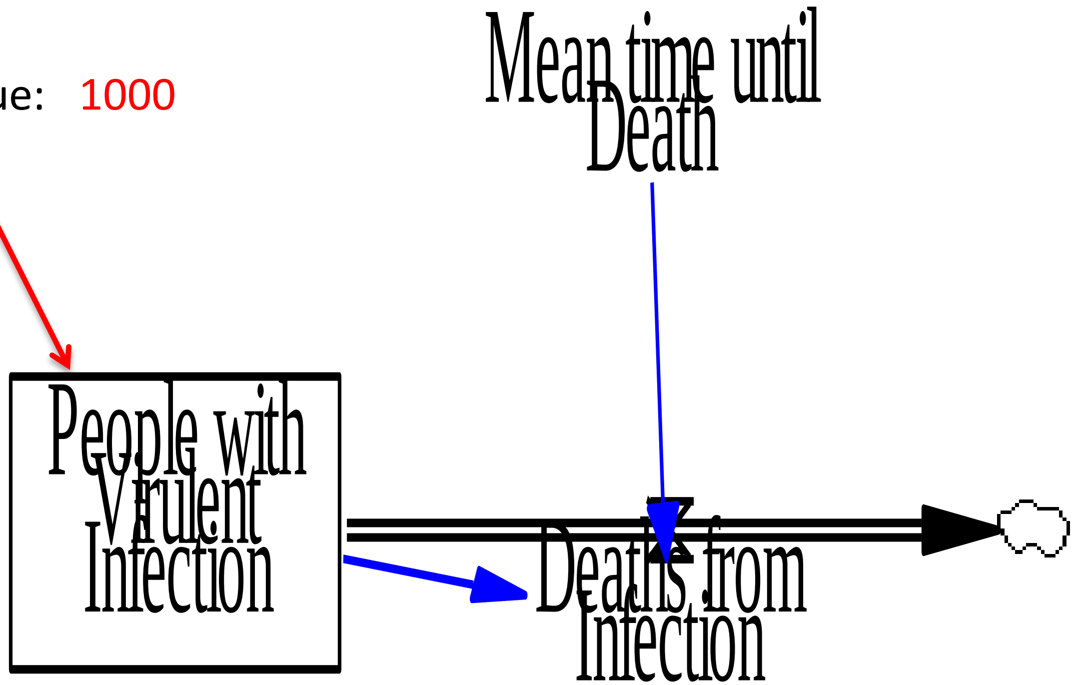
Simple First-Order Decay (Create this in Vensim!)

Use Initial Value: 1000

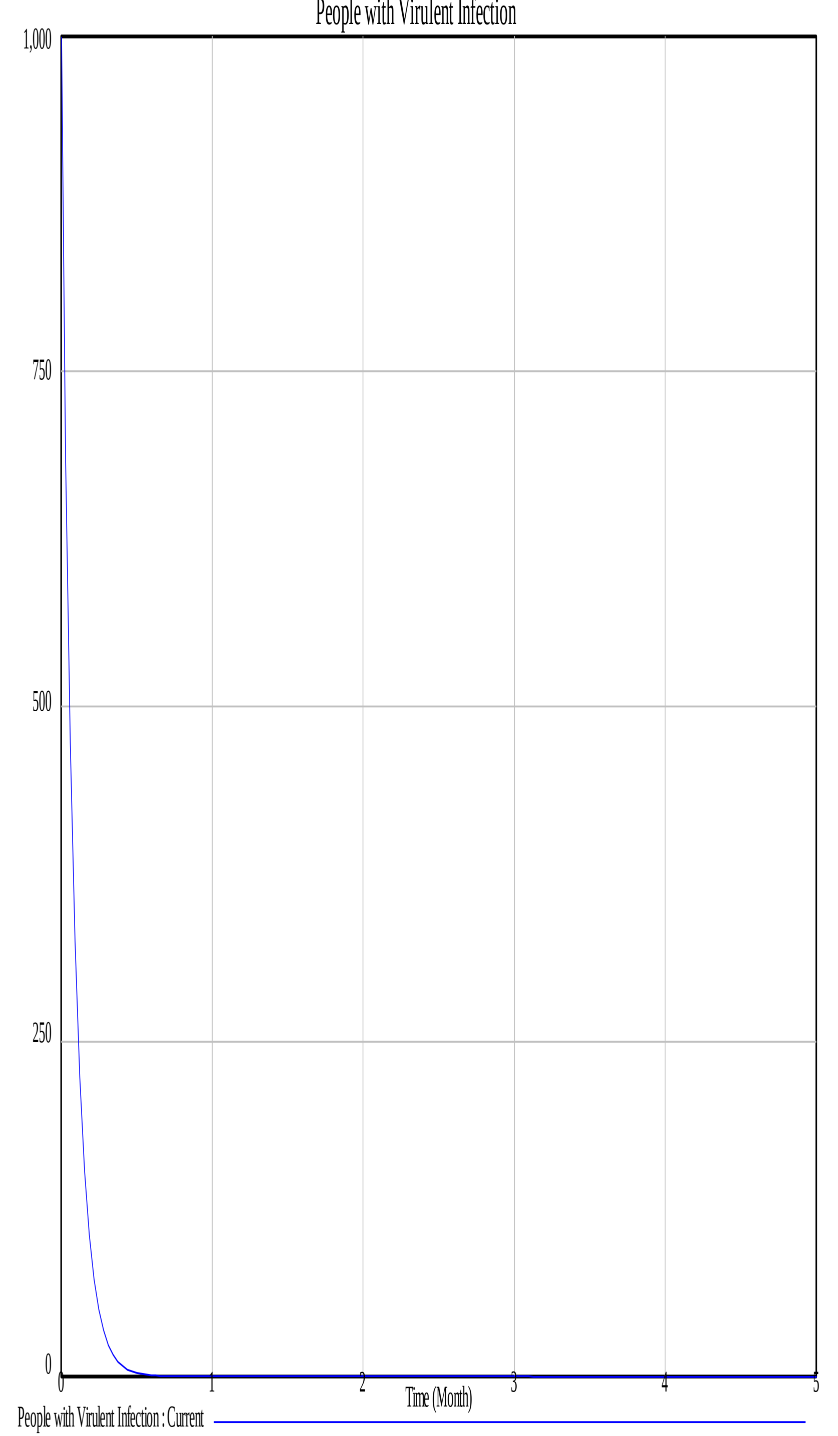


Use Formula: People with Virulent Infection/Mean time until Death

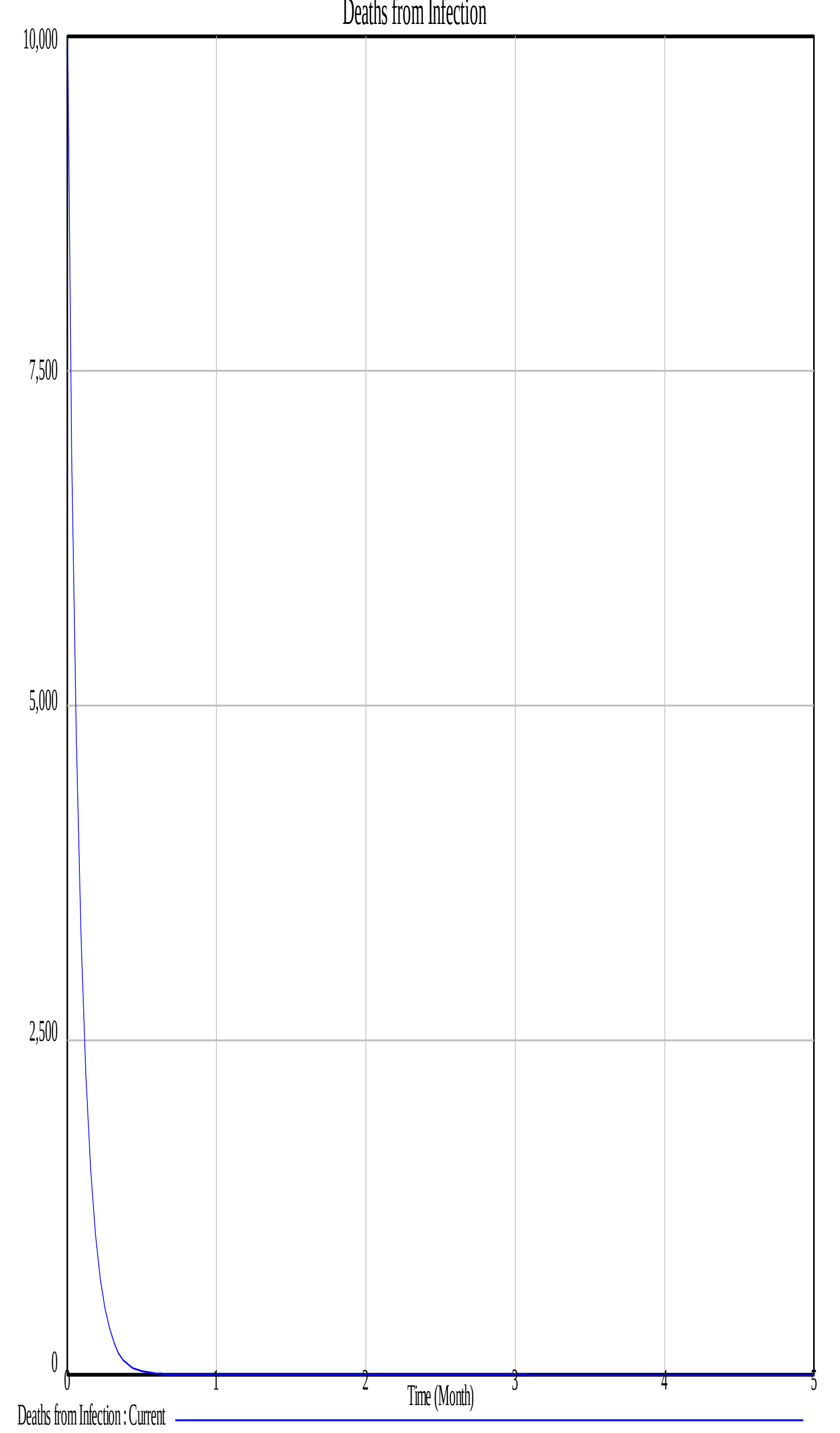
First Order Delays and Transition Processes

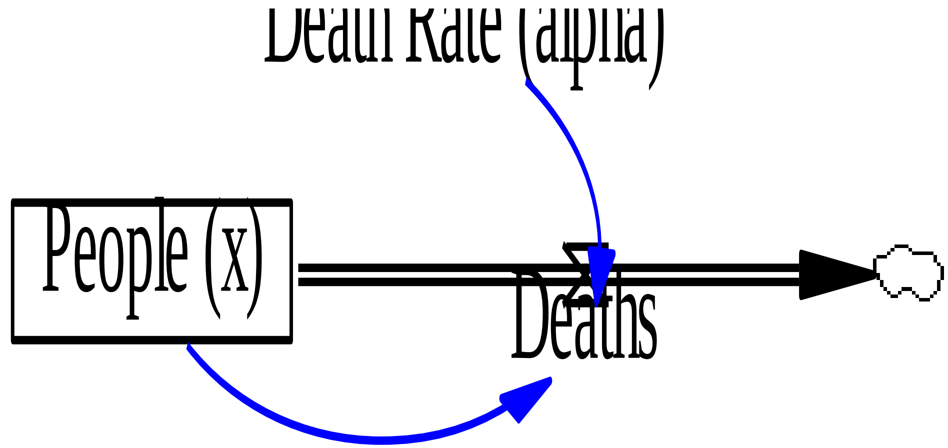
- We can think of first order delays as representing a deterministic approximation to a population experiencing a memoryless (Poisson) stochastic transition process
- The system is "memoryless" because the chance of e.g. a person leaving in the next unit of time is independent of how long they've been there!
- The probability distribution of residence time in the stock is exponentially distributed

Dynamics of Stock?



Dynamics of (Rate of) Death Flow?





- Alpha is per-time-unit likelihood of death
  - Chance of death over small Δt is $\alpha\Delta t$
  - If x people are at risk, # dying over Δt is x*(Likelihood of death over Δt)=$x(\alpha\Delta t)$= $x\alpha\Delta t$
  - When people die, they flow out => cause a negative change in x.
  - We denote the change in x over the time Δt as Δx

    Thus Δx= $-x\alpha\Delta t$
- As x is depleted (becomes smaller), Δx becomes smaller as well (for a fixed Δt)

Approximate Dynamics

Suspose

Suppose

$x(0)=1000$

$\Delta t=1$

$\alpha=.2$

Time (t)	Stock Value (x)	Change in stock (Δx) -x*Alpha*DeltaT
0	1000	-200
1	800	-160
2	640	-128
3	512	-102.4
4	409.6	-81.92
5	327.68	-65.536

Flow Rate Dynamics

- The total change in x over the time Δt is Δx

  Thus $\Delta x = -x\alpha\Delta t$

  – This might be 10 people over a timeframe of .1 year (~36.5 days)

- The *rate of change* of x over given time Δt is $\Delta x/\Delta t$

This is just the sum of all of the flows

For system, $\Delta x/\Delta t = (-x\alpha\Delta t)/\Delta t = -x\alpha = -$People*DeathRate

Because x (People) changes, this flow rate changes over the course of the time we are observing

Suppose time is measured in years; then for our example above, $\Delta x/\Delta t = 10/.1 = 100$ people per year

Approximate Dynamics: Net Flow Rate

Reminder: Suppose

Initial x=1000

$\Delta t=1$

$\alpha=.2$

Time (t)	Stock Value (x)	Change in stock (Δx) -x*Alpha*DeltaT	Net Flow Rate=$\Delta x/\Delta t$ Here, $\Delta t=1$, so $\Delta x/\Delta t=\Delta x/1=\Delta x$
0	1000	-200	-200
1	800	-160	-160
2	640	-128	-128
3	512	-102.4	-102.4
4	409.6	-81.92	-81.92
5	327.68	-65.536	-65.536

Why is This Approximate?

- Our previous graphs used a value of Δt=1
- In calculating the change (Δx) from t to t+Δt (here, t+1), we are assuming that the flow rate (people/year) *stays constant in that time*
  - Recall: In general, this flow rate will be determined by the value of stocks
  - So in assuming that the flow rate remains constant, we were basically assuming that the values of the stocks stay constant over time Δt
    - For our system, given that the value of the stock x (People) declines by around 20% per time unit, this is not a very good assumption!

How Can We Reduce the Error? Try a Smaller Δt

- Let's work forward for ½ of a year at a time instead of for a full year

  $x(0)=1000$

  $\Delta t=.5$

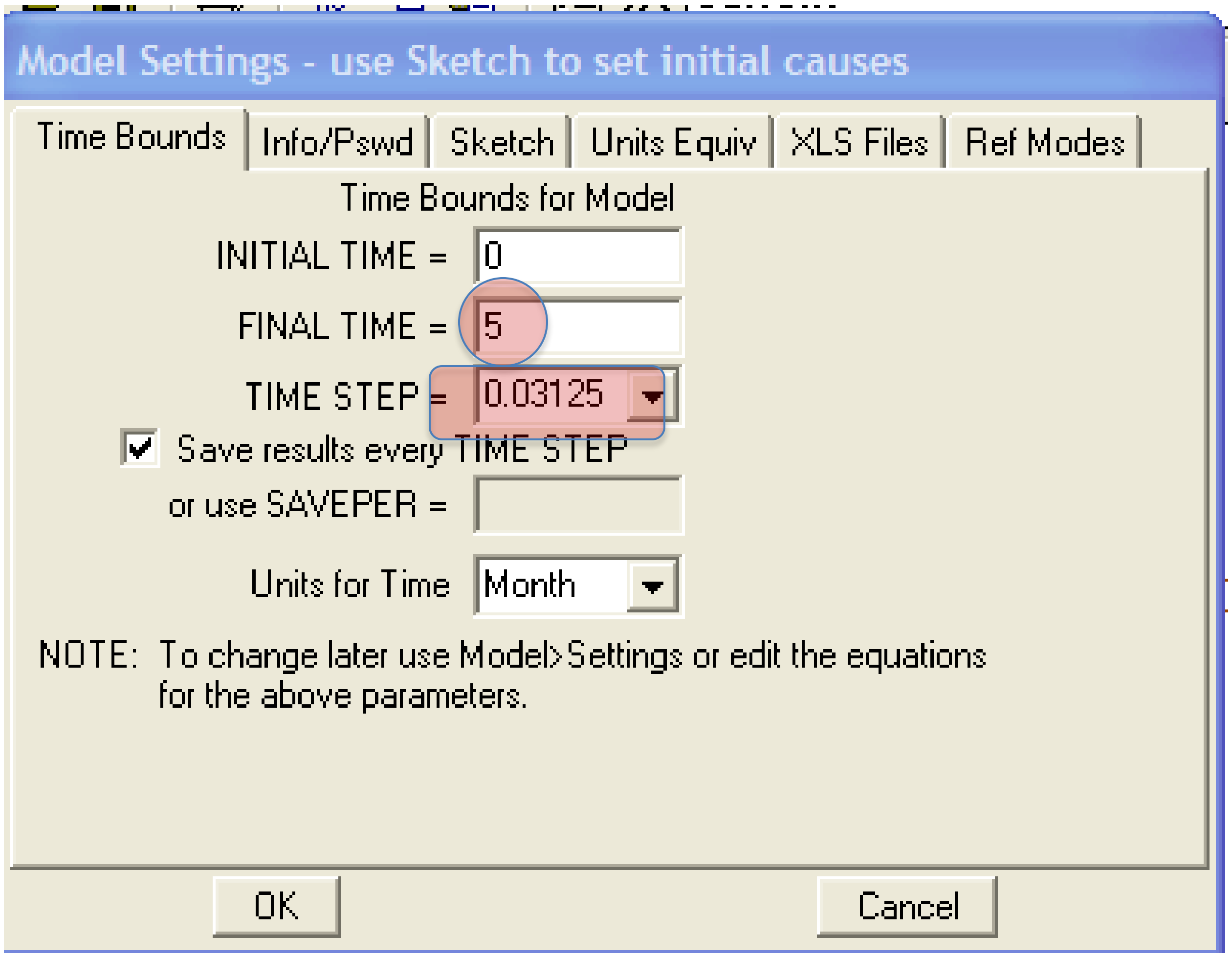
  $\alpha=.1$

Time (t)	Stock Value (x)	Change in stock (Δx) -x*Alpha*DeltaT	Net Flow Rate=Δx/Δt Here, Δt=1, so Δx/Δt=Δx/1=Δx
0	1000	-100	-200
0.5	900	-90	-180
1	810	-81	-162
1.5	729	-72.9	-145.8
2	656.1	-65.6	-131.2
2.5	590.5	-59.0	-118.1
3	531.4	-53.1	-106.3
3.5	478.3	-47.8	-95.7
4	430.5	-43.0	-86.1
4.5	387.4	-38.7	-77.5
5	348.7	-34.9	-69.7

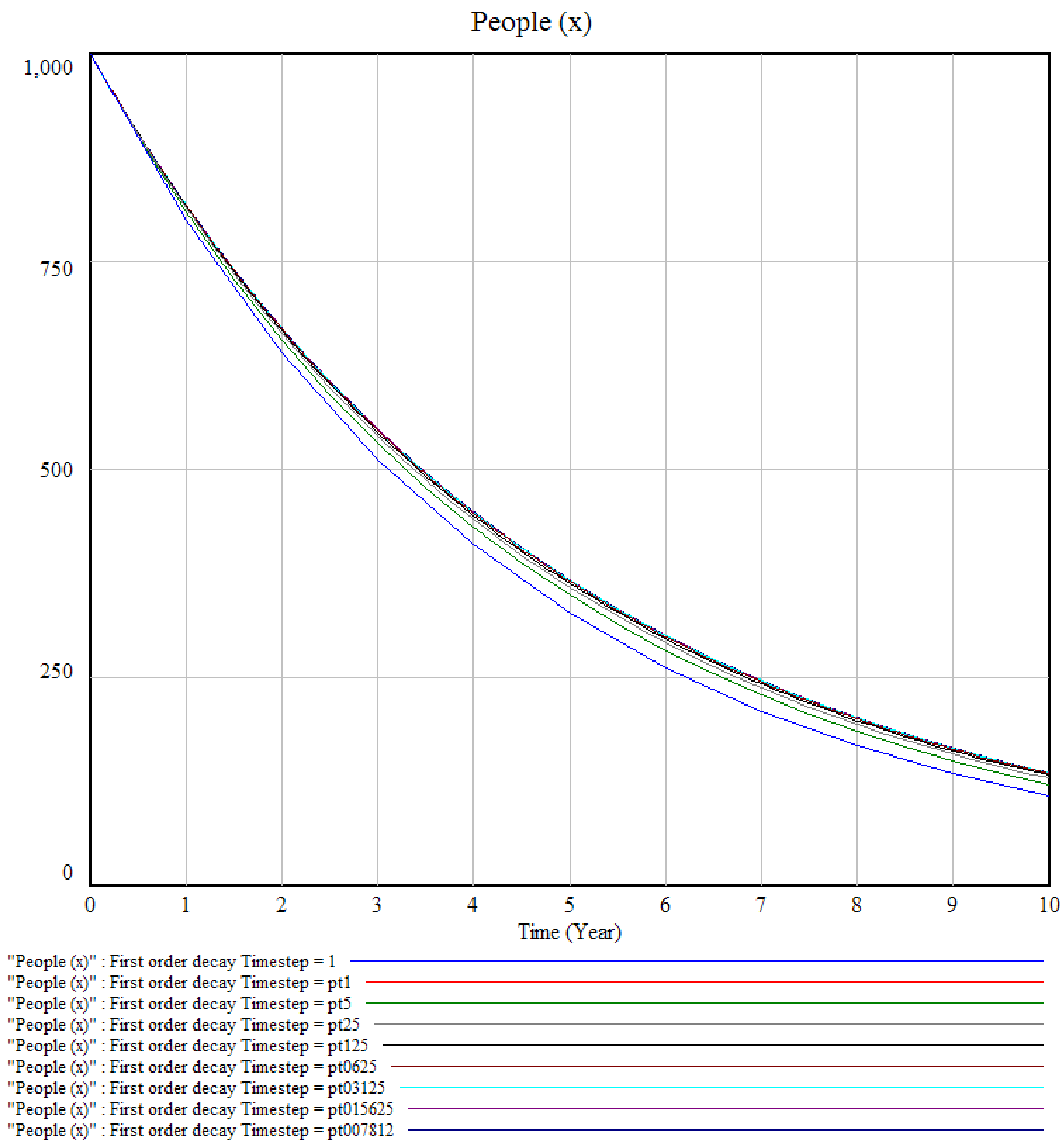
Approximate Dynamics: Net Flow Rate

	Δt=1	Δt=.5	Δt=.25
Time (t)	Stock Value (x)	Stock Value (x)	Stock Value (x)
0	1000	1000	1000
1	800	810	814.5
2	640	656.1	663.4
3	512	531.4	540.4
4	409.6	430.5	440.1
5	327.68	348.7	358.5

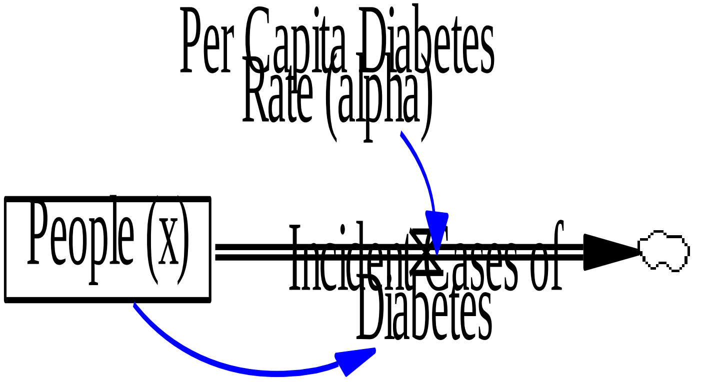
Vensim has a Step Size!
(Set via Model Menu/Settings Item)



Impact of Step Size on Simulation



Continuous Mathematics (Calculus!)



- Alpha is per-time-unit likelihood of death
  - Chance of death over small dt is αdt
  - If x people are at risk, # dying over dt is x*(Likelihood of death over Δt)=x(αdt)= xαdt
  - When people die, they flow out => cause a negative change in x.
  - We denote the change in x over the time dt as Δx

    Thus dx= -xαdt
- As x is depleted (becomes smaller), dx becomes smaller as well (for a fixed dt)

Flow Rate Dynamics: Continuous

- The total change in x over the time dt is dx

  Thus dx= $-x\alpha dt$

  – This might be 10 people over a timeframe of .1 year (~36.5 days)

- The *rate of change* of x over given time dt is dx/dt

  This is just the sum of all of the flows!

  For system, dx/dt =($-x\alpha dt$)/dt=- αx=-People*DeathRate

  Because x (People) changes, this flow rate changes over the course of the time we are observing

- We will sometimes write dx/dt as $\dot{x}$

$$\frac{dx}{dt} = \dot{x} = -\alpha x$$

The Concept of “Analytic” Solutions

- The model structure describes system behaviour *implicitly*
  - *This indicates how short term changes (flows) depends on the state of the system*
  - *This does not explicitly state how the system evolves*
- Analytic (“closed form”, “exact”) solutions describe system behaviour as an *explicit function of time*
  - *E.g.* $a+b*t+c*t^2$, $a+b*t$, $a*\sin(t)$, $e^{\alpha t}$
- For many systems we will be dealing with (nonlinear systems), an analytic solution *is simply not derivable*
  - Even when an analytic solution is possible, it is often most convenient to deal with simulations for most needs

An Exact Solution to Our Problem

- The state equation formulation of our system is

$$\frac{dx}{dt} = \dot{x} = -\alpha x$$

This is a linear differential equation with constant coefficients – a type of system that can be solved exactly.

Solution Procedure

$$\frac{dx}{dt} = -\alpha x$$

- Suppose we start x at time 0 with initial value x(0), and we want to find the value of x at time T
- Assuming that x does not start at 0, it will never reach exactly 0, so we can divide the left side by it, and multiply the right side by dt

$$\frac{dx}{x} = -\alpha dt$$

- Integrating both sides

$$\int_{t=0}^{t=T} \frac{dx}{x} = \int_{t=0}^{t=T} -\alpha dt$$

Completion of Derivation

$$\int_{t=0}^{t=T} \frac{dx}{x} = \int_{t=0}^{t=T} -\alpha\, dt = -\alpha \int_{t=0}^{t=T} dt$$

$$\ln x \Big|_{t=0}^{t=T} = -\alpha t \Big|_{t=0}^{t=T}$$

$$\ln x(T) - \ln x(0) = -\alpha T$$

$$\ln x(T) = \ln x(0) - \alpha T$$

$$x(T) = e^{\ln x(0) - \alpha T} = e^{\ln x(0)} e^{-\alpha T} = x(0) e^{-\alpha T}$$

So the stock x declines as a negative exponential in time T

i.e. # of people remaining in the stock goes down exponentially w/time

Fraction of Original People Still in Stock or Who have Left

- Assuming no inflows, the fraction of people still in the stock at time T is just

  (# of people in the stock at time T)/(initial # of people in the stock)=

$$\frac{x(T)}{x(0)} = \frac{x(0)e^{-\alpha T}}{x(0)} = e^{-\alpha T}$$

- Given that people either stay in the stock or leave, the fraction that have left by time T=

$$1 - \frac{x(T)}{x(0)} = 1 - e^{-\alpha T}$$

At Time=1

- At time t=1, we have a fraction $e^{-\alpha\cdot 1} = e^{-\alpha}$ in the stock, and a fraction $1-e^{-\alpha}$ who have left
- Note: By its Taylor Expansion

$$e^{-\alpha t} = \sum_{i=0}^{\infty} \frac{(-\alpha t)^i}{i!} = 1 + (-\alpha t) + \frac{(-\alpha t)^2}{2!} + \frac{(-\alpha t)^3}{3!} + \cdots$$

$$= 1 - \alpha t + \frac{(\alpha t)^2}{2} + \cdots$$

- For small αt, the higher order terms are very small, and this will be approximately $1-\alpha t$
- So by time 1 for small α, approx 1-α will remain after, and a fraction of α will have departed

Mean Time to Transition

- People are leaving via the flow
- Suppose we wish to determine the mean (average) time for a given person in the stock to leave
- Recall: A mean for a continuous probability distribution p(t) is given by

$$\int_{t=-\infty}^{t=\infty} tp(t)dt$$

- Since $p(t)dt$ is the probability that will leave between t and t+dt, this is just the continuous version of

$$E\left[q(a)\right] = \sum_{a \in \{\text{Possible values of a}\}} aq(a)$$

Mean Time to Leave

- p(t)dt here is the likelihood of a person leaving exactly between time t &dt+t
  - We start the simulation at t=0, so p(t)=0 for t<0
  - For t>0, P(leaving exactly between time t and dt+t)=P(leaving exactly between time t and t+dt|Still have not left by time t)P(Still have not left by time t)

For T>0, P(Still have not left by time t)= $e^{-\alpha T}$

For P(leaving exactly between time t and t+dt|Still have not left by time t)

Recall: For us, probability of leaving in a time dt always= αdt

Thus P(leaving exactly between time t and t+dt|Still have not left by time t)= αdt

P(t)dt=P(leaving exact b.t. time t &dt+t)= $\left(e^{-\alpha T}\right)\left(\alpha dt\right) = \alpha e^{-\alpha T} dt$

Derivation of Mean

- P(t)dt=P(leaving exactly between time t &dt+t)=

$$\left(e^{-\alpha T}\right)\left(\alpha dt\right) = \alpha e^{-\alpha T} dt$$

- Now that we have found the function p(t), we must do the integral $\int_{t=-\infty}^{t=\infty} tp(t)dt$ to derive the mean

- Here $E[p(t)] = \int_{t=-\infty}^{t=\infty} tp(t)dt = \int_{t=0}^{t=\infty} tp(t)dt = \int_{t=0}^{t=\infty} t\alpha e^{-\alpha T} dt$

$$= \alpha \int_{t=0}^{t=\infty} te^{-\alpha T} dt$$

Recall: Integration by Parts

- We have $E\left[p(t)\right]=\alpha\int_{t=0}^{t=\infty} te^{-\alpha T}dt=\alpha\left(\int_{t=0}^{t=\infty} te^{-\alpha T}dt\right)$
- To solve the term in brackets, we will use integration by parts
- Integration by parts exploits the following/l

$$\frac{d(uv)}{dt}=u\frac{dv}{dt}+v\frac{du}{dt}$$

$$d(uv)=udv+vdu$$

$$\int d(uv)=\int udv+\int vdu$$

$$uv=\int udv+\int vdu$$

and thus

$$\int udv=uv-\int vdu$$

Recall: Integration by Parts

- To solve $\int_{t=0}^{t=\infty} te^{-\alpha T}dt$ we will use integration by parts
- Here

$$u = t \Rightarrow du = \frac{du}{dt}dt = 1dt = dt$$

$$dv = e^{-\alpha T}dt \Rightarrow v = \int e^{-\alpha T}dt = \frac{-e^{-\alpha T}}{\alpha}$$

- From the previous page, we know

$$\int_{t=0}^{t=\infty} te^{-\alpha T}dt = \int_{t=0}^{t=\infty} udv = uv - \int_{t=0}^{t=\infty} vdu = \left(t\frac{-e^{-\alpha T}}{\alpha}\right)\Bigg|_{t=0}^{t=\infty} - \int_{t=0}^{t=\infty}\left(\frac{-e^{-\alpha T}}{\alpha}\right)dt$$

$$= \left(\frac{-te^{-\alpha T}}{\alpha}\right)\Bigg|_{t=0}^{t=\infty} + \frac{1}{\alpha}\int_{t=0}^{t=\infty} e^{-\alpha T}dt = (0-0) + \frac{1}{\alpha}\left(\frac{-1}{\alpha}e^{-\alpha T}\right)\Bigg|_{t=0}^{t=\infty} =$$

$$\frac{1}{\alpha}\left(0 - \frac{-1}{\alpha}\right) = \frac{1}{\alpha^2}$$

Thus

- The mean time (the *delay associated with a first order delay)* is thus given by

$$E\left[p(t)\right] = \alpha \int_{t=0}^{t=\infty} te^{-\alpha T} dt = \alpha \left(\int_{t=0}^{t=\infty} te^{-\alpha T} dt \right)$$

$$= \alpha \left(\frac{1}{\alpha^2} \right) = \frac{1}{\alpha}$$

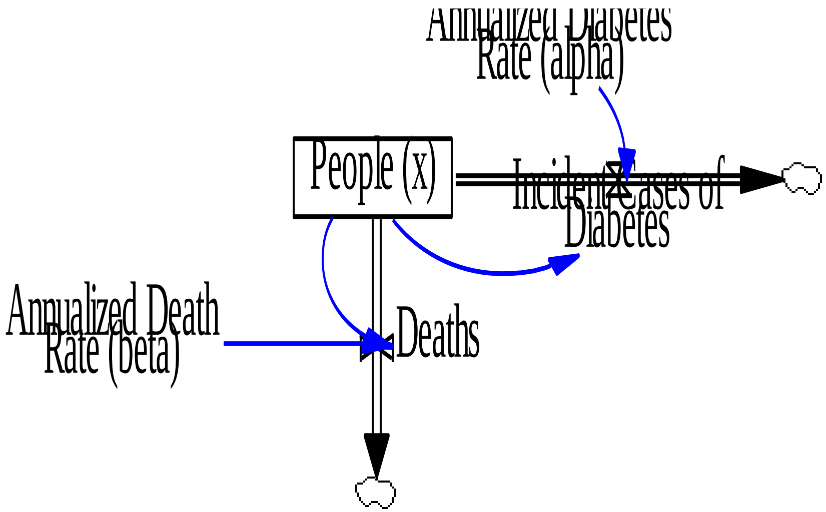
- So e.g. if we have an annualized rate of diabetes incident, the mean time to develop diabetes (independent of other risks) is just the reciprocal of that rate (i.e. 1 over that rate)

Computer Exercise: Simulating a First Order Delay

- Create a first order delay
- Feed in a “step function” that rises suddenly at time 10.
- How does the output from the stock change over time?

Competing Risks

- Suppose we have another outflow from the stock. How does that change our mean time of proceeding specifically down flow 1 (here, developing diabetes)?



Competing Risks Stock Trajectory Solution Procedure

$$\frac{dx}{dt} = -\alpha x - \beta x = -(\alpha + \beta) x$$

- Suppose we start x at time 0 with initial value x(0), and we want to find the value of x at time T
- This is just like our previous differential equation, except that "α" has been replaced by "(α+β)"
  - The solution must therefore be the same as before, with the appropriate replacement
  - Thus

$$x(T) = x(0)e^{-(\alpha+\beta)T}$$

Mean Time to Leave: Competing Risks

- p(t)dt here is the likelihood of a person leaving *via flow 1* (e.g. developing T2DM) exactly between time t &dt+t
  - We start the simulation at t=0, so p(t)=0 for t<0
  - For t>0, P(leaving on flow 1 exactly between time t &dt+t)=P(leaving on flow 1 exactly between time t &t+dt|Still have not left by time t)P(Still have not left by time t)

For T>0, P(Still have not left by time T)= $e^{-(\alpha+\beta)T}$

For P(leaving exactly between time t and t+dt|Still have not left by time t)

Recall: For us, probability of leaving in a time dt always=αdt

Thus P(leaving exactly between time t and t+dt|Still have not left by time t)= αdt

P(t)dt=P(leaving exact b.t. time t &dt+t)

$$= \alpha e^{-(\alpha+\beta)T} dt$$

Mean Time to Transition via Flow 1: Competing Risks

- By the same procedure as before, we have

$$E[p(t)] = \alpha \int_{t=0}^{t=\infty} t e^{-(\alpha+\beta)T} dt$$

- Using the formula we derived for the integral expression, we have

$$E[p(t)] = \frac{\alpha}{(\alpha+\beta)^2}$$

- Note that this correctly approaches the single-flow case as $\beta \to 0$

Equilibrium Value of a First-Order Delay

- Suppose we have flow of rate i into a stock with a first-order delay out
  - This could be from just a single flow, or many flows
- The value of the stock will approach an equilibrium where inflow=outflow

Equilibrium Value of 1st Order Delay

- Recall: Outflow rate for 1st order delay=αx
  - Note that this depends on the value of the stock!
- Inflow rate=i
- At equilibrium, the level of the stock must be such that inflow=outflow
  - For our case, we have

$$\alpha x = i$$

Thus $x = i/\alpha$

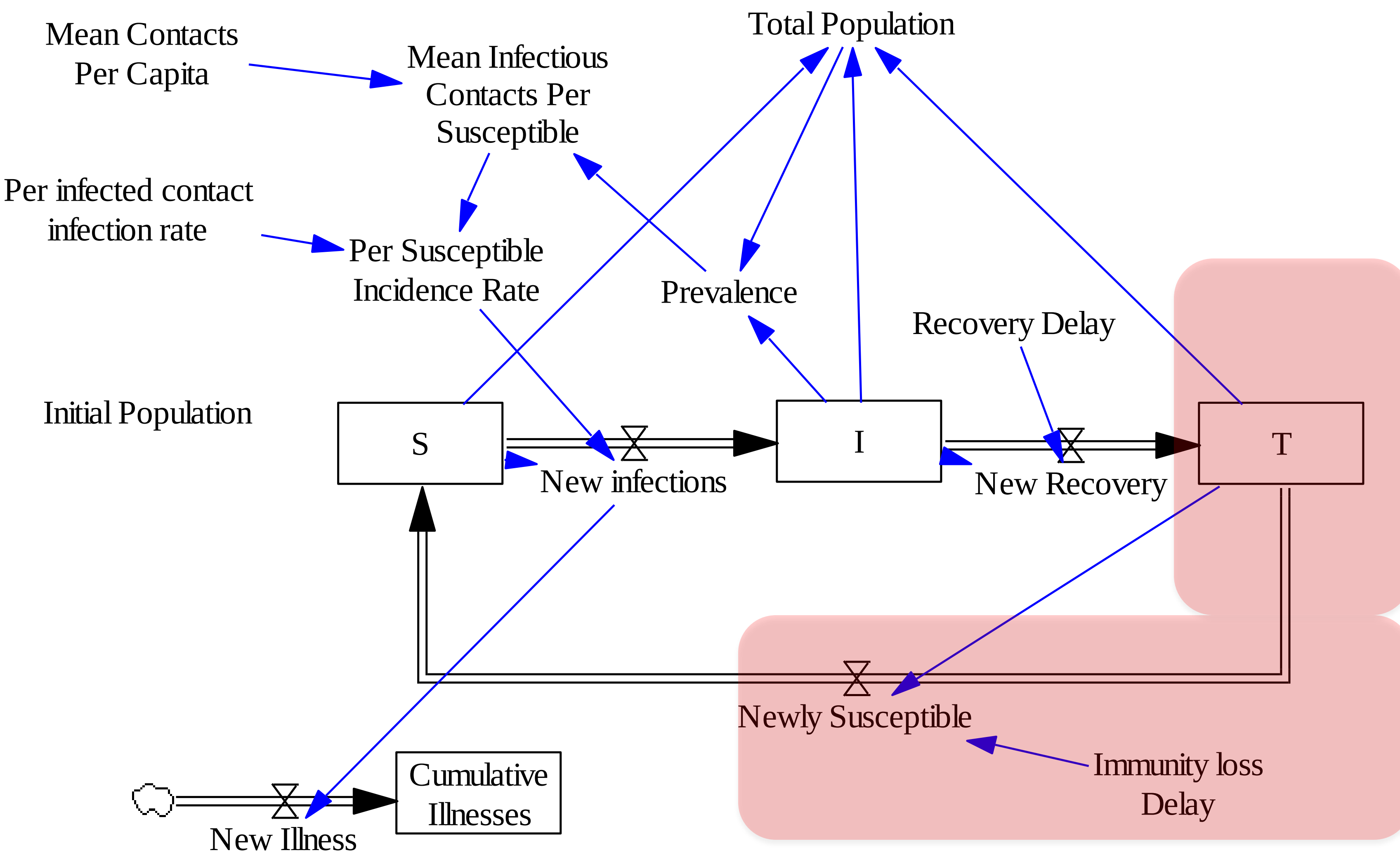
The lower the chance of leaving per time unit (or the longer the delay), the larger the equilibrium value of the stock must be to make outflow=inflow

Computer Exercise: Simulating a First Order Delay

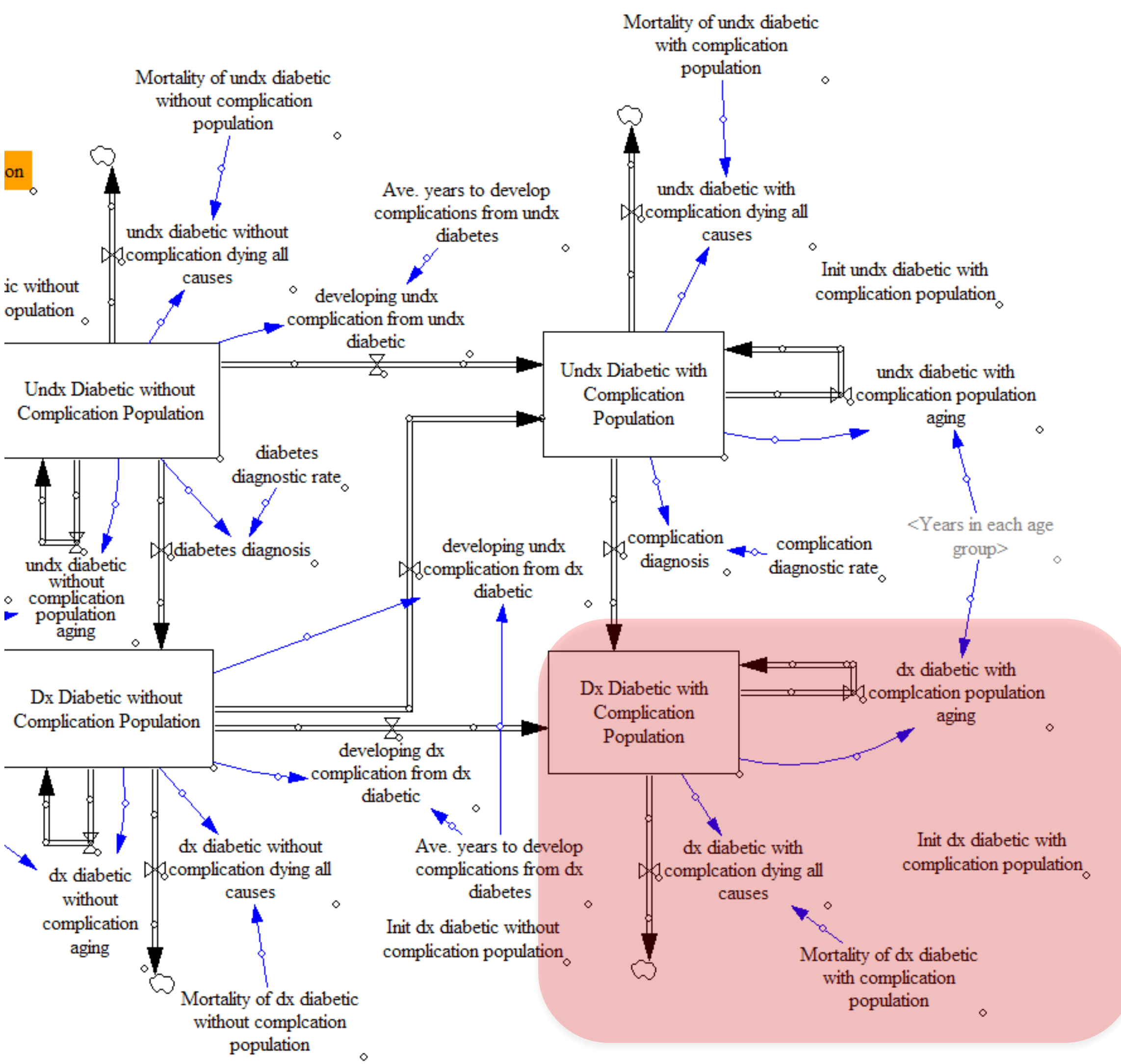
- Create a first order delay
- Feed in a “step function” that rises suddenly from 0 to 20 at time 10

  Use formula if then else(Time > 10, 20, 0)
- Questions to ponder
  - How does the output from the stock change over time?
  - How does the equilibrium value of the stock vary with chance of proceeding (alpha)?

First Order Delays in Action: Simple SIT Model

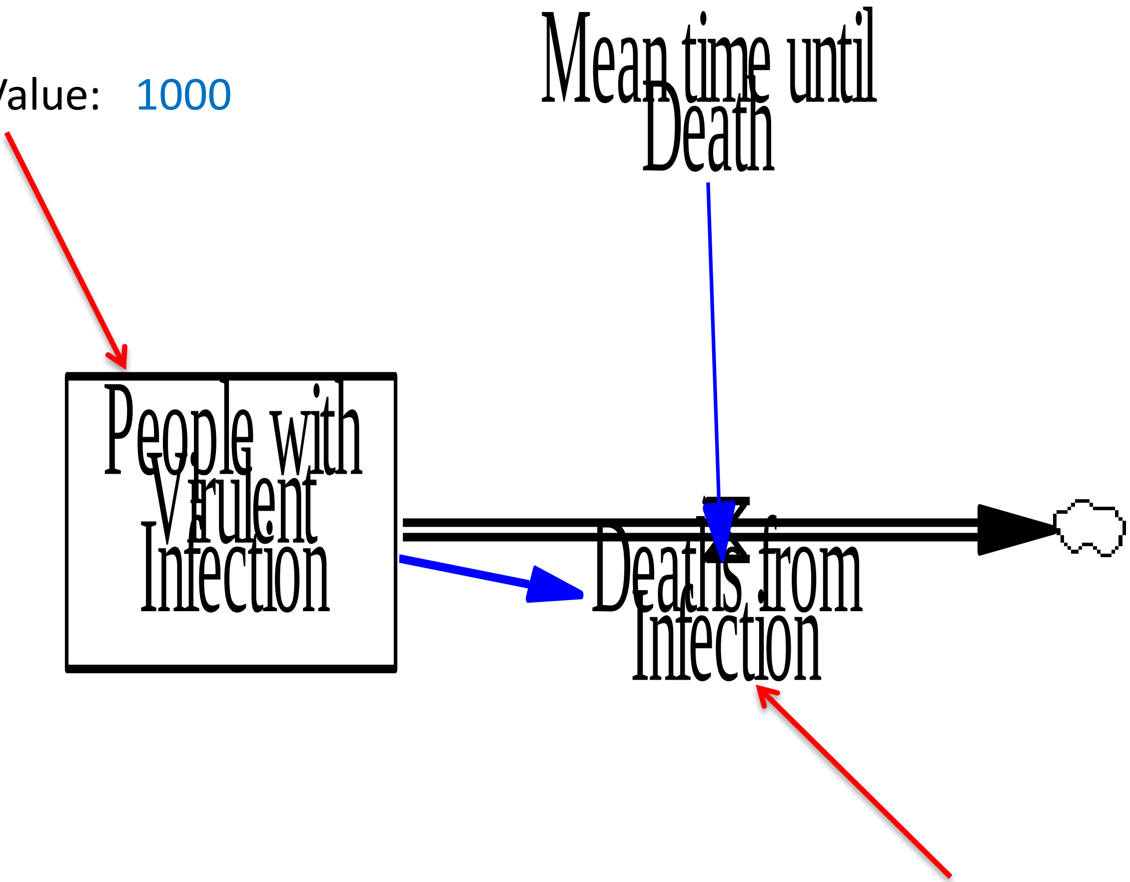


First Order Delays in Action: Simple SIT Model



Recall: Simple First-Order Decay

Use Initial Value: 1000



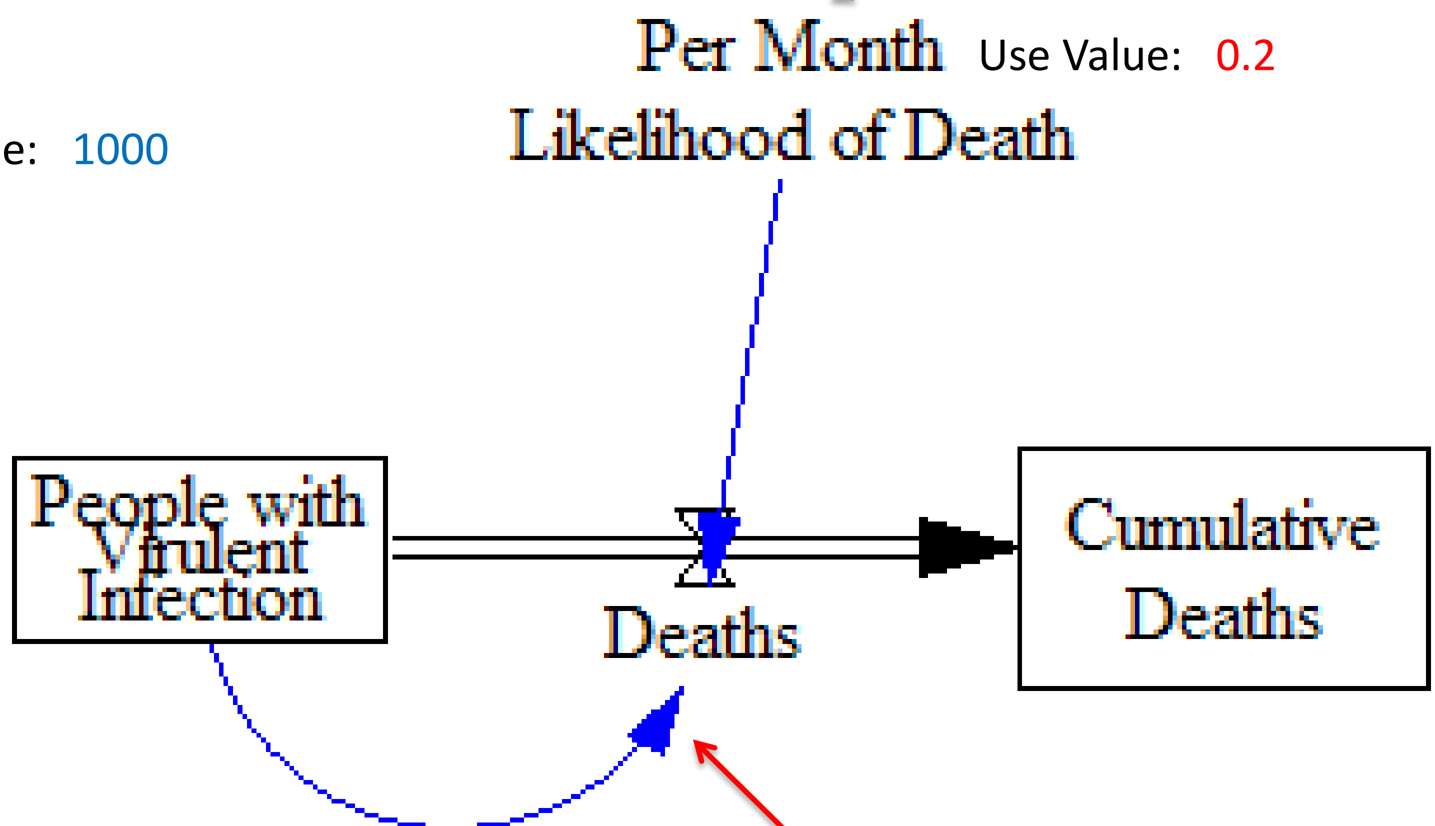
Use Formula: People with Virulent Infection/Mean time until Death

First-Order Decay (Variant of Last Time)

Recall: How does this relate to the mean time until death?

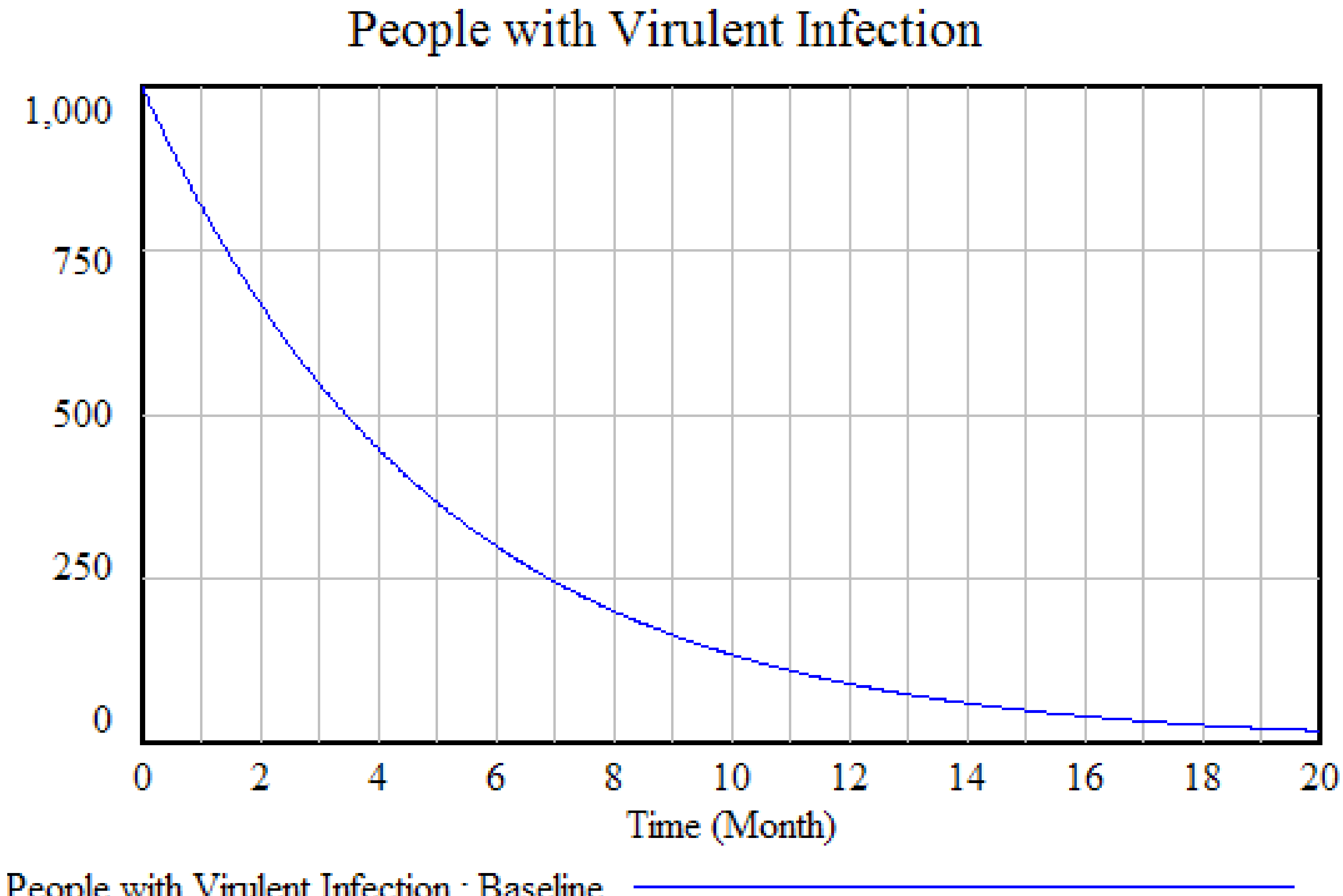
Use Value: 0.2

Use Initial Value: 1000

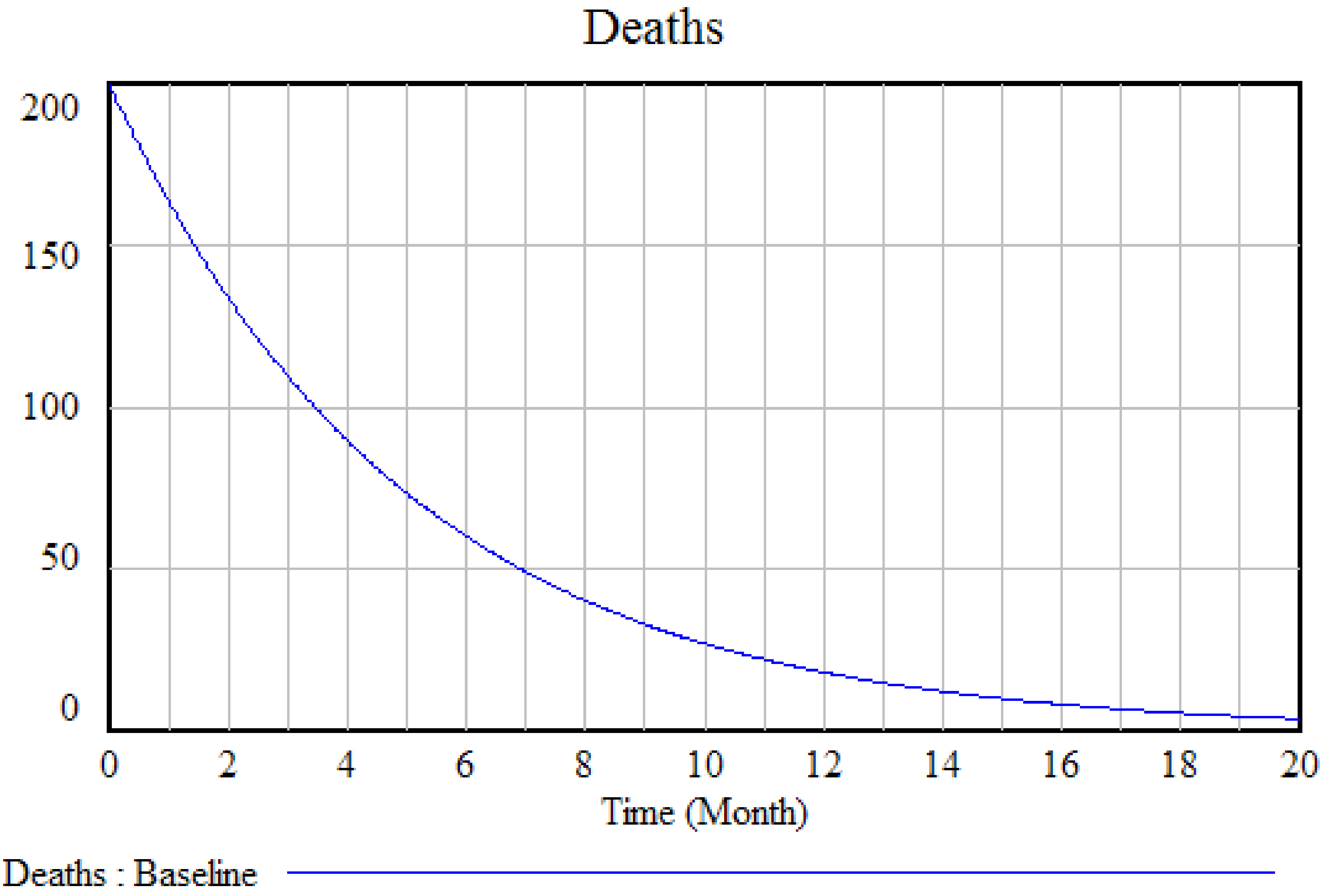


Use Formula: People with Virulent Infection*Per Month Likelihood of Death

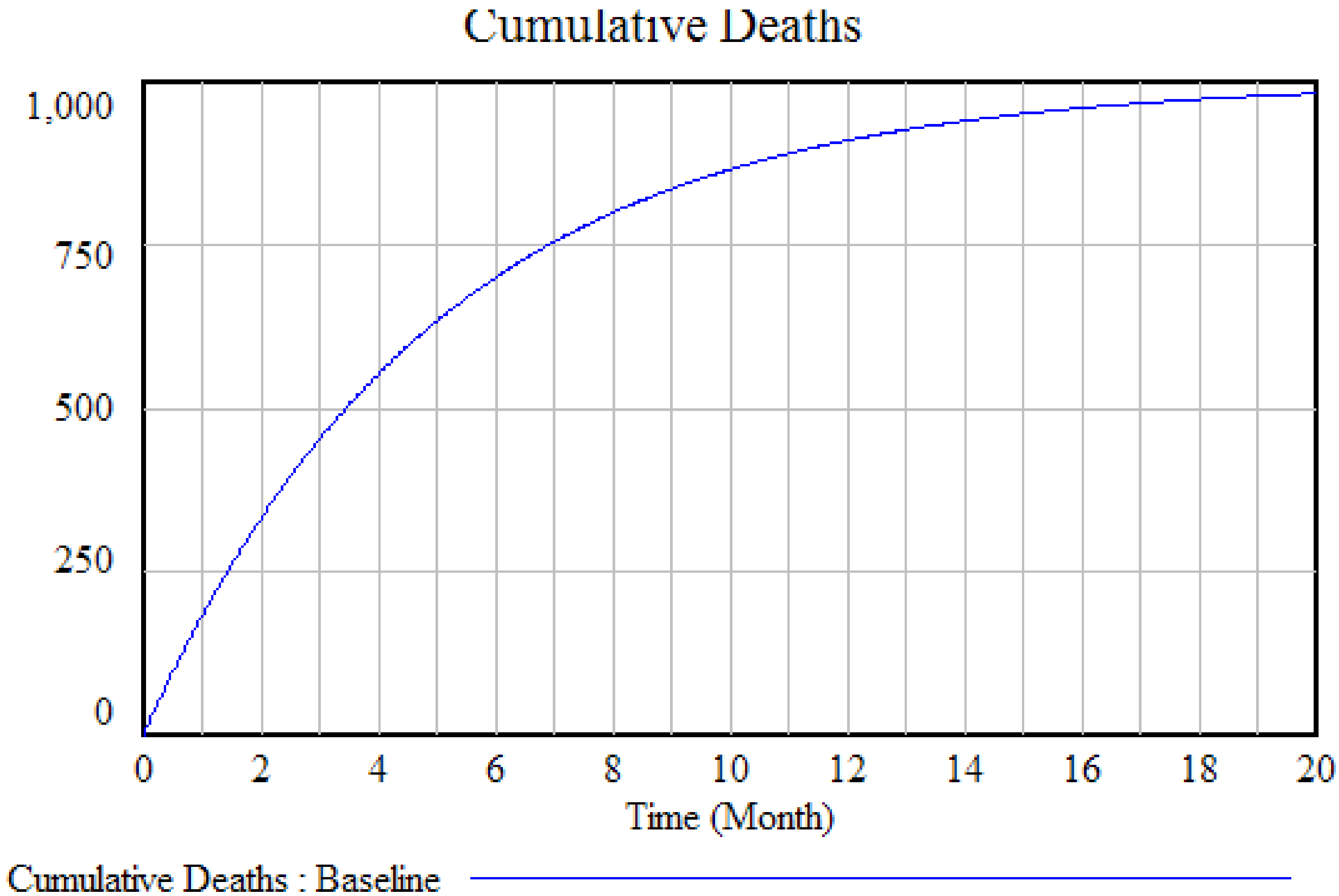
People in Stock



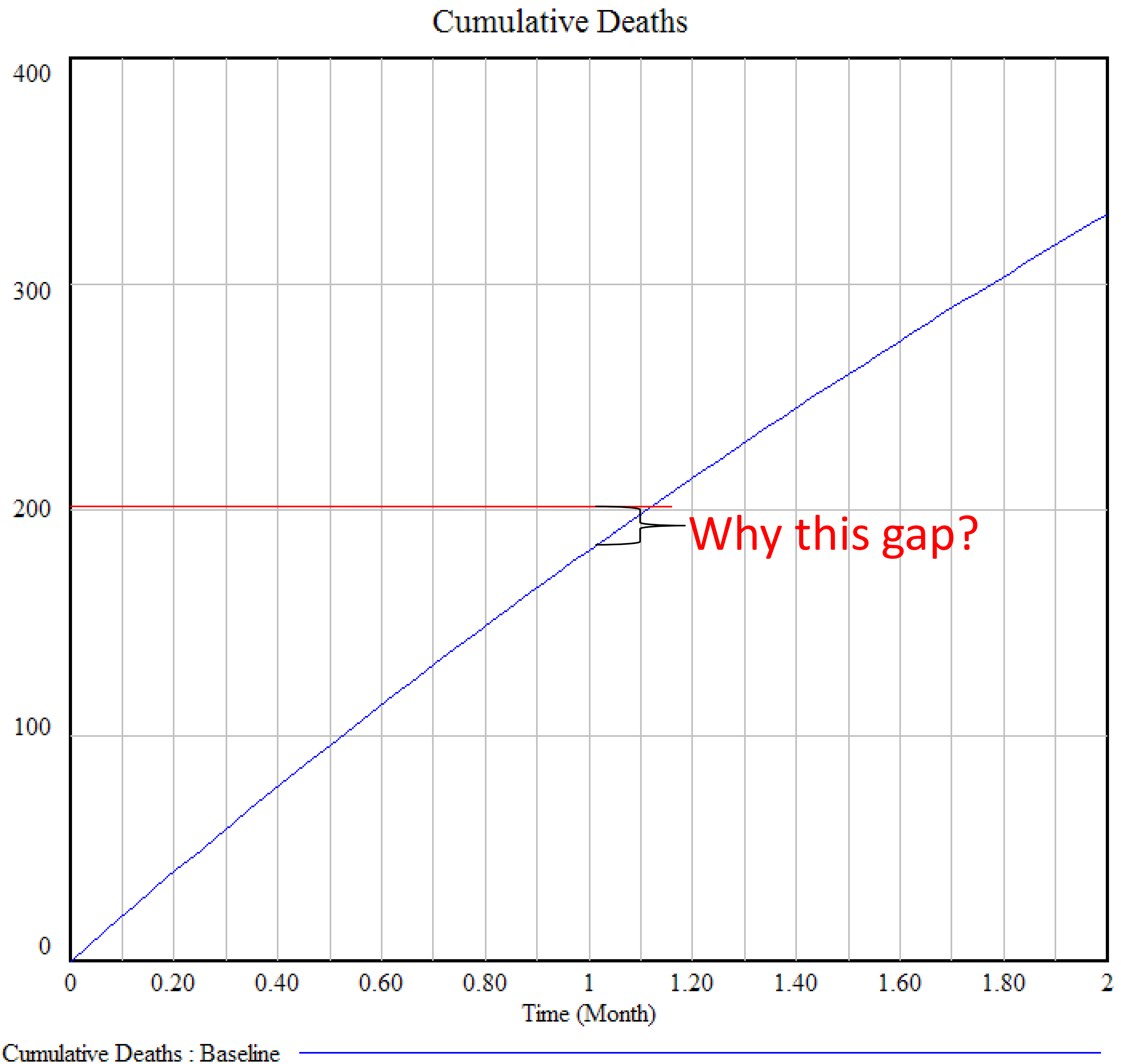
Flow Rate of Deaths



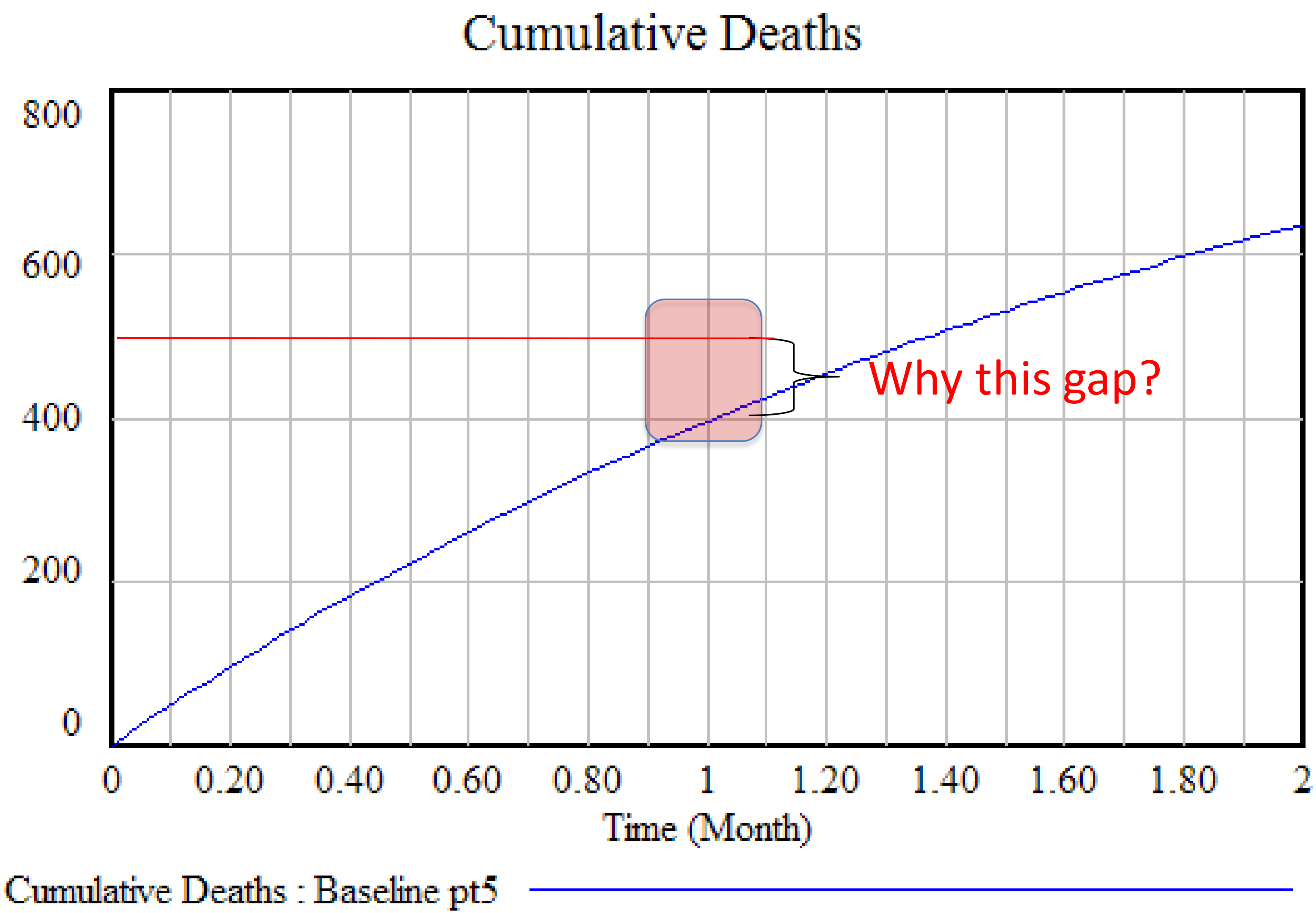
Cumulative Deaths



Closeup


Cumulative Deaths
400
300
200
100
0
0
0.20
0.40
0.60
0.80
1
1.20
1.40
1.60
1.80
2
Time (Month)
Cumulative Deaths : Baseline
Why this gap?

50% per Month Risk of Deaths



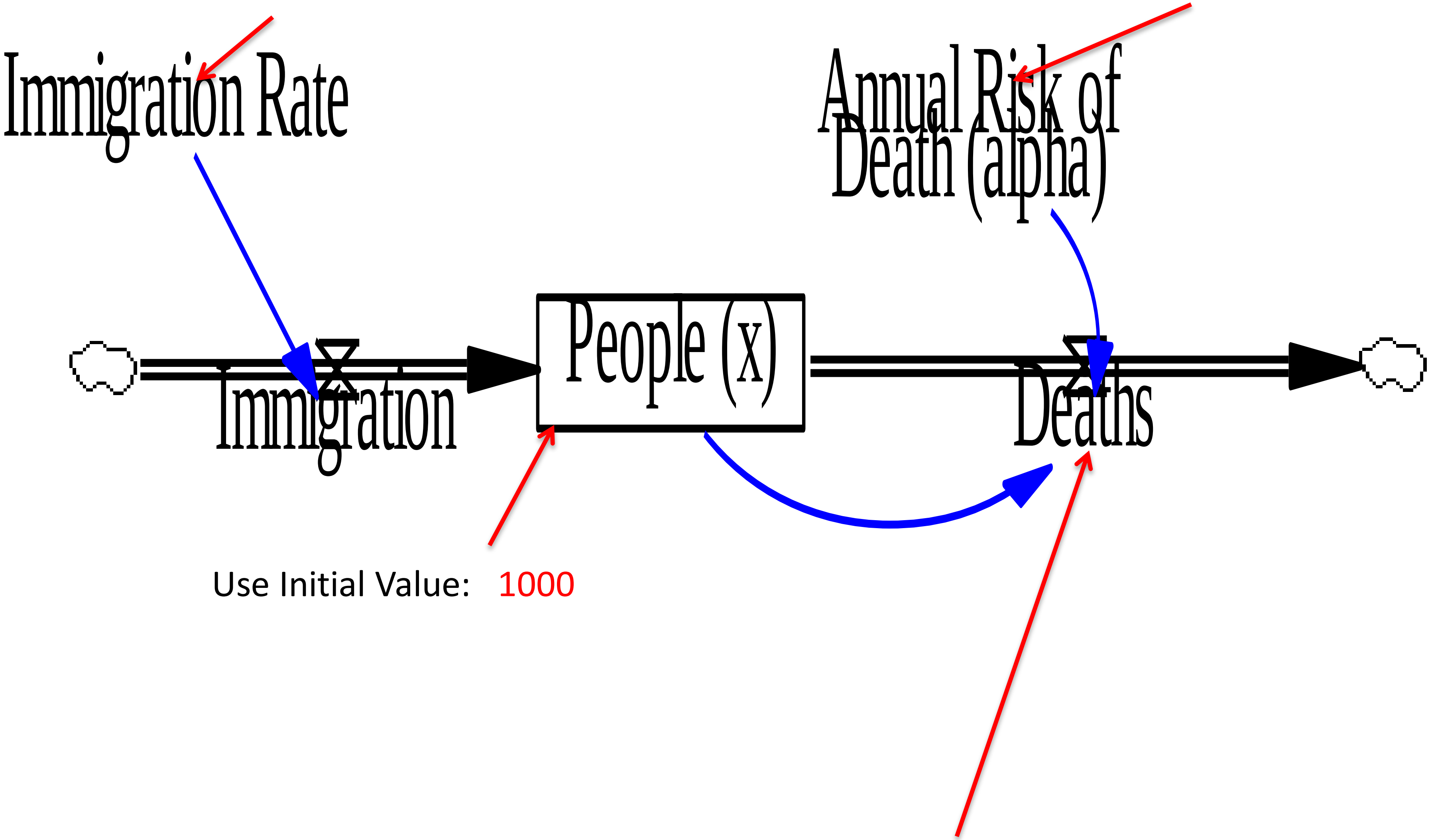
Answer: The "Gap" is Present Because not all 1000 people are at risk for a month!

- The value of the stock is declining over the first month
- The rate of death indicates that 20% of the population will die per month
- While we may have been expecting 200 people (20% of the 1000) to die, this (erroneously) assumes that all 1000 were at risk for the entire month
  - In fact, because the stock was declining, there were considerably fewer people at risk, meaning that we have fewer deaths
- If we had maintained 1000 people in the stock for the 1st month, 1000 people would have died!

Recall: First Order Delay

Use Value: 0

Use Value: 0.05



Use Initial Value: 1000

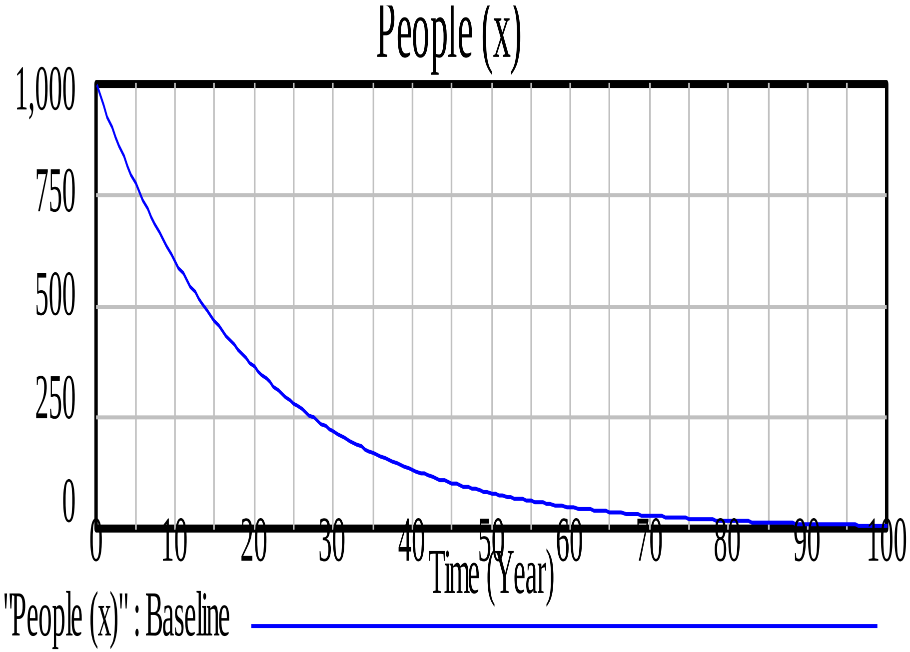
Use Formula: People (x) * Annual Risk of Death (alpha)

Questions

- What is behaviour of stock x?
- What is the mean time until people die?
- Suppose we had a constant inflow – what is the behaviour then?

Answers

- Behaviour Of Stock



- Mean Time Until Death

  Recall that if coefficient of first order delay is α , then mean time is $1/\alpha$ (Here, 1/0.05 = 20 years)

Equilibrium Value of a First-Order Delay

- Suppose we have flow of rate i into a stock with a first-order delay out
  - This could be from just a single flow, or many flows
- The value of the stock will approach an equilibrium where inflow=outflow

Equilibrium Value of 1st Order Delay

- Recall: Outflow rate for 1st order delay=αx
  - Note that this depends on the value of the stock!
- Inflow rate=i
- At equilibrium, the level of the stock must be such that inflow=outflow
  - For our case, we have

$$\alpha x = i$$

Thus $x = i/\alpha$

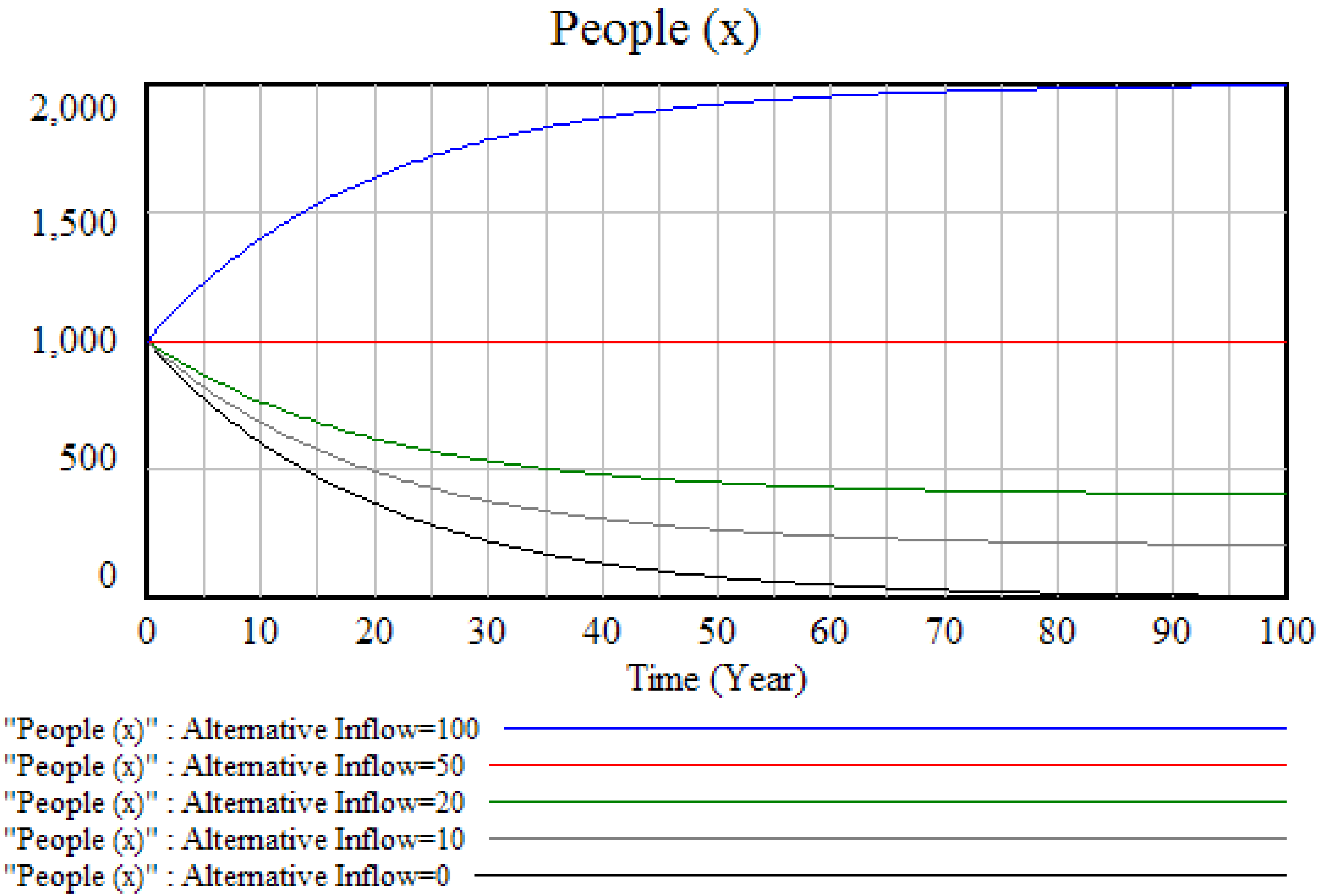
(equivalently, x = i * Mean time to Transition)

The lower the chance of leaving per time unit (or the longer the delay), the larger the equilibrium value of the stock must be to make outflow=inflow

Scenarios for First Order Delay: Variation in Inflow Rates

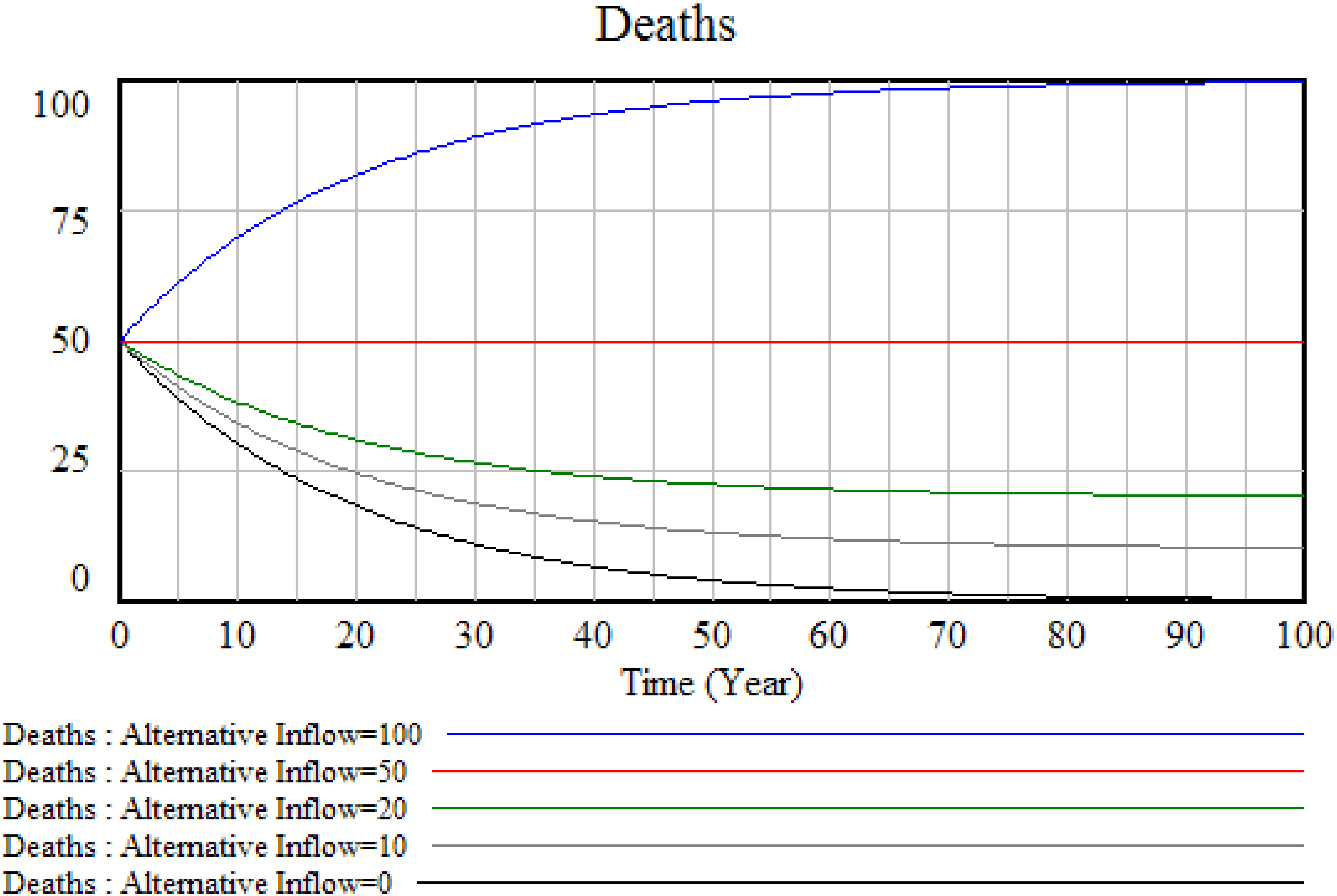
- For different immigration (inflows) (what do you expect?)
  - Inflow=10
  - Inflow=20
  - Inflow=50
  - Inflow=100
  - Why do you see this “goal seeking” pattern?
  - What is the “goal” being sought?

Behaviour of Stock for Different Inflows



Why do we see this behaviour?

Behaviour of *Outflow* for Different Inflows

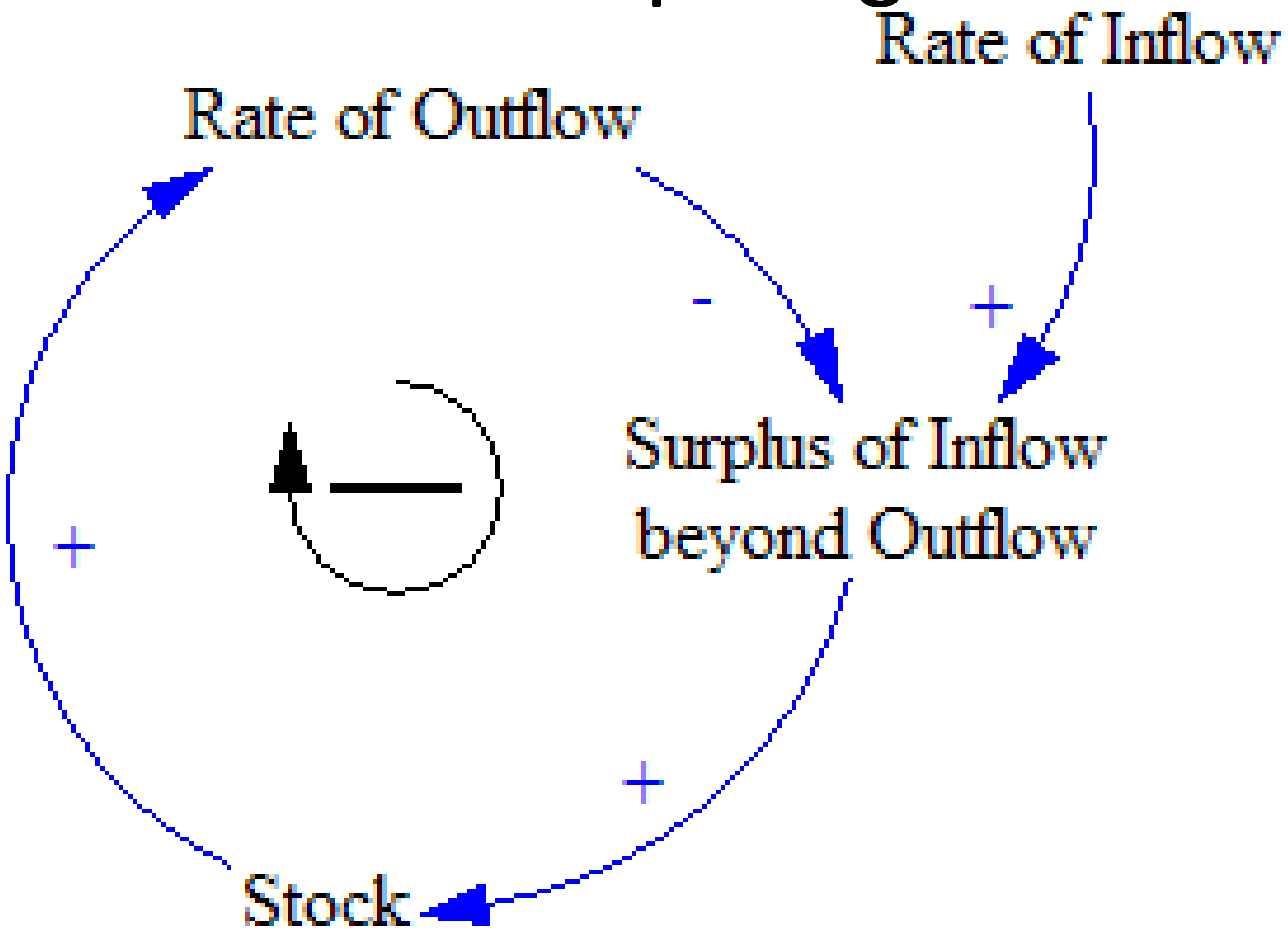


Why do we see this behaviour? Imbalance (gap) causes change to stock (rise or fall) $\Rightarrow$ change to outflow to lower gap **until outflow=inflow**

Goal Seeking Behaviour

- The goal seeking behaviour is associated with a negative feedback loop
  - The larger the population in the stock, the more people die per year
- If we have more people coming in than are going out per year, the stock (and, hence, outflow!) rises until the point where inflow=outflows
- If we have fewer people coming in than are going out per year, the stock declines (& outflow) declines until the point where inflow=outflows

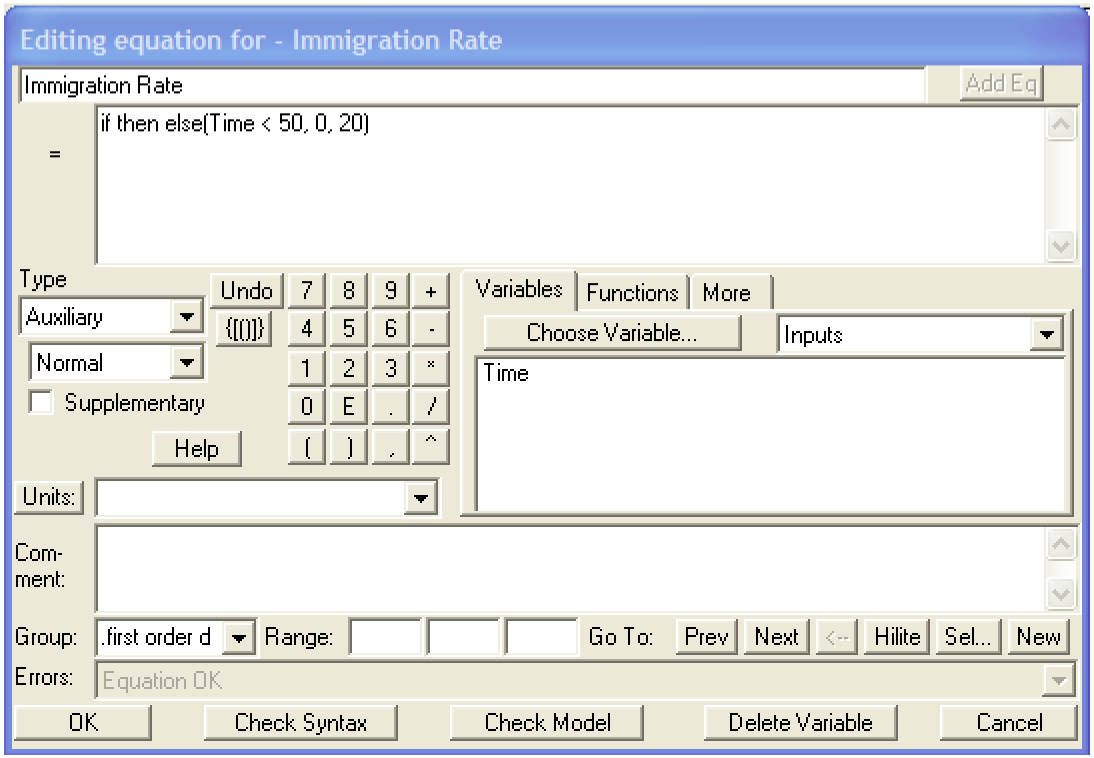
As a Causal Loop Diagram



What does this tell us about how the system would respond to a sudden change in immigration?

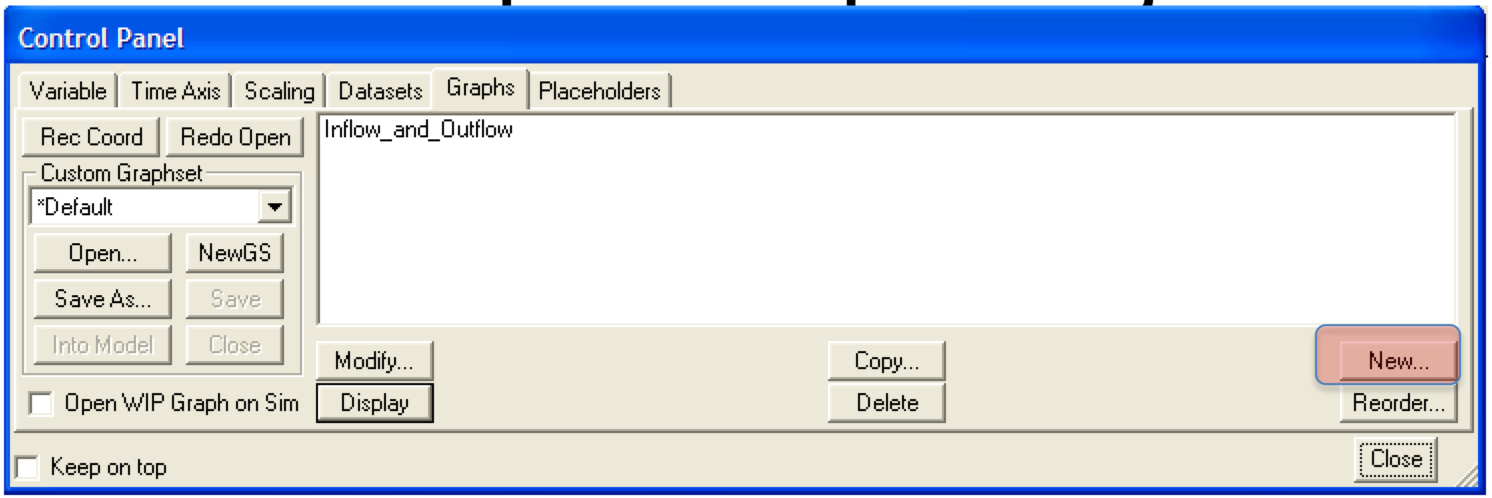
Response to a Change

- Feed in an immigration “step function” that rises suddenly from 0 to 20 at time 50

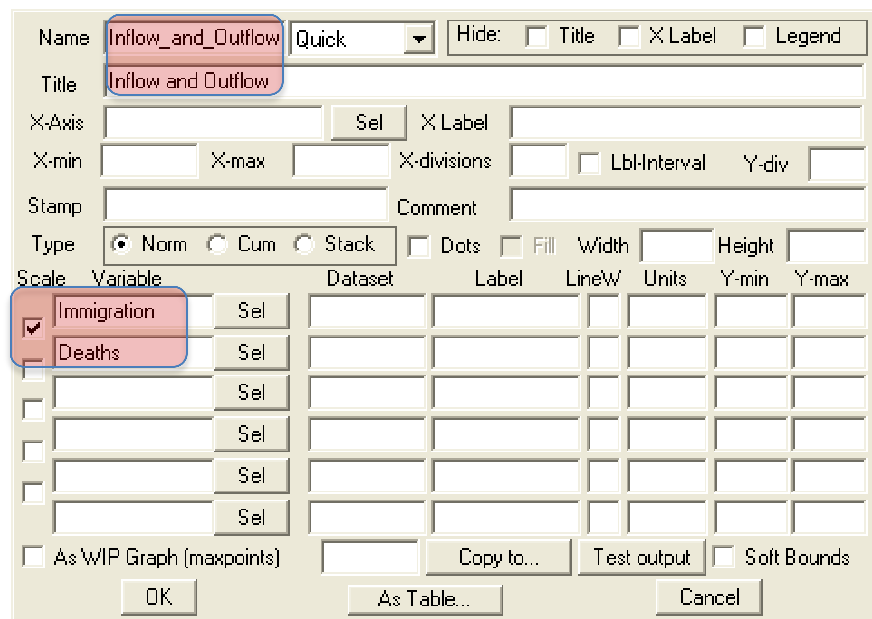


- Set the Initial Value of Stock to 0
- How does the stock change over time?

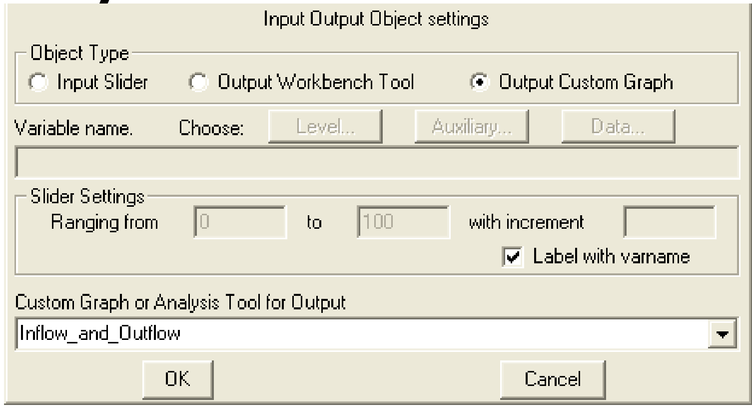
Create a Custom Graph & Display it as an Input-Output Object

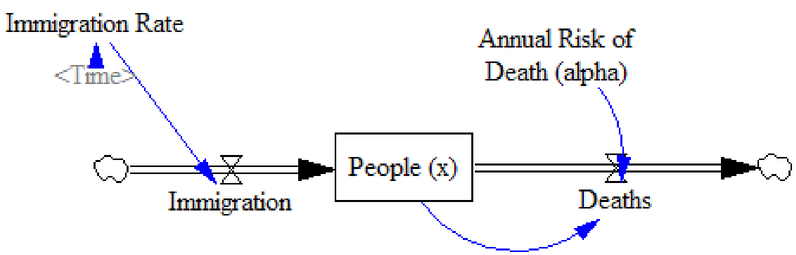


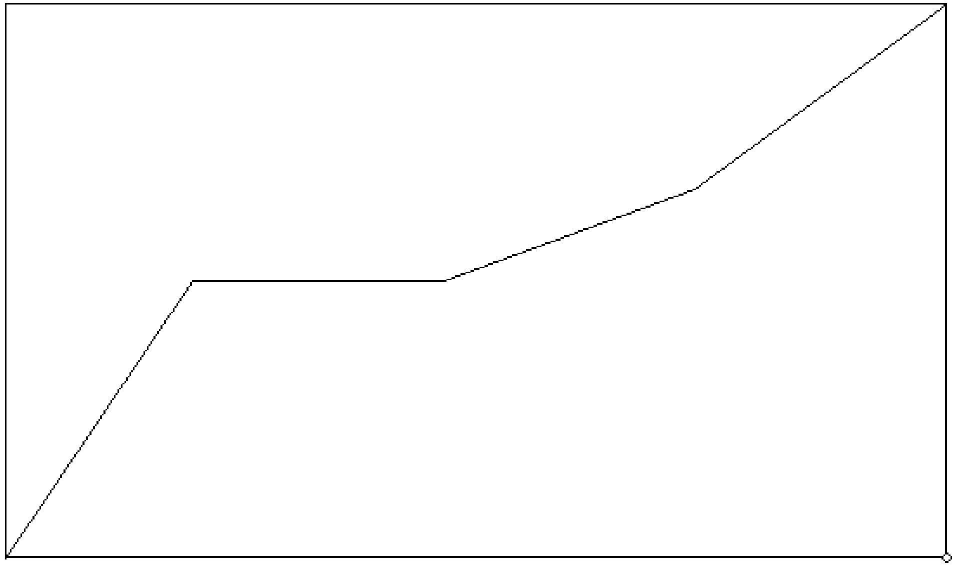
- Editing



Create Input-Output Object (for Synthesim)

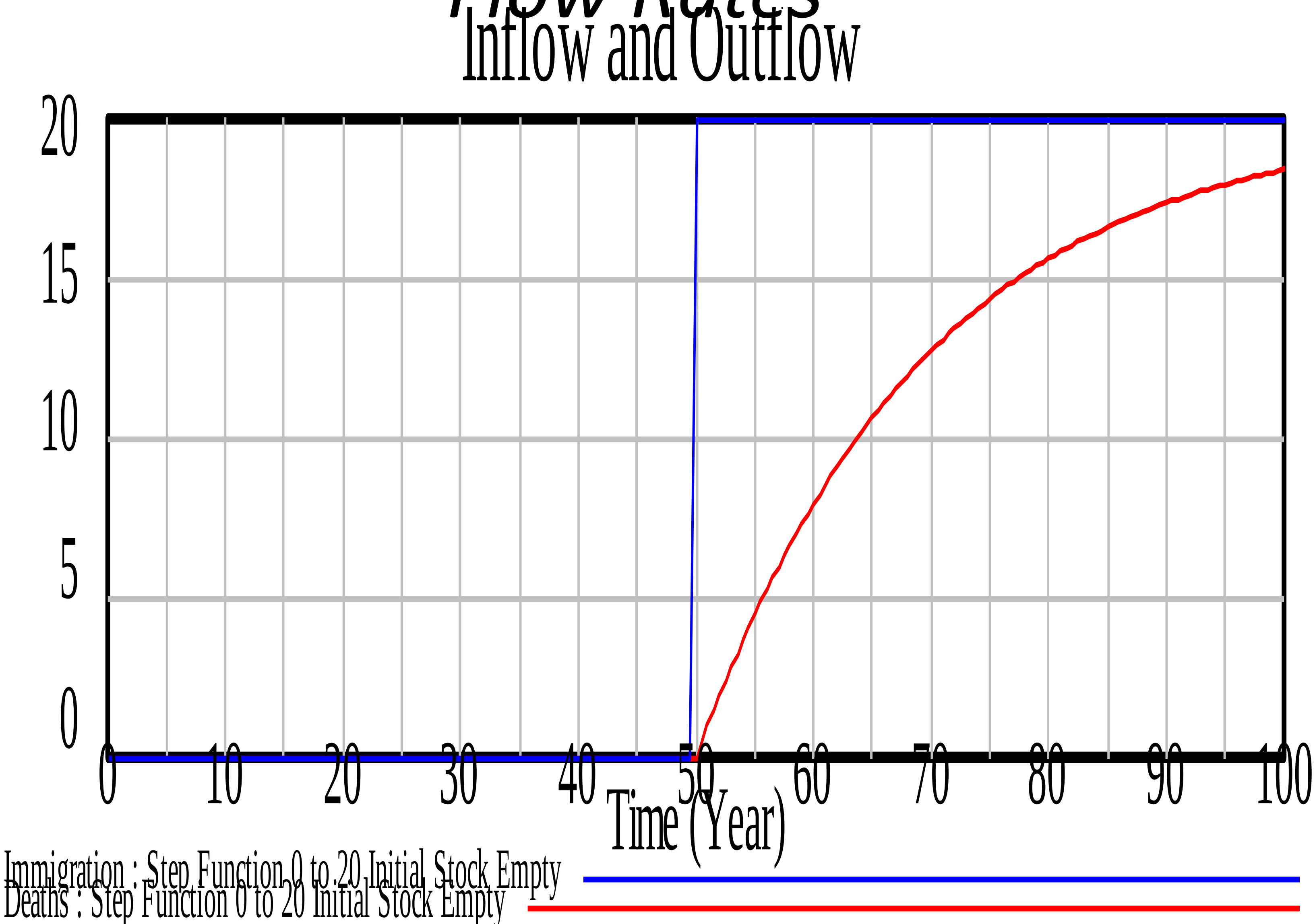






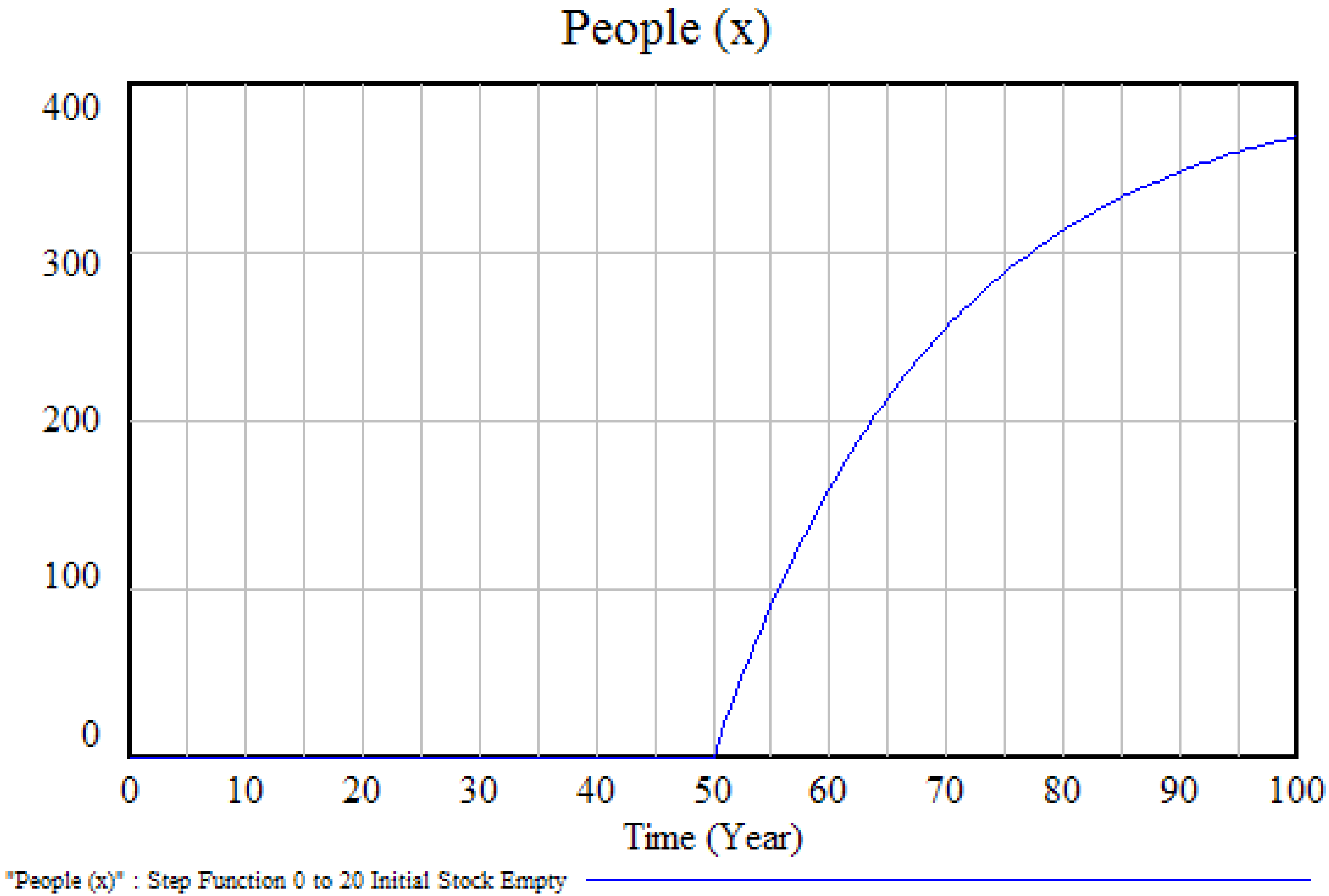
Stock Starting Empty

Flow Rates



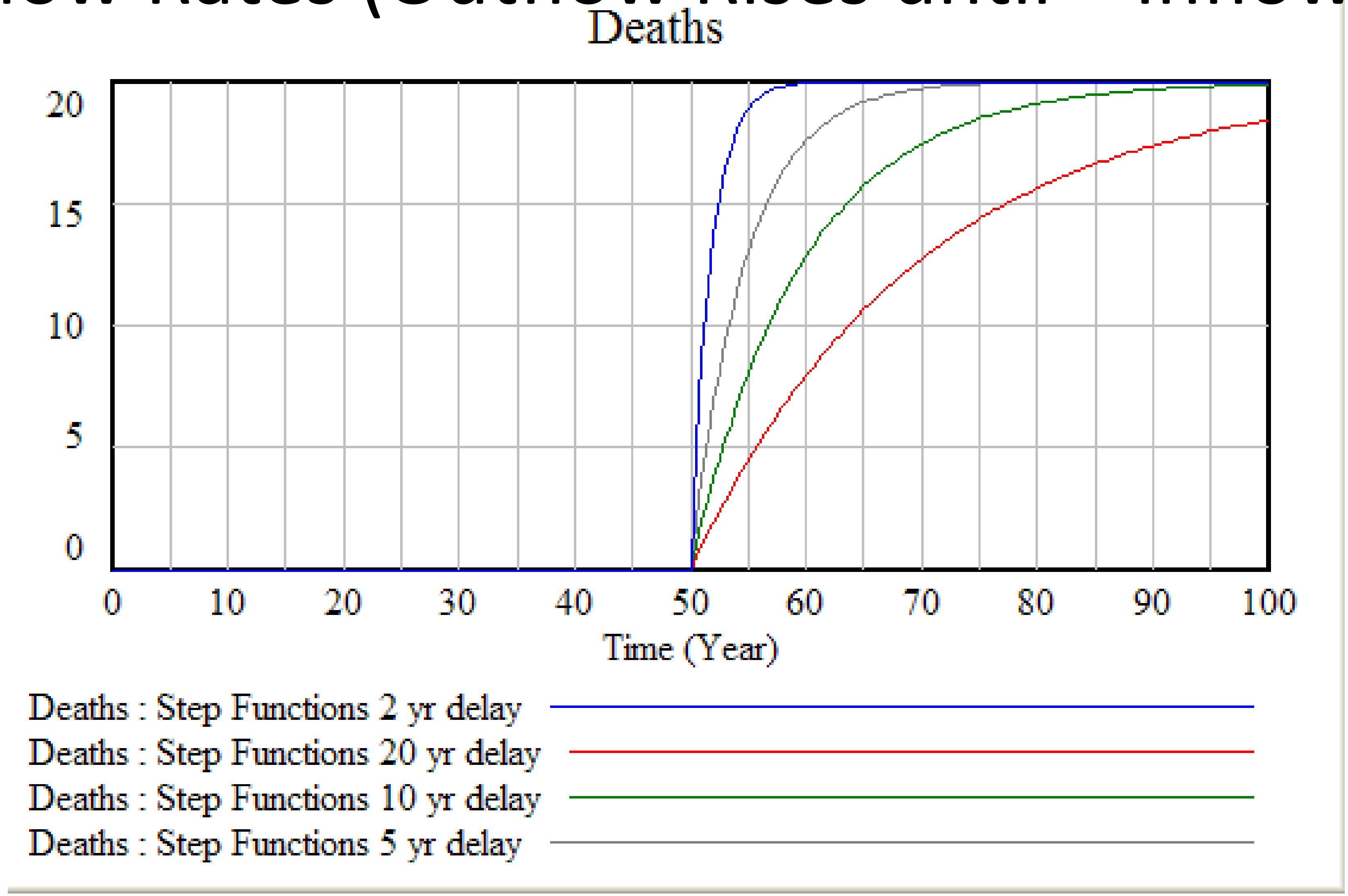
How would this change with alpha?

Stock Starting Empty?
Value of *Stock* (Alpha=.05)



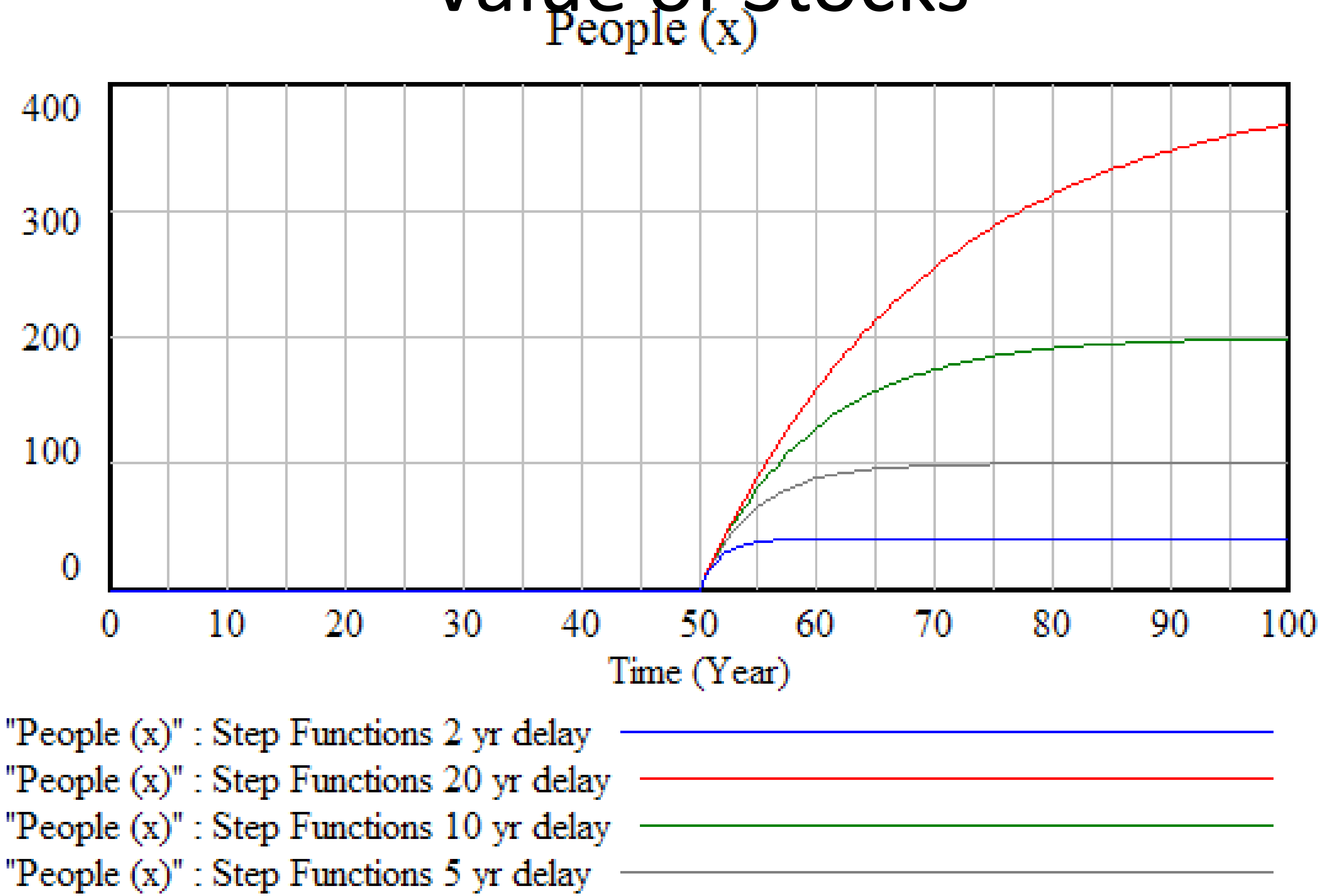
How would this change with alpha?

For Different Values of (1/) Alpha
Flow Rates (Outflow Rises until = Inflow)



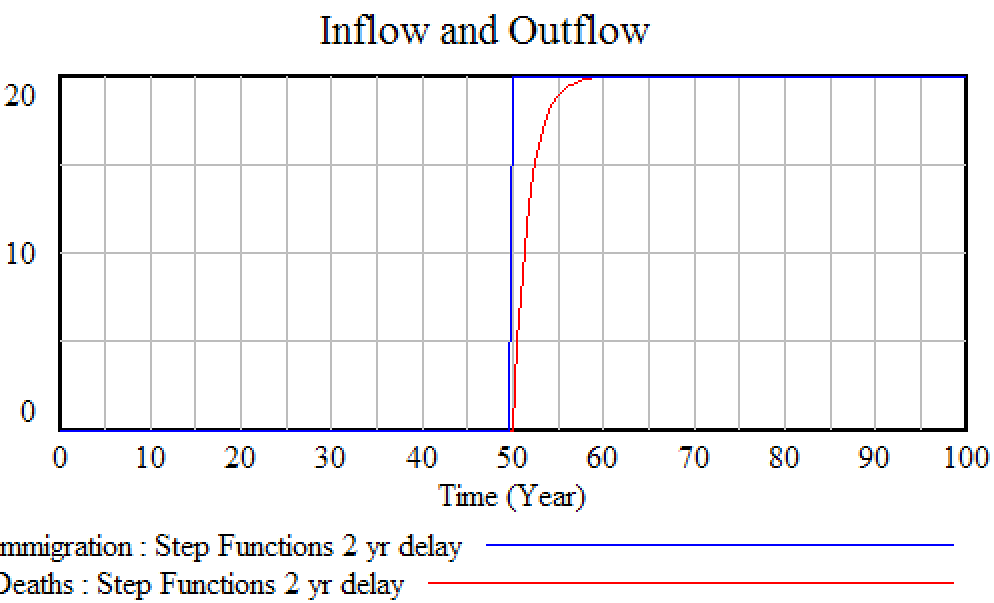
This is for the *flows*. What do stocks do?

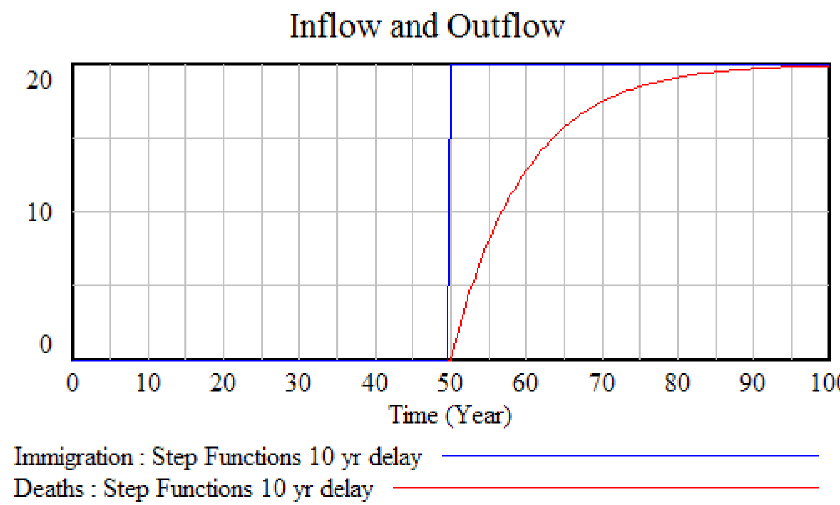
For Different Values of (1/) Alpha
Value of Stocks

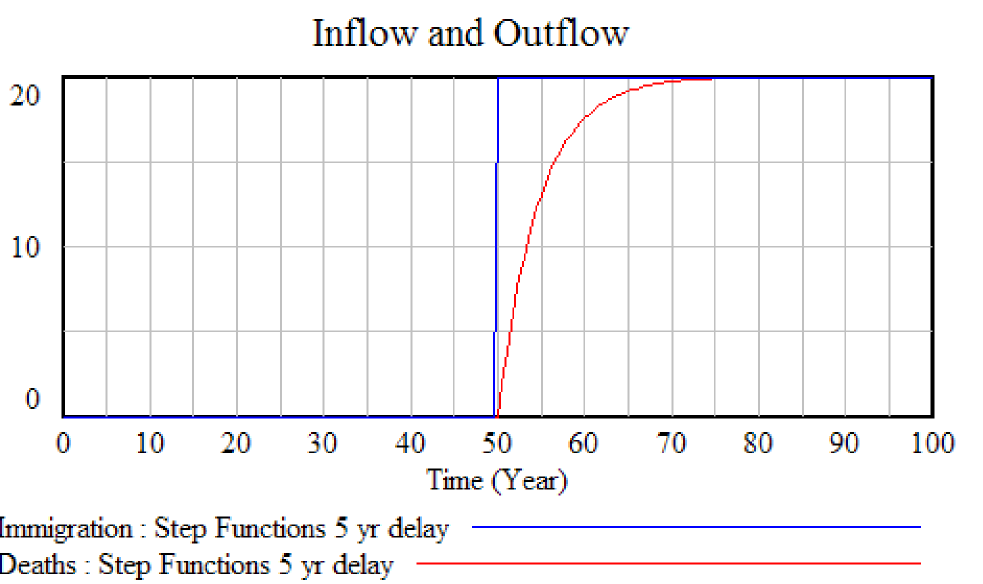


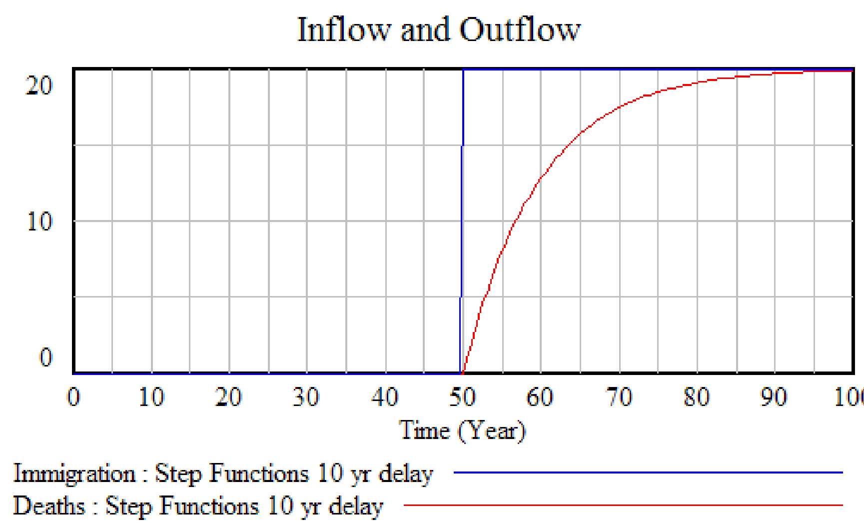
Why do we see this behaviour? A longer time delay (or smaller chance of leaving per unit time) requires *x* to be *larger* to make outflow=inflow

Outflows as Delayed Version of Inputs



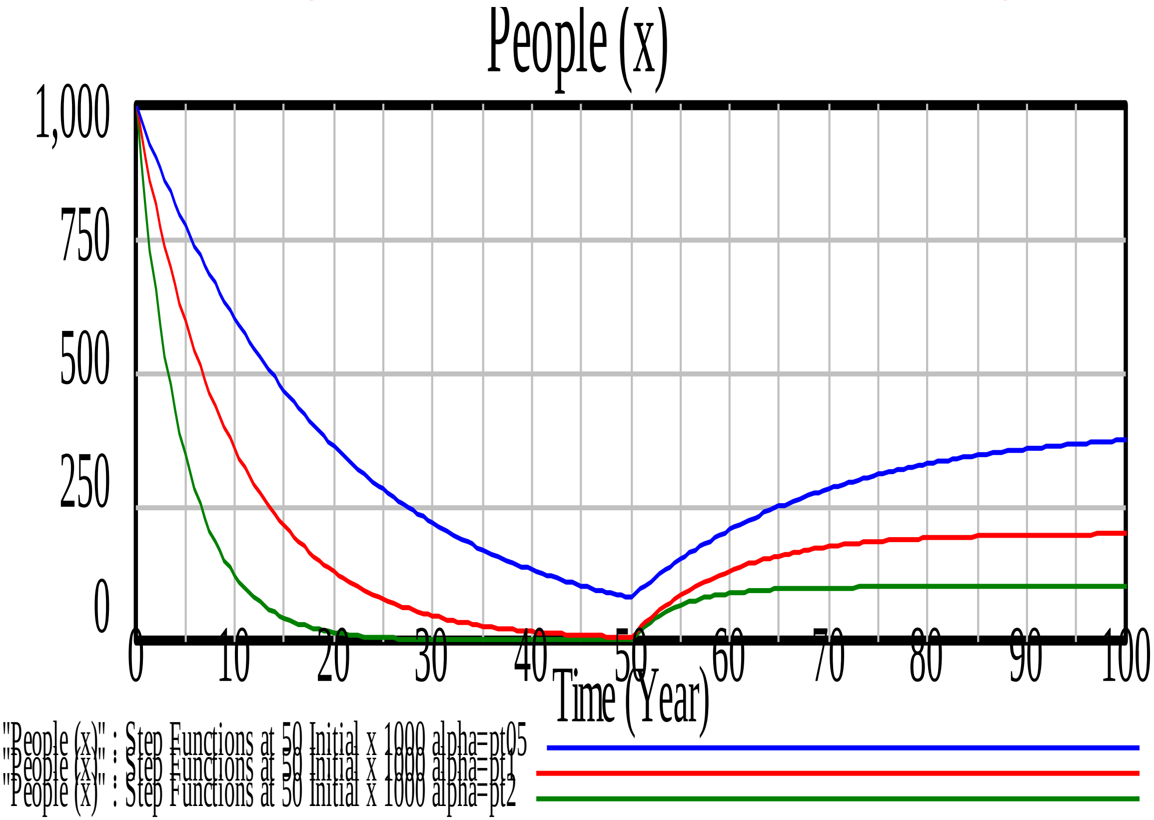


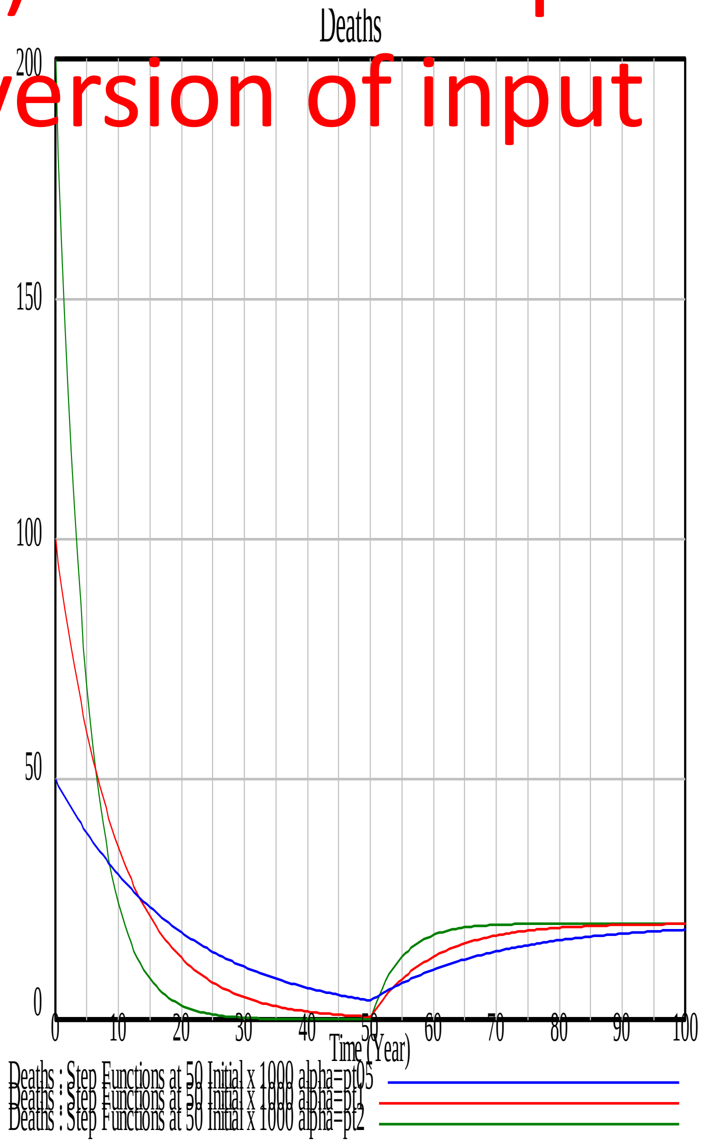




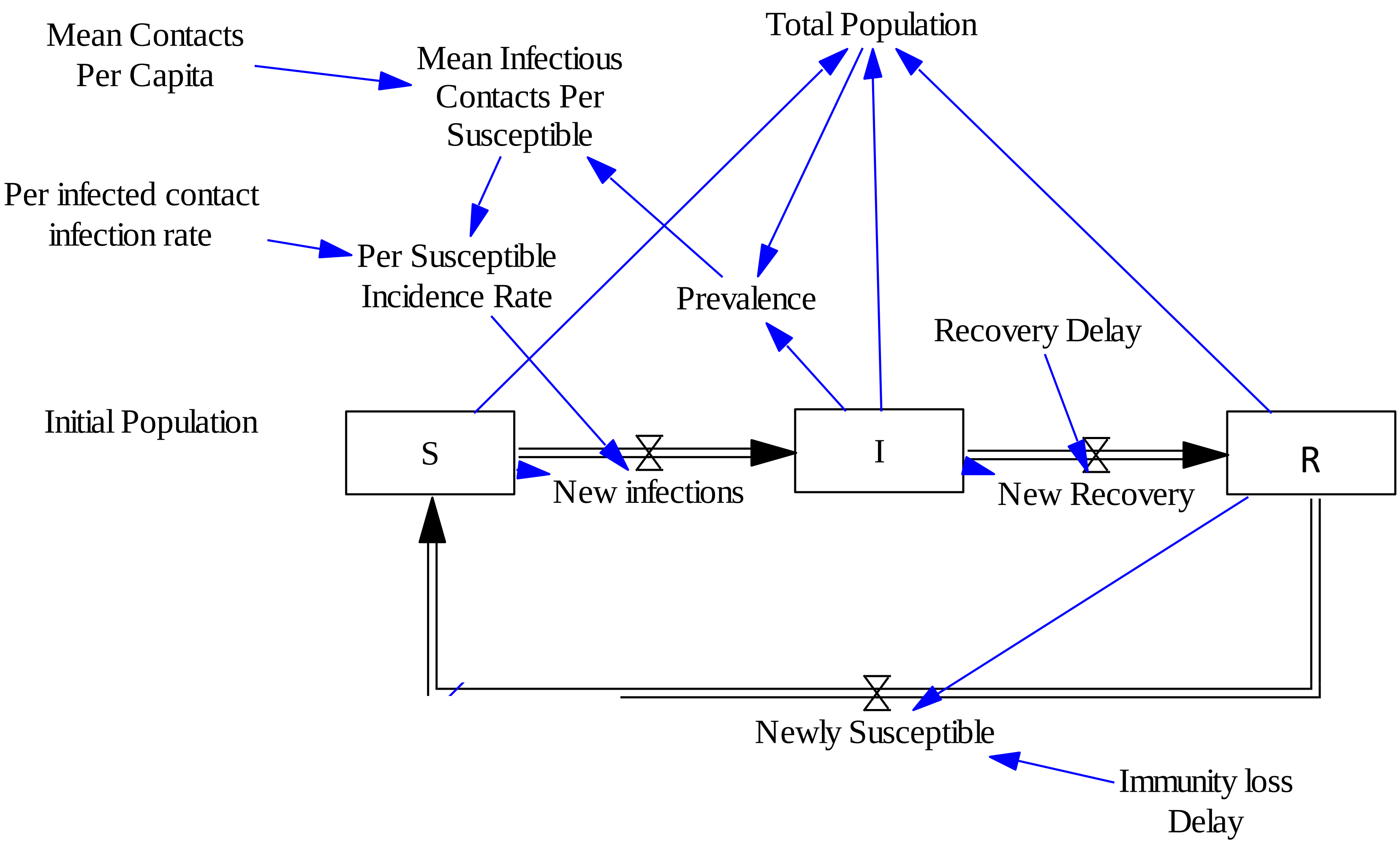
What if stock doesn't start empty?

Decays at first (no inflow) & then output responds with delayed version of input

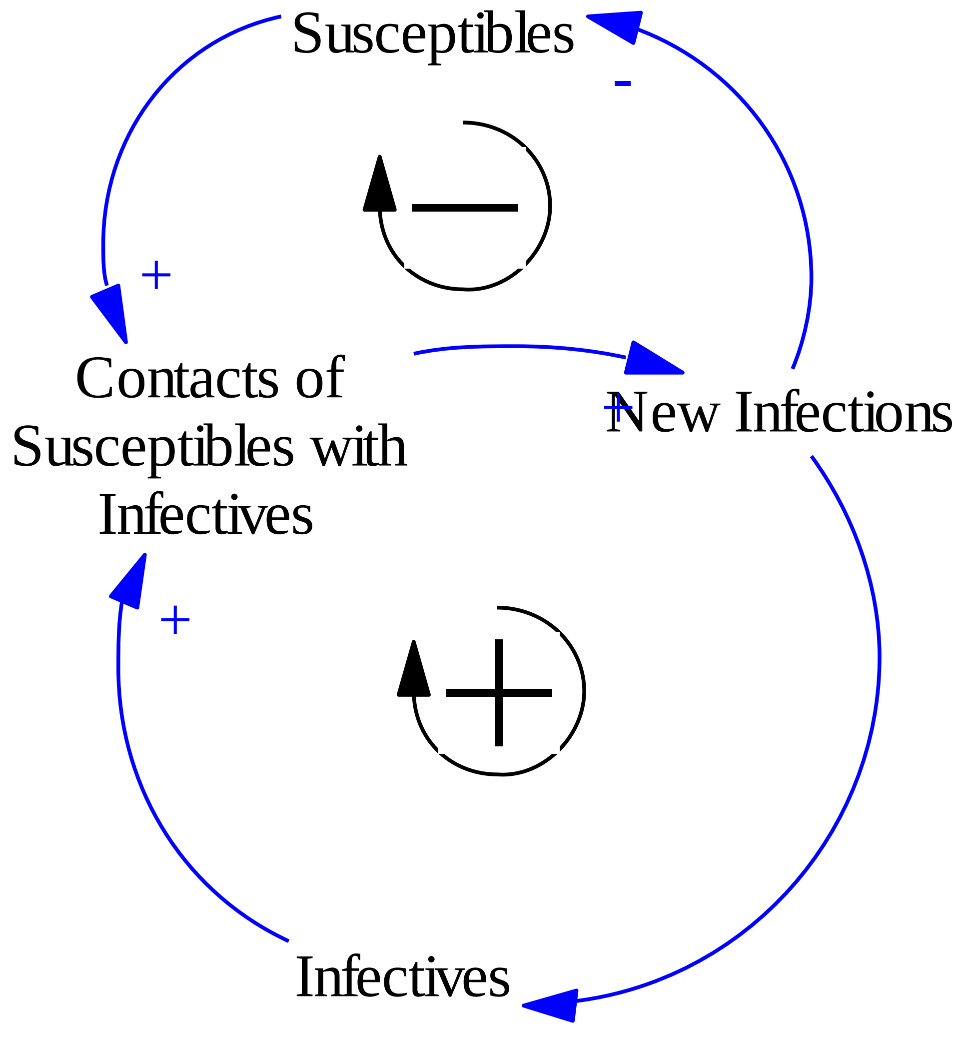




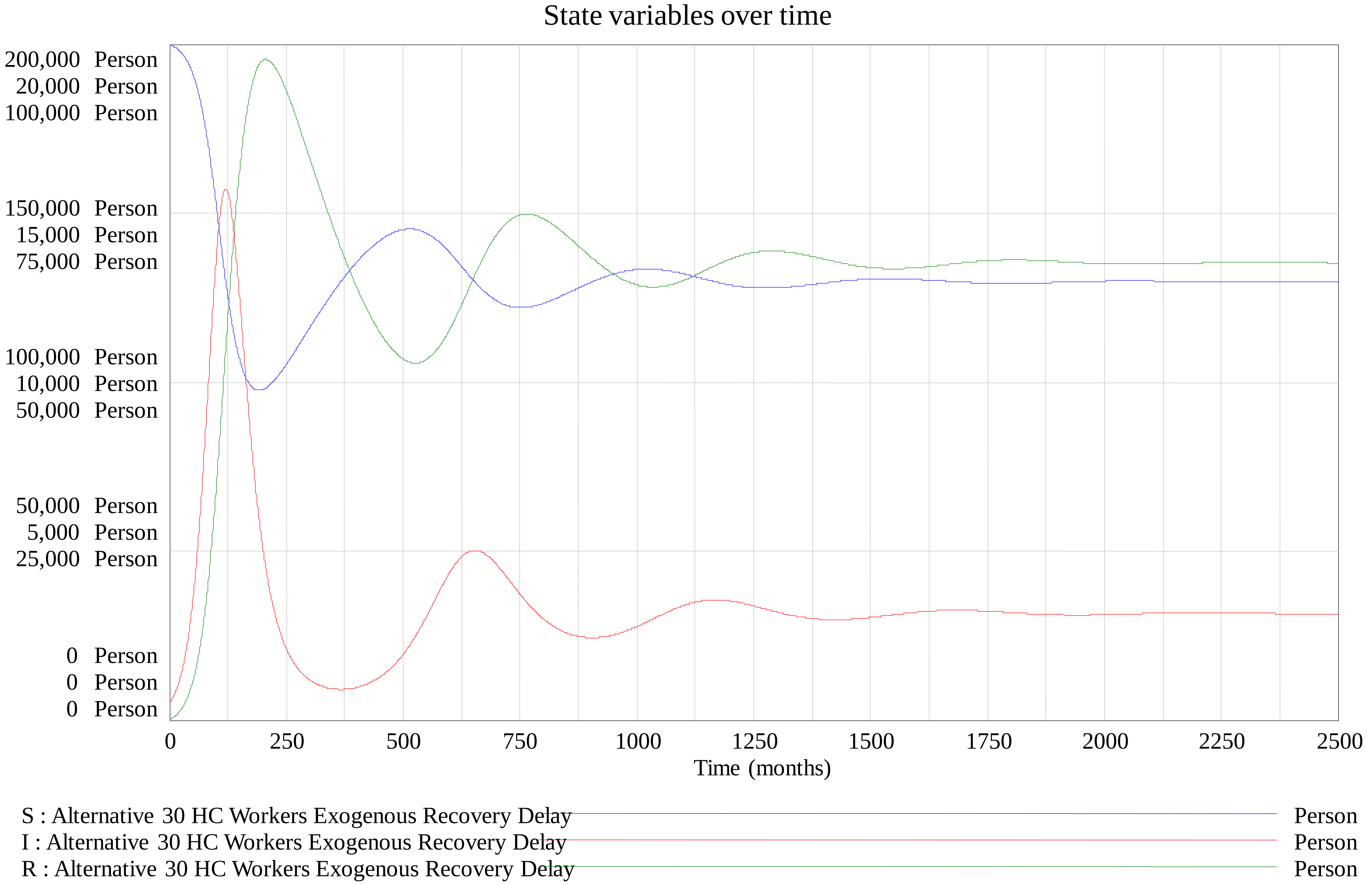
Simple SIT Model



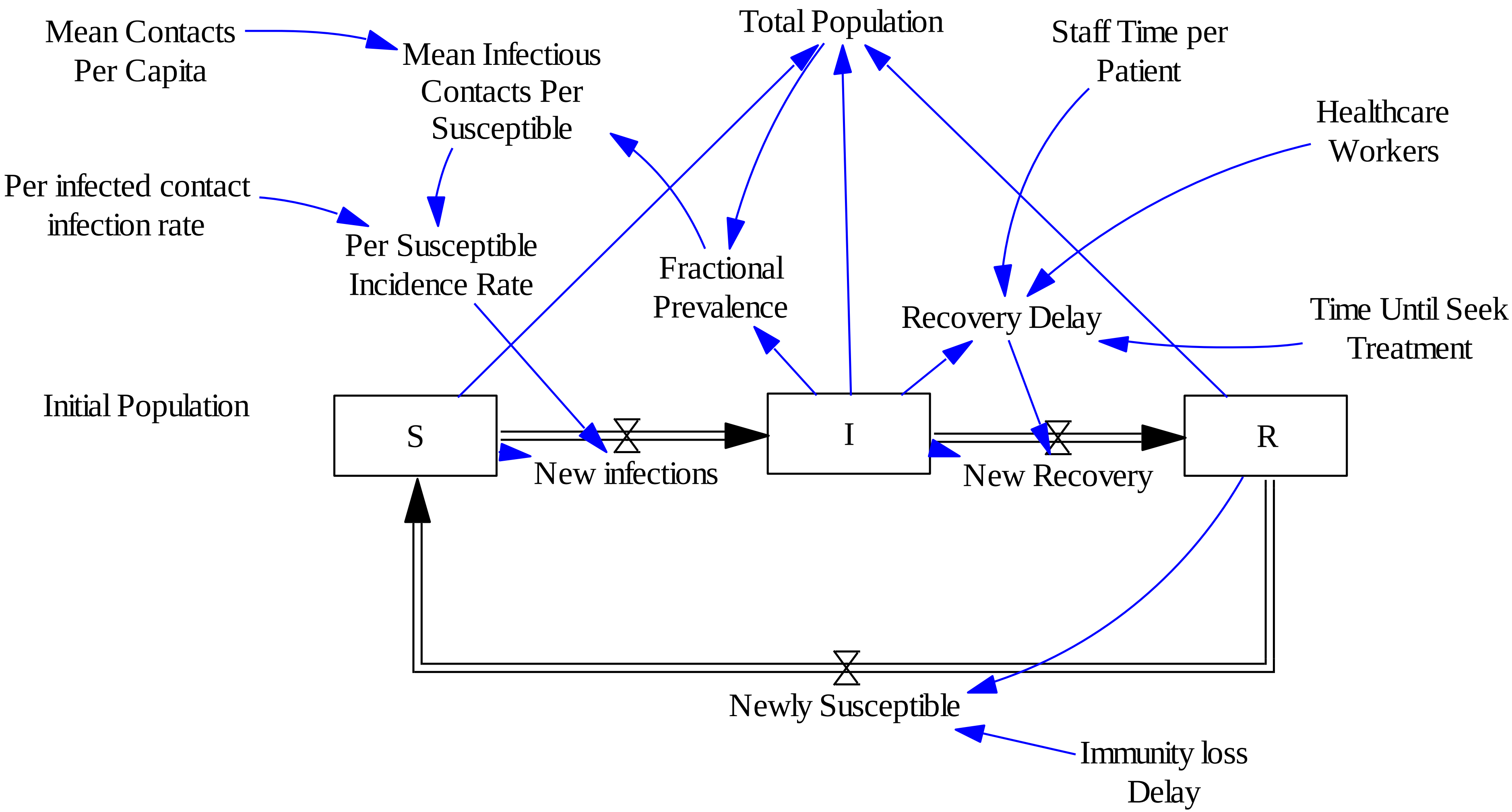
Classic Feedbacks



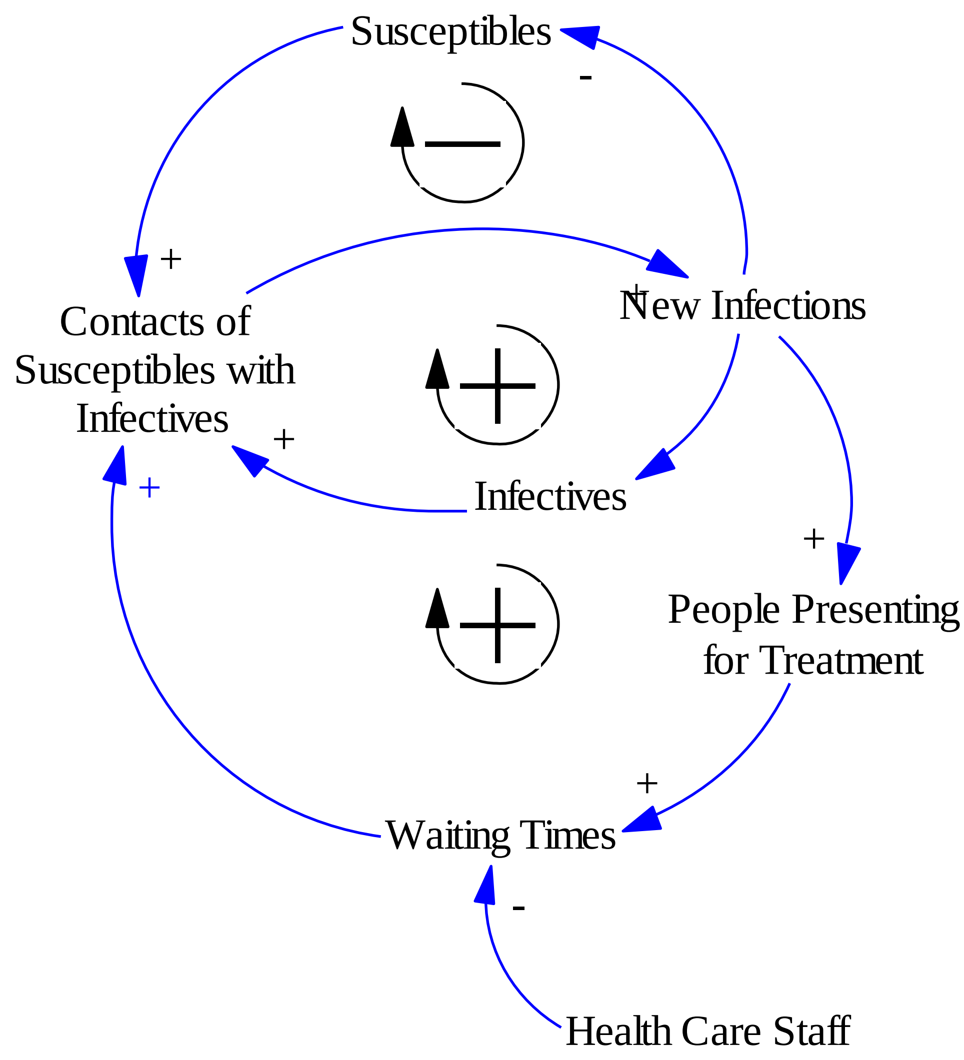
Dynamics


State variables over time
200,000 Person
20,000 Person
100,000 Person
150,000 Person
15,000 Person
75,000 Person
100,000 Person
10,000 Person
50,000 Person
50,000 Person
5,000 Person
25,000 Person
0 Person
0 Person
0 Person
0
250
500
750
1000
1250
1500
1750
2000
2250
2500
Time (months)
S : Alternative 30 HC Workers Exogenous Recovery Delay Person
I : Alternative 30 HC Workers Exogenous Recovery Delay Person
R : Alternative 30 HC Workers Exogenous Recovery Delay Person

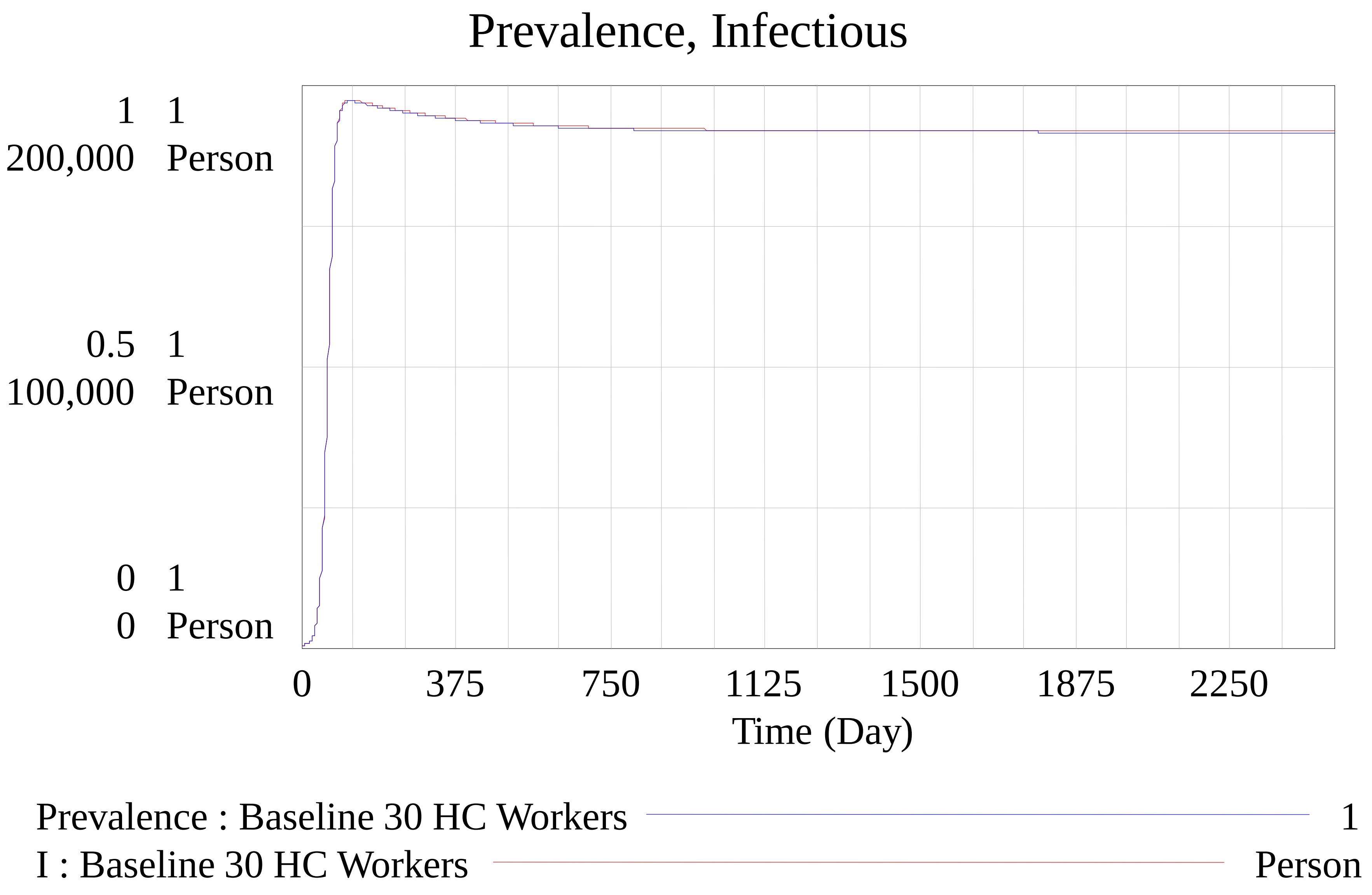
Broadening the Model Boundaries: Endogenous Recovery Delay



Broadening the Model Boundaries: Endogenous Recovery Delay



A Different Behaviour Mode



Structure as Shaping Behaviour

- System structure is defined by
  - Stocks
  - Flows
  - Connections between them
- Nonlinearity: The behaviour of the whole is more than the sum of the behaviour of the parts
  - "Emergent" behaviour would not be anticipated from simple behaviour of each piece in turn
- Stock and flow structure (including feedbacks) of a system determines the qualitative behaviour modes that the system can take on