

# Dimensional Reasoning & Dimensional Consistency Testing

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# Talk Outline

- Motivations
- Dimensional Systems
- Dimensional Analysis
- Examples
- Discussion

# Motivations

- General
  - Dimensional analysis (DA) critical historically for
    - Scoping models
    - Formulating models
    - Validating models
    - Calibrating models
  - Systems modeling community has made important but limited use of DA
  - Strong advantages from & opportunities for improved DA use
- Specific
  - Performance concerns for public health models

# Dimensions and Units

- *Dimensions* describe semantic category of referent
  - e.g. Length/Weight/Pressure/Acceleration/etc.
  - Describe referent
  - Independent of size (or existence of) measure
  - No conversions typical between dimensions
  - A given quantity has a unique dimension
- *Units* describe references used in performing a particular measurement
  - e.g. Time:  $\mu$ Seconds/Weeks/Centuries
  - This is *metadata*: Describes measured value
  - Relates to a *particular dimension*
  - Describe measurement of referent
  - Dimensional constants apply between units
  - A given quantity can be expressed using many units
  - Even dimensionless quantities can have units

# Units & Dimensions

- Frequency
  - Dimension:  $1/\text{Time}$
  - Units: 1/Year, 1/sec, etc.
- Angle
  - Dimension: “Dimensionless” (1, “Unit”)
  - Units: Radians, Degrees, etc.
- Distance
  - Dimension: Length
  - Units: Meters/Fathoms/Li/Parsecs

# Dimensional Homogeneity: Distinctions

- Adding items of different dimensions is semantically incoherent
  - Fatally flawed reasoning
- Adding items of different units but the same dimension *is* semantically sensible but numerically incorrect
  - Requires a conversion factor

# Structure of Dimensional Quantities

- Dimensional quantity can be thought of as a pair (value,  $m$ ) where  $\text{value} \in \mathfrak{R}$  and  $m \in \mathfrak{R}^d$
- Quantity's dimension/units can be represented as
  - Products of powers of “reference” dimensions/units  
Rate of water flow:  $L^3T^{-1}$
  - Vectors in a  $d$  dimensional vector space (of ref. dimens.)
    - Each index in the vector represents the exponent for that reference dimension/unit
- Dimension dictates the value scaling needed for *unit conversion*
  - A *dimensionless* quantity holds the same value regardless of measurement system
- Dimensional quantities have operations that are related to but more restricted than for e.g.  $\mathfrak{R}$

# A Particularly Interesting Dimensionality: “Unit” Dimension

- Recall: dimensions associated with quantities can be expressed as “product of powers”
- We term quantities whose exponents are all 0 as being of “unit dimension”
- Another term widely used for this is “Dimensionless”
  - This is somewhat of a misnomer, in that these quantities do have a dimension – just a very special one
    - Analogy: calling something of length 0 “lengthless”
- Such quantities are independent of unit choice



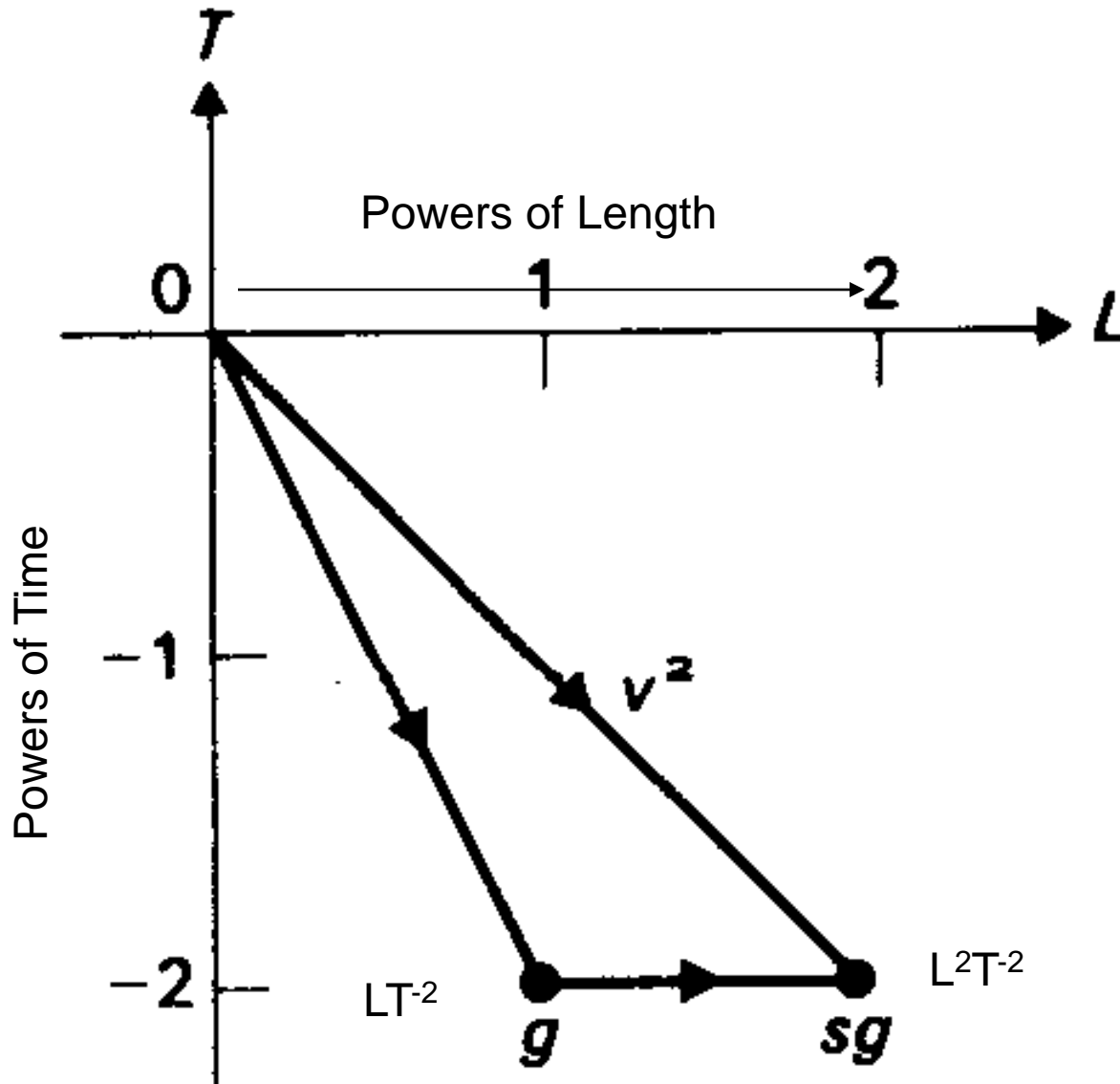
# Dimensionality & Unit Choice

- Exponent for dimension dictates the numerical value scaling required by *unit conversion*
  - Consider  $x=1 \text{ \$/ft}$  and  $y=1 \text{ \$/ft}^2$ 
    - Consider converting from feet to meters
      - $x=1 \text{ \$/ft} * (1\text{ft}/1\text{m}) \approx 3.208 \text{ \$/m}$
      - $y= 1 \text{ \$/ft}^2 * (1\text{ft}/1\text{m})^2 \approx 10.764 \text{ \$/m}^2$
- ***A dimensionless quantity maintains the same numeric value regardless of measurement system***
  - Cf: Fraction = .1 (Unit Dimension)
  - $100 \text{ ft}^2/1000 \text{ ft}^2 =.1$

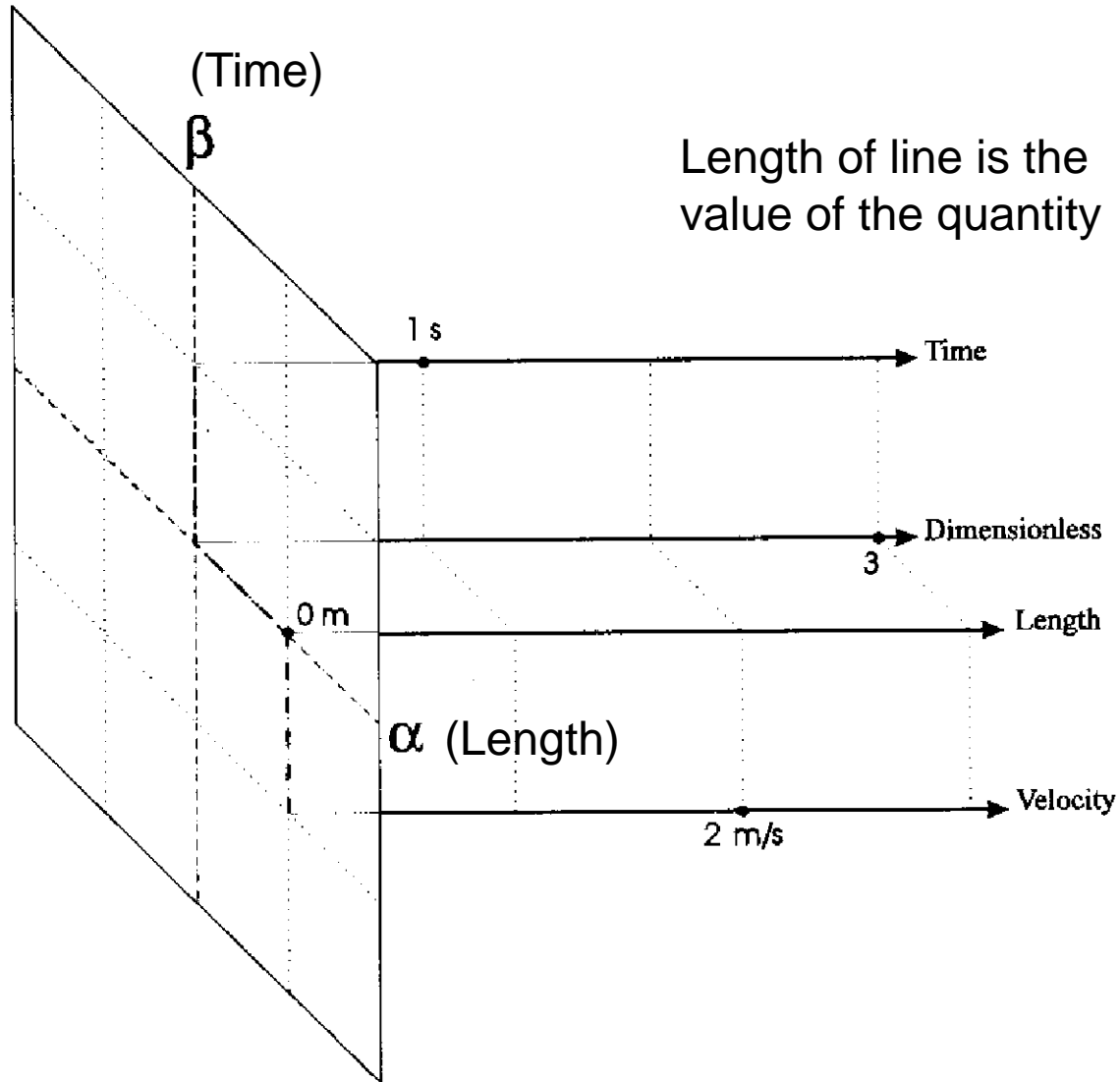
# Common Quantities of Unit Dimension

- Fractions of some quantity
- Likelihoods (probabilities)

# Dimensional Space



# Quantities in Dimension Space



Treating all quantities as dimensionless loses information (projects purely onto the z dimension)

# Stock-Flow Dimensional Consistency

- Invariant: Consider a stock and its inflows and outflows. For any flow, we must have  
$$[\text{Flow}] = [\text{Stock}] / \text{Time}$$
- This follows because the Stock is the integral of the flow
  - Computing this integral involves summing up many timesteps in which the value being summed is the flow multiplied by time.

# Seeking Hints as to the Dimension Associated w/a Quantity

- How is it computed in practice?
  - What steps does one go through to calculate this?  
Going through those steps with dimensions may yield a dimension for the quantity
- Would its value need to be changed if we were to change diff units (e.g. measure time in days vs. years)?
- Is there another value to which it is converted by some combination with other values?
  - If so, can leverage knowledge of dimensions of those other quantities

# Computing with Dimensional Quantities

- To compute the dimension (units) associated with a quantity, perform same operations as on numeric quantities, but using dimensions (units)
- We are carrying out the same operations in parallel in the numerics and in the dimensions (units).
  - With each operation, we can perform it twice
    - Once on the numerical values
    - Once on the associated dimensions

# Dimensional Homogeneity

- There are certain computations that are dimensionally inconsistent are therefore meaningless
- Key principle: Adding together two quantities whose dimensions differ is dimensionally “inhomogeneous” (inconsistent) & meaningless
- By extension

$a^b$  is only meaningful if  $b$  is dimensionless

$$\text{Derivation: } a^b = ((a/e)e)^b = (a/e)^b e^b = (a/e)^b (1 + b + b^2/2 + b^3/3 + \dots)$$

*The expression on the right is only meaningful if  $[b]:1$*



# Dimensional Notation

- Within this presentation, we'll use the notation  
[x]: D to indicate quantity x is associated with dimension D
- For example,  
[x]: \$  
[y]: Person/Time  
[z]: 1

# Example

$$\frac{a+(b*c)}{d}$$

Suppose further that

[a]: Person

[b]: Person/Time

[c]: Time

[d]: \$

To compute the dimensions, we proceed from “inside out”, just as when computing value

- $[b*c]=[b]*[c]=$   
 $(\text{Person}/\text{Time})*\text{Time}=\text{Person}$
- $[a+(b*c)]=[a]+[b*c]=\text{Person} + \text{Person}=\text{Person}$
- Thus, the entire expression has dimension  
 $[a+(b*c)/d] = [a+(b*c)/d]/[d]$   
 $=\text{Person}/\$$

# Lotka Volterra model

- Variables Dimensions  $\dot{H} = -\beta HF + \alpha H$   
[ $\beta$ ]: 1/(Fox \* Time)  
[ $\gamma$ ]: 1/(Hare \* Time)  $\dot{F} = \gamma HF - \delta F$   
[ $\delta$ ],[ $\alpha$ ]: 1/Time
- Cf: Frequency of oscillations: [ $\lambda$ ] : (1/Time)
  - Clearly cannot depend on  $\beta$  or  $\gamma$ , because
    - These parameters would introduce other dimensions
    - Those dimensions could not be cancelled by any other var.
- The exponent of Time in [ $\lambda$ ] is -1
- By symmetry, the period must depend on both  $\alpha$  and  $\delta$ , which suggests

$$\sqrt{\alpha\delta}$$

# Classic SIR model

$$\dot{S} = -cS \left( \frac{I}{S+I+R} \right) \beta$$

$$\dot{I} = cS \left( \frac{I}{S+I+R} \right) \beta - \frac{I}{\mu}$$

$$\dot{R} = \frac{I}{\mu}$$

- Variables Dimensions

[S]=[I]=[R]: Person

[ $\beta$ ]: 1 (A likelihood!)

[c]: (Person/Time)/Person=1/Time

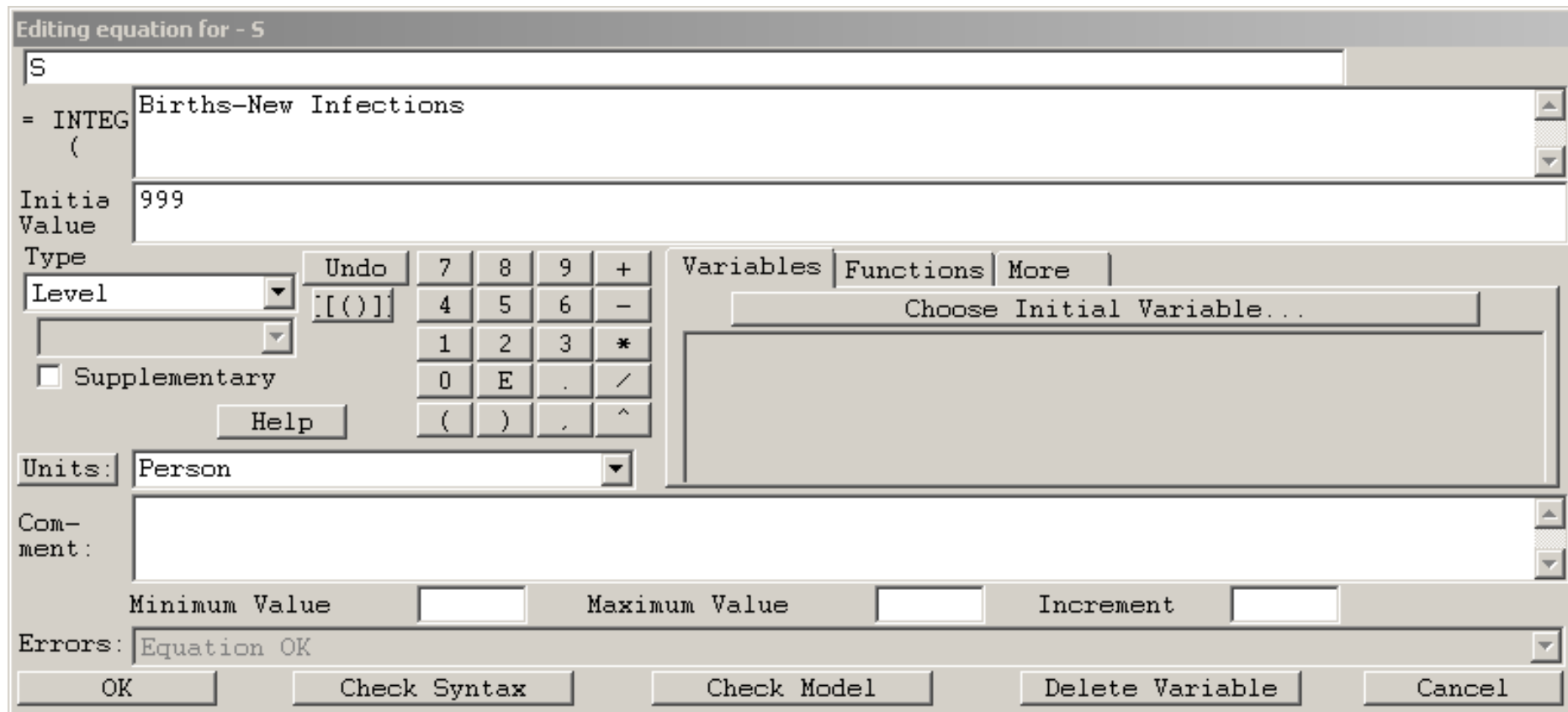
(Just as could be calculated from data on contacts by  $n$  people over some time interval)

[ $\mu$ ]: Time

Note that the force of infection  $\lambda = c \left( \frac{I}{S+I+R} \right) \beta$  has units 1/Time, which makes sense

- Firstly, multiplying it by S must give rate of flow, which is Person/Time
- Secondly, the reciprocal of such a transition hazard is just a mean duration in the stock, which is a Time => dimension must be 1/Time

# Indicating Units Associated with a Variable in Vensim



# Accessing Model Settings

The screenshot displays the Vensim software interface for a model named "Dimensional Analysis Example.mdl". The "Settings..." menu is open, showing options such as "Check Model (Ctrl+T)", "Units Check (Ctrl+U)", "Reform and Clean", "Compare to...", "Simulate (Ctrl+R)", "Start SyntheSim (Ctrl+B)", "Reality Check", "Stop Simulation", "Import Dataset...", and "From .dat format...".

The main workspace shows a compartmental model diagram with three states: S (Susceptible), I (Infected), and R (Recovered). The diagram includes the following elements:

- Births:** An inflow into the S compartment, controlled by a valve and labeled "Births".
- Force of Infection:** A flow from S to I, controlled by a valve and labeled "New Infections".
- New Recoveries:** A flow from I to R, controlled by a valve and labeled "New Recoveries".
- Total Population:** A stock at the bottom, which is the sum of S, I, and R.
- Parameters and Stocks:** "Birth Rate" (stock), "Likelihood of Transmission per Discordant Contact" (stock), "Prevalence of Infection" (stock), and "Mean time Until Recovery" (stock).
- Inputs:** "Contacts per Year" and "<Total Population>" are inputs to the "Prevalence of Infection" stock.

Blue arrows indicate the flow of information from the stocks and parameters to the flows and other stocks in the model.

# Choosing Model Time Units

The screenshot displays the Vensim software interface. The main window title is "Vensim: Dimensional Analysis Example.mdl Var:5". The menu bar includes "File", "Edit", "View", "Layout", "Model", "Options", "Windows", and "Help". The toolbar contains various icons for file operations and simulation control. The "Current" model is selected.

The "Model Settings - use Sketch to set initial causes" dialog box is open, showing the "Time Bounds" tab. The settings are as follows:

- Time Bounds for Model
  - INITIAL TIME = 0
  - FINAL TIME = 100
  - TIME STEP = 0.007812!
- Save results every TIME STEP
- or use SAVEPER =
- Units for Time: Year (selected in the dropdown menu)
- Integration Type: Year (selected in the dropdown menu)

A note at the bottom of the dialog box states: "NOTE: To change later, click on the Units Equiv tab or edit the equations for the above parameter." Buttons for "OK" and "Cancel" are visible.

The background diagram is a stock-and-flow model. It features a stock labeled "Total Population" at the bottom. Three flows contribute to this stock: "Births" (inflow), "New Infections" (inflow), and "New Recoveries" (inflow). The "Births" flow is controlled by a control variable "Birth Rate". The "New Recoveries" flow is controlled by a control variable "Mean time Until Recovery". A stock labeled "R" is shown at the top right, with a flow from it to "New Recoveries".

# Setting Unit Equivalence

Model Settings - use Sketch to set initial causes

Time Bounds | Info/Pswd | Sketch | Units Equiv | XLS Files | Ref Modes

\$.Dollar,Dollars,\$s  
Day,Days  
Hour,Hours  
Month,Months  
Person,People,Persons  
Unit,Units  
Week,Weeks  
Year,Years

Use strictest testing

Delete Selected

Modify Selected

Add Editing

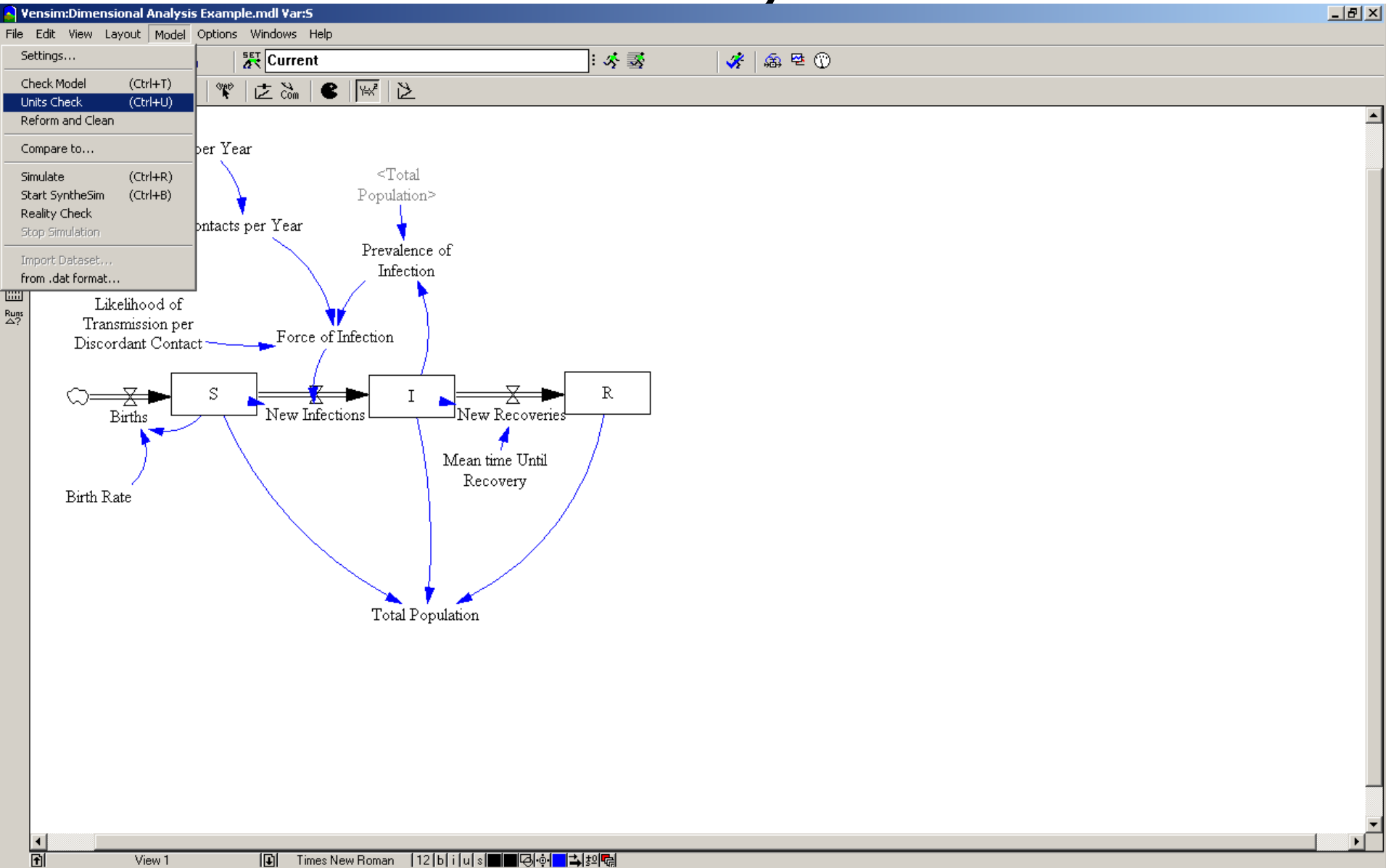
Replace these with the New Model default synonyms

Make these synonyms the New Model default synonyms

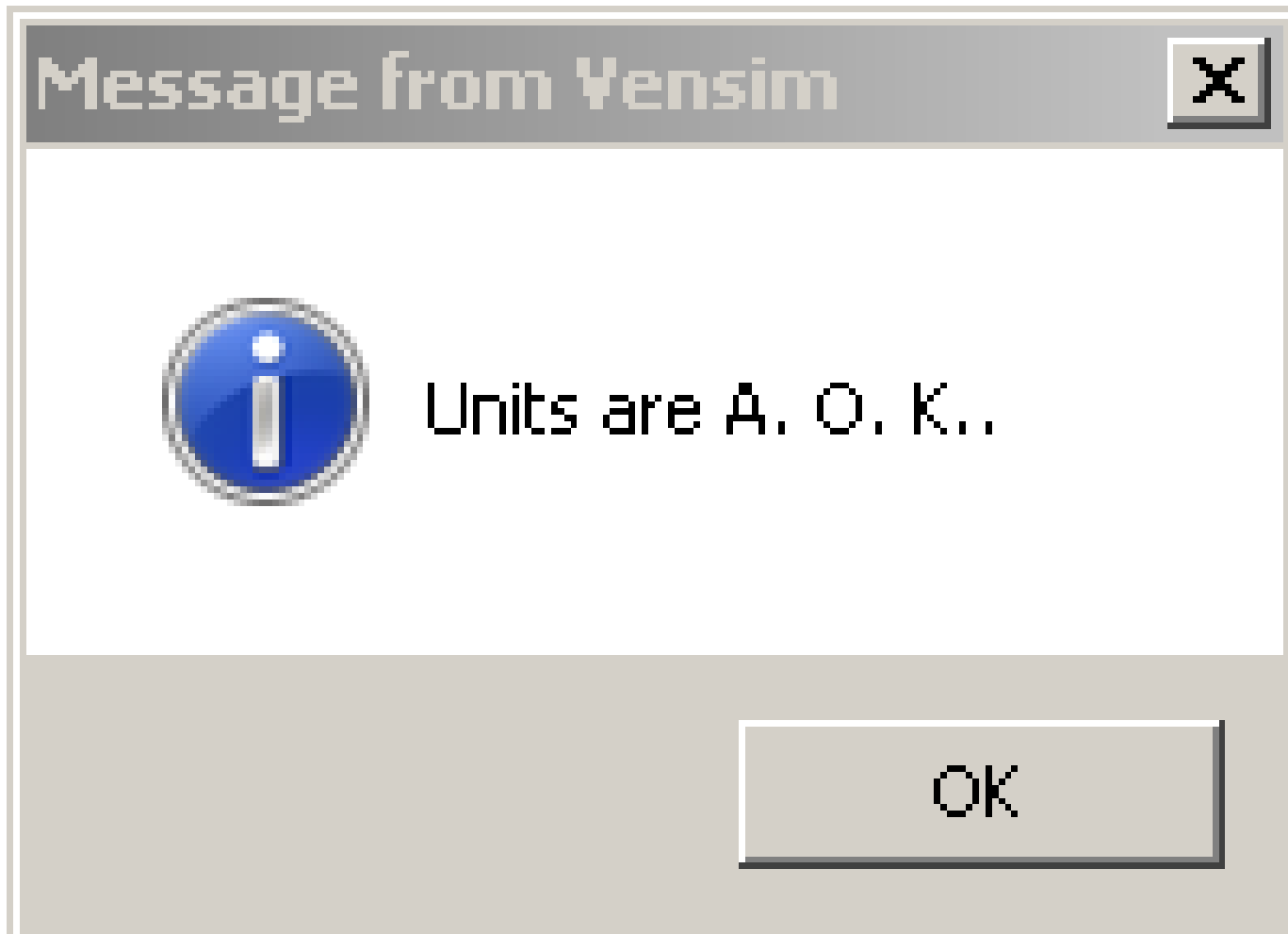
OK Cancel



# Requesting a Dimensional Consistency Check



# Confirmation of Unit Consistency



# Indication of (Likely) Dimensional Inconsistency

The screenshot shows the Vensim software interface with a 'Units Checking' dialog box open. The dialog box contains the following text:

```
*****  
Error in units for the following equation:  
Contacts per Year =  
  Contacts per Day  
    * Days per Year  
Contacts per Year --> 1/Year  
Contacts per Day --> Contact/Day  
Days per Year --> Days/Year  
  
Analysis of units error:  
Right hand and left hand units do not match  
Contacts per Year  
Has Units: Dimensionless/Year  
Contacts per Day  
  * Days per Year  
Has Units: Contact/Year
```

A smaller dialog box titled 'Stop from Vensim' is also open, displaying a red 'X' icon and the message: 'There were 1 unit errors discovered.' with an 'OK' button.

Blue arrows point from the 'Contacts per Year' and 'Contacts per Day' lines in the 'Units Checking' dialog to the 'Total Population' label in the background, indicating the source of the error.

# Vensim Interface

- Vensim will perform dimensional simplification via simple algebra on dimensional expressions
  - E.g. Person/Person is reduced to 1
- In some vensim modes, when the mouse hovers over a variable, Vensim will show a pop-up “tab tip” that shows the dimension for that variable
- Vensim can check many aspects of dimensional consistency of a model

# Vensim Capabilities

- Associate variables with units
- Define new units (beyond built-in units)  
e.g. Person, Deer, Bird, Capsule
- Define unit equivalence  
e.g. “Day”, “Days”