One-Dimensional Magnetotelluric Inversion with Radiation Boundary Conditions

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Abstract

We present an algebraic method of solving the magnetotelluric inverse problem for the case of one-dimensional conductivity profiles in the class D^+. We show that the typically examined Dirichlet boundary conditions are a limiting case of the radiative boundary conditions introduced by Srnka and Crutchfield. By examining the analogous inverse inhomogeneous string problem studied by Kreĭn we demonstrate the usefulness of the conductivity class D^+. Results of the inversion procedure are presented, as well as a discussion of the continued fraction expansions resulting from the more general boundary conditions. The presentation presupposes no knowledge of magnetotellurics.

1 Introduction

Scientists have made inferences about the electrical structure of the Earth based on measurements of the electric and magnetic field at its surface since the end

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of the 19th century [25]. The magnetotelluric method is based on measurements made at a single location on the surface of the Earth; as first noted by Cagniard [9], Tikhonov [32], and Rikitake [30] the ratio of the electric and magnetic field at the surface contains information about the conductivity structure of the Earth.

One–dimensional magnetotellurics is one of the few geophysical inverse problems that is exactly solvable [34]. Exact solutions have been given by, for example, Bailey [2], Weidelt [34], and Parker [27]. The various approaches require different degrees of smoothness of the conductivity profile; the modification of the Gelfand–Levitan inversion given by Weidelt requires C^2 conductivity profiles, whereas the approach of Parker examines conductivity profiles in the class D^+, i.e., weighted sums of Dirac deltas. This paper argues that the class D^+ is a natural setting for the magnetotelluric inverse problem, and that profiles in D^+ can be meaningfully interpreted.

After deriving the equations that govern the electric (and magnetic) fields inside of the Earth, we describe the inversion process for conductivity profiles of class D^+ and show some examples of the inversion procedure. The class D^+ has been shown by Parker to contain the optimal solution to the magnetotelluric problem for finite datasets [27]. Conductivity profiles in D^+ have attracted some degree of criticism in the magnetotelluric literature on the basis of being either geophysically unreasonable or insufficiently smooth [28, 14, 29]. In this paper we argue that such concerns are unwarranted, and that the class D^+ is indeed worthy of the large amount of attention given to it in the literature.

2 Background Assumptions

To derive a model for the behaviour of the electric field within the Earth we make several simplifying assumptions. It is assumed that the atmospheric current of the Earth can be modeled as a uniform current sheet of infinite extent at a given height h above the surface of the Earth; see [18] for a discussion of this assumption. We assume that the conductivity σ of the Earth is a function of the depth z only. We also assume that the Earth is composed of linear dielectrics, meaning the elementary form of Ohm's law $\boldsymbol{J} = \sigma(z)\boldsymbol{E}$ holds within the Earth. On physical grounds we assume that $\sigma(z)$ is nonnegative for all z. Finally we assume that the electrical permittivity ϵ and magnetic permeability μ within each layer can be treated as being equal to the permittivity and permeability of free space, ϵ_0 and μ_0. For numerical computations we use SI units, so that $\mu_0 = 4\pi \times 10^{-7}$ NA^{-2}, and $\epsilon_0 = (\mu_0 c_0^2)^{-1}$, where c_0 is the speed of light in vacuum.

3 Governing Equation of the Electric Field

We choose coordinates in which the origin is contained within the Earth at a depth L. We use a rectangular coordinate system, with the homogeneous current sheet directed along the x-axis.[1] The current sheet is located at a height h above the surface of the Earth, so it can be expressed as

$$\boldsymbol{J}(x,y,z,t) = J(t)\delta(z-(h+L))\hat{\boldsymbol{x}}. \tag{1}$$

An application of Faraday's law shows that the current sheet (1) induces an electric field directed parallel to the y-axis. Ampére's law shows that the induced electric field induces a magnetic field within the Earth along the x-axis. Subsequent induced fields are directed in the same directions, implying that $\boldsymbol{E}(z,t) = E(z,t)\hat{\boldsymbol{y}}$ and $\boldsymbol{H}(z,t) = H(z,t)\hat{\boldsymbol{x}}$. The orthogonality of the electric and magnetic fields suggests a plane wave solution to Maxwell's equations.

Maxwell's equations in linear media take the form

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_f}{\epsilon_0}, \tag{2a}$$

$$\nabla \times \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t}, \tag{2b}$$

$$\nabla \cdot \boldsymbol{H} = 0, \tag{2c}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_f + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}, \tag{2d}$$

where ρ_f is the free charge density and $\boldsymbol{J}_f$ is the free current. The continuity equation, which is a consequence of equations (2a) and (2d), is

$$\nabla \cdot \boldsymbol{J}_f = -\frac{\partial \rho_f}{\partial t} \tag{3}$$

and has the clear interpretation of conservation of charge. Substituting Ohm's law ($\boldsymbol{J}_f = \sigma(z)\boldsymbol{E}$) into (3) gives

$$\nabla\sigma(z)\cdot\boldsymbol{E} + \frac{\sigma(z)}{\epsilon_0}\rho_f = -\frac{\partial \rho_f}{\partial t}. \tag{4}$$

By assumption $\nabla\sigma(z)$ points in the $\hat{\boldsymbol{z}}$ direction, so $\nabla\sigma(z)\cdot\boldsymbol{E} = 0$, implying that

$$\rho_f(z,t) = \rho_f(z,0)e^{-\sigma(z)t/\epsilon_0}. \tag{5}$$

For most physically reasonable Earth compositions $\sigma(z)$ is between 10^{-4} and 10^{0} S/m, and as previously mentioned $\epsilon_0 = 8.85\times10^{-12}$ Ss/m, so for all t large enough equation (5) shows that $\rho_f(z,t)\approx 0$. For the remainder of the paper we will assume $\rho_f(z,t) = 0$.

Putting a plane wave ansatz $\boldsymbol{E} = E(z)e^{i\omega t}\hat{\boldsymbol{y}}$, $\boldsymbol{H} = H(z)e^{i\omega t}\hat{\boldsymbol{x}}$ into (2b) and (2d) and using Ohm's law gives

$$\nabla\times\boldsymbol{E} = -i\omega\mu_0\boldsymbol{H}, \tag{6}$$

$$\nabla\times\boldsymbol{H} = \sigma(z)\boldsymbol{E} + i\omega\epsilon_0\boldsymbol{E}. \tag{7}$$

[1] This research was conducted with exploratory magnetotellurics in mind, in which case the depths of investigation are sufficiently small for the curvature of the Earth to be neglected. The methods described are equally valid for greater depths; to take into account the curvature of the Earth the transformations described by Weidelt in [34] can be used.

We make what is known as the *quasistatic approximation* by neglecting displacement current term $i\omega\epsilon_0\boldsymbol{E}$ of (7). This is justified by comparing the magnitude of the displacement current and $\sigma(z)\boldsymbol{E}$. For reasonable ω (waves with period between 10^{-4} and 10^5 seconds), $\sigma(z)/\omega\epsilon_0 \gtrsim \mathcal{O}(10^6)$. Equation (7) becomes

$$\nabla \times \boldsymbol{H} = \sigma\boldsymbol{E}. \tag{8}$$

To arrive at a differential equation for $\boldsymbol{E}$ we take the curl of (2b), use the vector identity $\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$, and substitute (8) into the left–hand side of (8) to obtain

$$\frac{d^2E(z)}{dz^2} = i\omega\mu_0\sigma(z)E(z). \tag{9}$$

As has been noted by several authors, this is essentially the equation for a vibrating string with an inhomogeneous mass distribution [27, 3]. The inverse problem for a vibrating string has been thoroughly studied, particularly by M.G. Kreĭn in the 1950s [4, 19], and more recently in connection with the Camassa-Holm equation [6]. The relationship between the inhomogeneous string and magnetotelluric inverse problems is discussed further in Section 7.

4 Boundary Conditions

The boundary conditions are based upon the requirement that

$$\lim_{z\to-\infty} E(z) = 0,$$

which corresponds to the physical condition that there can be only a finite amount of energy deposited into the Earth by the electric field.

We suppose that for $z < 0$ the Earth is composed of a homogeneous medium with constant conductivity $\sigma_B^2 \in (0, \infty)$. The general solution of (9) for $z < 0$ can therefore be written as a sum of exponentials

$$E(z) = A_1 e^{\sqrt{i\omega\mu_0\sigma_B^2}z} + A_2 e^{-\sqrt{i\omega\mu_0\sigma_B^2}z}. \tag{10}$$

The requirement that $\lim_{z\to-\infty} E(z) = 0$ implies that $A_2 = 0$, provided we take the root of i with positive real part. Evidently

$$\frac{dE(z)}{dz} = \sqrt{i\omega\mu_0}\sigma_B E(z), \tag{11}$$

so in particular

$$E(0) = E_0, \quad E'(0) = \sqrt{i\omega\mu_0}\sigma_B E_0, \tag{12}$$

and these are taken as the boundary conditions for (9). By convention we choose $E_0 \in \mathbb{R}^+$.

Note that by setting $\sigma_B = 0$ in equation (11) we get Neumann boundary conditions. Alternatively, by letting $\sigma_B \to \infty$, we get Dirichlet boundary conditions. The Dirichlet conditions are the most studied in the magnetotelluric literature; see for example [27, 34, 2, 3]. The more general boundary conditions (12), called the *radiation boundary conditions*, are studied in [31] as well as in a forthcoming paper of Beals, Sattinger, and Szmigielski [5].

5 Spectral Properties

The spectrum of (9) is of critical importance in what follows. In this section we present relevant results from a larger project in preparation [5]. The results we report here are those which are central to numerical studies of the magnetotelluric problem.

To simplify later expressions we set $\lambda^2 = i\omega\mu_0$, and for clarity we explicitly include the dependence of the electric field on λ by writing $E(z) = E(z;\lambda)$. We note that $E(z;\lambda)$ is necessarily an entire function of order one in λ; see [33]. Letting primes represent differentiation with respect to z, (9) becomes

$$E''(z;\lambda) = \lambda^2\sigma(z)E(z;\lambda). \tag{13}$$

The boundary conditions are

$$E(0;\lambda) = E_0, \quad E'(0;\lambda) = \sigma_B E_0 \lambda, \tag{14}$$

with $E_0 \in \mathbb{R}$. We search for eigenvalues λ^* such that $E(L;\lambda^*) = 0$ and $\lambda^\dagger$ such that $E'(L;\lambda^\dagger) = 0$; recall that the right endpoint $z = L$ lies at the surface of the Earth.

Proposition 1. *The real parts of all eigenvalues are negative, and any eigenvalues with nonzero imaginary parts occur in conjugate pairs.*

Proof. Multiplying (13) by $\overline{E(z;\lambda)}$, multiplying the conjugate of (13) by $E(z;\lambda)$, taking the difference between the two equations and then integrating from 0 to L gives

$$\begin{aligned}(\lambda^2 - \overline{\lambda}^2)\int_0^L \sigma(z)|E(z;\lambda)|^2\,dz &= \int_0^L \left(\overline{E(z;\lambda)}E''(z;\lambda) - E(z;\lambda)\overline{E''(z;\lambda)}\right)dz \\ &= \left(\overline{E(z;\lambda)}E'(z;\lambda) - E(z;\lambda)\overline{E'(z;\lambda)}\right)\Big|_0^L. \end{aligned}\tag{15}$$

Applying the boundary conditions (14) gives

$$\begin{aligned}&\overline{E(L;\lambda)}E'(L;\lambda) - E(L;\lambda)\overline{E'(L;\lambda)} \\ &\qquad = (\lambda - \overline{\lambda})\left(\sigma_B|E_0|^2 + (\lambda + \overline{\lambda})\int_0^L \sigma(z)|E(z;\lambda)|^2\,dz\right). \end{aligned}\tag{16}$$

Suppose $\mathfrak{Im}(\lambda^*) \neq 0$. Then setting $\lambda = \lambda^*$ (16) becomes

$$0 = 2i\,\mathfrak{Im}(\lambda^*)\left(\sigma_B|E_0|^2 + 2\,\mathfrak{Re}(\lambda^*)\int_0^L \sigma(z)|E(z;\lambda^*)|^2\,dz\right),$$

which implies $\mathfrak{Re}(\lambda^*) < 0$. The symmetric occurrence of $E(L;\lambda)$ and $E'(L;\lambda)$ in (16) implies that $\mathfrak{Re}(\lambda^\dagger) < 0$ as well.

The boundary conditions (14) and differential equation (13) imply that $\overline{E(z;\lambda)} = E(z;\overline{\lambda})$. In particular for $\lambda^* \in \mathbb{R}$ $E(z;\lambda)$ is real. Multiplication of (13) by $E(z;\lambda)$ and integration from 0 to L gives

$$\lambda^2 \int_0^L \sigma(z) E(z;\lambda)^2 \, dz + \int_0^L E'(z;\lambda)^2 \, dz = E'(L;\lambda)E(L;\lambda) - |E_0|^2 \sigma_B \lambda. \quad (17)$$

Substitution of λ^* into (17) when $\lambda^* \in \mathbb{R}$ combined with the non-negativity of σ_B implies that $\lambda^* < 0$. The argument for $\lambda^\dagger \in \mathbb{R}$ is essentially identical. □

In the case of Dirichlet or Neumann boundary conditions at $z = 0$, more can be said about the spectrum.

Proposition 2. *For Dirichlet or Neumann boundary conditions the spectrum of* (13) *is purely imaginary, and all nonzero eigenvalues occur in conjugate pairs.*

Proof. We explicitly write down the boundary conditions. For the Neumann problem $\sigma_B = 0$, so

$$E(0;\lambda) = E_0, \quad E'(0;\lambda) = 0, \quad (18)$$

and for the Dirichlet problem, which formally corresponds to $\sigma_B = \infty$,

$$E(0;\lambda) = 0, \quad E'(0;\lambda) = E_0'. \quad (19)$$

We show first that the spectrum is purely imaginary for Dirichlet or Neumann boundary conditions. Let λ^* belong to the spectrum. Either of the boundary conditions (18) or (19) turn equation (15) into

$$({\lambda^*}^2 - \overline{\lambda^*}^2) \int_0^L \sigma(z) |E(z;\lambda)|^2 \, dz = 0, \quad (20)$$

which implies that λ^* is either purely real or purely imaginary.

Both the Neumann and Dirichlet boundary conditions imply that equation (17) becomes

$$\int_0^L E'(z;\lambda^*)^2 \, dz = -{\lambda^*}^2 \int_0^L \sigma(z) E(z;\lambda^*)^2 \, dz, \quad (21)$$

which implies that $\lambda^* \in i\mathbb{R}$, as for purely real and purely imaginary eigenvalues $\overline{E(z;\lambda)} = E(z;\lambda)$, so that $-{\lambda^*}^2$ is positive. □

Proposition 3. *For the cases of Dirichlet and Neumann boundary conditions the spectrum of* (13) *is simple, i.e., both the geometric and algebraic multiplicities of eigenvalues are one.*

Proof. Differentiate (13) with respect to λ and multiply the result by $E(z;\lambda)$, and subtract from this (13) multiplied by $E_\lambda(z;\lambda)$, where the subscript denotes differentiation with respect to λ. The result of this is

$$E(z;\lambda)E''_\lambda(z;\lambda) - E_\lambda(z;\lambda)E''(z;\lambda) = 2\lambda\sigma(z)E(z;\lambda)^2. \tag{22}$$

Recognizing that the left hand side of (22) is a total derivative, integration from 0 to L gives

$$(E(z;\lambda)E'_\lambda(z;\lambda) - E_\lambda(z;\lambda)E'(z;\lambda))|_0^L = 2\lambda\int_0^L \sigma(z)E(z;\lambda)^2\,dz. \tag{23}$$

Setting $\lambda = \lambda^*$ and using either the Neumann boundary conditions (18) or the Dirichlet boundary conditions (19) equation (23) specializes to

$$-E_\lambda(L;\lambda^*)E'(L;\lambda^*) = 2\lambda^*\int_0^L \sigma(z)E(z;\lambda^*)^2\,dz. \tag{24}$$

For all nonzero eigenvalues λ^* the right hand side of (24) is nonzero. This implies that both factors on the left hand side are nonzero, and as $E(z;\lambda)$ is an entire function of λ this implies that all eigenvalues λ^* of $E(z;\lambda)$ are simple. □

The following result can be utilized to simplify computations later:

Proposition 4. *For the Dirichlet and Neumann problems the residues b_j of the function $E'(L;\lambda)/E(L;\lambda)$ are purely imaginary, and residues corresponding to conjugate eigenvalues are conjugate.*

Proof. As mentioned at the beginning of section 5 $E(L;\lambda)$ is an entire function of order 1. Therefore we can write

$$\frac{E'(L;\lambda)}{E(L;\lambda)} = \frac{E'(L;\lambda)}{E_0\prod_j(1-\lambda/\lambda_j)}. \tag{25}$$

Since the zeroes of $E(L;\lambda)$ are all simple, we can also write (25) in a partial fraction decomposition:

$$\frac{E'(L;\lambda)}{E(L;\lambda)} = \sum_j \frac{a_j}{\lambda-\lambda_j}. \tag{26}$$

This gives

$$a_j = -\frac{\lambda_j E'(L;\lambda_j)}{E_0\prod_{\substack{k\\k\neq j}}(1-\lambda_j/\lambda_k)}. \tag{27}$$

Note also that

$$-\lambda_j E_\lambda(L;\lambda_j) = \prod_{\substack{k\\k\neq j}}(1-\lambda_j/\lambda_k). \tag{28}$$

Insert (27) into (28), and put the result and (21) into (24) to get

$$a_j = \lambda_j\frac{E'(L;\lambda_j)^2}{2\int_0^L E'(z;\lambda_j)^2\,dz} \tag{29}$$

The positivity of the integral and $E'(L;\lambda_j)^2$ gives the result. □

6 Discretization of the Conductivity Profile

For the remainder of the paper we focus on the case of conductivities in the class D^+, that is, conductivity profiles which are sums of weighted Dirac deltas:

$$\sigma_n(z) = \sum_{j=1}^{n} \sigma_j \delta(z - z_j). \tag{30}$$

As mentioned in Section 1, the conductivity class D^+ has been viewed with some skepticism from a geophysical point of view in the magnetotelluric literature [28, 14, 29]. We share the view that D^+ is not a physical class of solutions, so a few words explaining the utility of D^+ may be in order. One reason why D^+ is a good class of conductivity profiles is that the theory concerning the existence of a solution for a given data set is understood; see Parker [27] and especially Yee & Paulson [35]. However, the main reason for viewing discrete conductivity profiles as adequate approximations to physical conductivity profiles comes from converting (13) to an integral equation for $E(z;\lambda)$:

$$E(z;\lambda) = E_0 + z\lambda\sigma_B E_0 + \lambda^2 \int_0^z \int_0^y \sigma(x) E(x;\lambda)\, dx dy. \tag{31}$$

Integration by parts converts this expression to

$$E(z;\lambda) = E_0 + z\lambda\sigma_B E_0 + \lambda^2 \int_0^z (z - x) E(x;\lambda)\sigma(x)\, dx, \tag{32}$$

which is identical to (13) for all continuous $\sigma(z)$.

It is natural to consider (32) as the fundamental equation of our problem for discontinuous conductivity profiles. We further re-express the equation as follows. Define the cumulative conductance $M(z)$ of the Earth by

$$M(z) = \int_0^z \sigma(z)\, dz. \tag{33}$$

This is a nondecreasing function of z. We simply refer to the cumulative conductance function as the conductance when there is no risk of confusion. Equation (32) becomes

$$E(z;\lambda) = E_0 + z\lambda\sigma_B E_0 + \lambda^2 \int_0^z (z - x) E(x;\lambda)\, dM(x). \tag{34}$$

At this point what has been done resembles work done by Parker [27]. We make the following additional points:

(i) The integral equation (34) makes sense for any nondecreasing function $M(x)$. Using standard techniques [1] one easily shows that the solution to this integral equation exists and is unique. Moreover $dM(x)$ is a measure (as the distributional derivative of a function of bounded variation) and as such it can be approximated in the weak topology by sums of Dirac deltas. Therefore D^+ is dense in the space of positive measure $\mathcal{M}^+(\mathbb{R})$. There is also a more elementary way of thinking about the role of $\mathcal{M}^+$. A conductance $M(z)$ corresponding to a conductivity profile $\sigma_n(z)$ of type (30) converts this integral equation into a Riemann sum; it creates an approximate solution. For any continuous function $f(z)$ and continuous conductance $\sigma(z)$, it is possible to show that there exists a sequence of step function conductances $\{M_n(z)\}_n$ such that

$$\lim_{n\to\infty}\int_0^L f(z)dM_n(z) = \int_0^L f(z)dM(z)$$

Even more importantly, the following continuity results holds

Theorem 1. *Let σ_n be a sequence of Borel measures converging weakly to a Borel measure σ and let M_n and M be their respective cumulative distribution functions. Let $\{E_n\}$ be the sequence of solutions to the integral equation* (34). *Then $E_n \to E$.*

Proof. Let us set $E - E_n = u_n$. Then u_n satisfies:

$$u_n(z;\lambda) = \lambda^2\left(\int_0^z (z-x)u_n(x;\lambda)dM_n(x) + \int_0^z (z-x)E(x;\lambda)d(M-M_n)(x)\right)$$

Because $E(x;\lambda)$ is absolutely continuous and $dM_n \to dM$ weakly, the second term converges uniformly to 0 on every compact set. Let us define

$$\alpha_n(z;\lambda) = \left|\int_0^z (z-x)E(x;\lambda)d(M-M_n)(x)\right|.$$

Then we have the following inequality:

$$|u_n(z;\lambda)| \le \alpha_n(z;\lambda) + |\lambda|^2\int_0^z (z-x)|u_n(x;\lambda)|dM_n(x).$$

Then, by the integral form of Grönwall's inequality [16],

$$|u_n(z;\lambda)| \le \alpha_n(z;\lambda)+|\lambda|^2\int_0^z (z-x)\alpha_n(x;\lambda)\exp(\int_x^z (z-s)dM_n(s))dM_n(x)$$

which, in view of the uniform convergence $\alpha_n \to 0$ and the weak convergence of dM_n implies uniform convergence $u_n \to 0$ on every compact set. □

(ii) Expressing the problem as an integral equation shifts the focus from the conductivity profile to the conductance. An exact inversion procedure recovers the function $M(z)$, from which the conductivity $\sigma(z)$ is easily computed for continuous $\sigma(z)$. Without significant *a priori* assumptions incomplete datasets force us to consider $M(z)$ that are step functions, in which case we cannot uniquely recover $\sigma(z)$ — we can only determine the average of $\sigma(z)$ in between the jumps of $M(z)$.

7 Inhomogeneous Strings

As mentioned in Section 3, equation (9) is essentially the same as the equation governing the motion of a vibrating inhomogeneous string. The inhomogeneous string inverse problem is to determine the cumulative mass of the string as a function of position based on observations of the string at one endpoint at various frequencies. Much work on the string problem has been performed by M.G. Kreĭn [21, 19, 20, 22, 23, 24], as well as I.S. Kac and M.G. Kreĭn [17]. An alternative perspective on the string problem is provided by Dym and McKean [13]; see also [11, 12] . Aside from the connection to the magnetotelluric problem, the inhomogeneous string problem also plays a role in constructing multi-peakon solutions to the Camassa-Holm equation [7, 6]. We do not dwell on the inhomogeneous string problem other than to make explicit the relation to the magnetotelluric problem, because this helps to clarify the roles of the objects involved in the magnetotelluric problem.

The linearized equation governing the motion of a vibrating string with mass density $\rho(z)$ under constant tension T is

$$Tv_{zz} = \rho(z)v_{tt}. \tag{35}$$

Harmonic modes of the string are obtained by setting $v(z,t) = u(z)\cos\nu t$. Substituting this into (35) results in a differential equation for $u(z)$:

$$Tu''(z;\nu) = -\nu^2\rho(z)u(z;\nu). \tag{36}$$

Initially one considers mass densities $\rho(z)$ that are nonnegative positive functions, often with additional constraints to facilitate analysis. For example, supposing that $\rho(z) \in C^2$, $\rho(z) > 0$, allows one to rewrite (36) in the Schrödinger (canonical) form [10]:

$$\tilde{u}_{zz} + V(z)\tilde{u} = \lambda\tilde{u}. \tag{37}$$

One can also consider highly singular mass densities $\rho(z)$ given by Radon measures by interpreting (36) as a distributional equation. This is equivalent to recasting (36) as a Stieltjes integral equation. This formulation of the problem is common in the string literature. For example, an inhomogeneous string of total length 2, tension $T = 1$, tied at both ends can be described by the following integral equation:

$$u(z) = -\nu^2\int_{-1}^{1} K(z,y)u(y)\,dM(y), \tag{38}$$

where $K(z,y) = (1+z)/(2-2y)$ and $M(y)$ is a non-decreasing, right-continuous function, describing the total mass accumulated on the interval $[-1, y]$. A particularly simple case is that of point masses $m_1, m_2, \ldots m_n$, situated at $z_1 < z_2 < \cdots < z_n$. Then for $y \in [z_k, z_{k+1})$ the cumulative mass is $M(y) = m_1 + \ldots + m_k$ and the integral equation is equivalent to a linear system of algebraic equations:

$$\widehat{U} = -\nu^2\widehat{K}\widehat{M}\widehat{U},$$

where

$$\widehat{U} = \begin{bmatrix} u(z_1) & u(z_2) & \cdots & u(z_n) \end{bmatrix}^T, \ \widehat{M} = \mathrm{diag}(m_1, m_2, \ldots, m_n), \ \widehat{K} = [K(z_i, z_j)].$$

Magnetotelluric Object	Inhomogeneous String Object
Conductivity $\sigma(z)$	Mass density $\rho(z)$
Point Conductance σ_i	Point mass m_i
Perfectly resistive layer	Massless wire segment

Table 1: Analogous objects in the magnetotelluric and inhomogeneous string inverse problems.

Table 1 summarizes the correspondence between the objects involved in the magnetotelluric problem and the inhomogeneous string problem. It is typical to think of mass (conductance) as the integral of the mass density (conductivity), but we would advocate thinking of mass density (conductivity) in the way it is ideally defined, namely as the derivative of the cumulative mass (conductivity). This derivative must be interpreted in the correct sense, as it is certainly possible to have discontinuous cumulative mass functions.

8 General Inversion for Discrete Conductivities

The central object for inversion is the response (or Weyl) function $W(L;\lambda) = E'(L;\lambda)/E(L;\lambda)$.[2] We describe how to perform the inversion for the general boundary conditions (14) and conductivity profiles in the class D^+. As a first step we reinterpret (13) for conductivities in the class D^+; the equations become

$$E''(z;\lambda) = 0, \quad z \neq z_j, \tag{39a}$$

$$E'(z_j^+;\lambda) - E'(z_j^-;\lambda) = \lambda^2 \sigma_j E(z_j;\lambda), \quad z \in \{z_1, z_2, \ldots, z_n\}. \tag{39b}$$

We explicitly label two more points: $z_0 = 0$ and $z_{n+1} = L$. The boundary conditions then become $E(z_0;\lambda) = E_0$, $E'(z_0;\lambda) = \lambda\sigma_B E_0$, and at an eigenvalue λ^* we have $E(z_{n+1};\lambda^*) = 0$. For convenience let $q_j = E(z_j;\lambda)$, $p_j = E'(z_j^+;\lambda)$, and $\ell_j = z_j - z_{j-1}$. We then have, from (39a) and (39b),

$$p_j \ell_{j+1} = q_{j+1} - q_j, \qquad p_{j+1} - p_j = \lambda^2 \sigma_{j+1} q_{j+1}. \tag{40}$$

[2]Many authors define the response function to be $c(\lambda) = 1/W(L;\lambda)$; the difference is immaterial in practice.

Observe that

$$\begin{aligned} W(L;\lambda) &= \frac{p_{n+1}}{q_{n+1}} \\ &= \lambda^2\sigma_{n+1} + \frac{p_n}{q_{n+1}} \\ &= \frac{p_n}{q_n + p_n\ell_{n+1}} \\ &= \frac{1}{\ell_{n+1} + q_n/p_n} \\ &= \cfrac{1}{\ell_{n+1} + \cfrac{1}{p_n/q_n}}, \end{aligned}$$

so that

$$W(L;\lambda) = \cfrac{1}{\ell_{n+1} + \cfrac{1}{\lambda^2\sigma_n + \cfrac{1}{\ell_n + \cfrac{1}{\ddots + \cfrac{1}{\lambda\sigma_B}}}}}. \tag{41}$$

The termination of the expansion occurs at q_0/p_0, which are the boundary values $E(0;\lambda)$ and $E'(0;\lambda)$. Use of the fact that the point z_{j+1} lies in perfectly insulating space has been made; i.e., $\sigma_{n+1} = 0$. For Neumann boundary conditions ($\sigma_B = 0$) and Dirichlet boundary conditions ($\sigma_B = \infty$) the continued fractions are particularly nice:

$$W_N(L;\lambda) = \cfrac{1}{\ell_{n+1} + \cfrac{1}{\lambda^2\sigma_n + \cfrac{1}{\ell_n + \cfrac{1}{\ddots + \cfrac{1}{\lambda^2\sigma_1}}}}} \tag{42}$$

$$W_D(L;\lambda) = \cfrac{1}{\ell_{n+1} + \cfrac{1}{\lambda^2\sigma_n + \cfrac{1}{\ell_n + \cfrac{1}{\ddots + \cfrac{1}{\ell_1}}}}} \tag{43}$$

9 Recovering the Standard Boundary Condition Continued Fractions

The problem (13) is highly overdetermined; see [5] for an expanded discussion. The overdetermination can be seen by recasting the recurrence relations in terms of a series of matrix multiplications applied to q_0 and p_0:

$$\begin{bmatrix} q_{n+1} \\ p_{n+1} \end{bmatrix} = T(\lambda^2) \begin{bmatrix} q_0 \\ p_0 \end{bmatrix}, \tag{44}$$

where

$$T(\lambda^2) = \begin{bmatrix} 1 & \ell_{n+1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \lambda^2\sigma_n & 1 \end{bmatrix} \begin{bmatrix} 1 & \ell_n \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 \\ \lambda^2\sigma_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \ell_1 \\ 0 & 1 \end{bmatrix}. \tag{45}$$

The initial vector $[q_0, p_0]^{\mathrm{T}}$ can be decomposed

$$\begin{bmatrix} q_0 \\ p_0 \end{bmatrix} = E_0 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda\sigma_B \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right), \tag{46}$$

which gives the final solution as

$$E_0 \begin{bmatrix} q_e(\lambda^2) + \lambda\sigma_B q_o(\lambda^2) \\ p_e(\lambda^2) + \lambda\sigma_B p_o(\lambda^2) \end{bmatrix} = T(\lambda^2) \begin{bmatrix} q_0 \\ p_0 \end{bmatrix}, \tag{47}$$

and this division emphasizes that $E(L;\lambda)$ and $E'(L;\lambda)$ are composed of even and odd polynomial parts. These separate parts solve problems associated with the same conductivity profile but different boundary conditions. See Table 2.

Function	Boundary Condition at 0	Boundary Condition at L
q_o	Dirichlet	Dirichlet
q_e	Neumann	Dirichlet
p_o	Dirichlet	Neumann
p_e	Neumann	Neumann

Table 2: Boundary conditions satisfied by the even and odd parts of the polynomials $q(\lambda)$ and $p(\lambda)$.

A theorem of Kreĭn [17], the conditions of which are satisfied due to the Hermite-Biehler theorem [15], guarantees that knowledge of any one of the pairs (q_o, q_e), (q_o, p_o), (q_e, p_e), and (p_o, p_e) is sufficient to determine nearly the entire conductance function. Our restriction to discrete conductivity profiles makes it simpler to explicitly demonstrate the determination than to invoke these theorems. Section 17 details how to perform the inversion using these polynomials.

The essential point is that the information regarding the conductivity profile above the uniform basement is contained in the matrix $T(\lambda^2)$, so the same information regarding the discrete conductivity profile can be recovered using any valid pair of functions, with the exception of the final length ℓ_{n+1}, which can only be determined using pairs of functions satisfying Dirichlet conditions.

We remark that the ability to perform the inversion using solely the electric or magnetic field (that is, one of the pairs (q_o, q_e) or (p_o, p_e)) is primarily of theoretical interest. This is because the inversion procedure applies to fields that are due to the electrical structure of the Earth, the so-called *internal fields*. The observed electrical field is composed of the internal field and the *external field*, which originates from sources external to the conductivity structure of the Earth. There is evidently a causal relationship between the internal and external fields, and this relationship is necessarily of the following form [26]

$$E_{\text{int}}(L;\lambda) = \tau(\lambda) E_{\text{ext}}(L;\lambda). \tag{48}$$

The same equation holds for the magnetic field. Therefore

$$E_{\text{obs}}(L;\lambda) = (1 + \tau^{-1}(\lambda)) E_{\text{int}}(L;\lambda), \tag{49}$$

and likewise for the magnetic field. The ratio of $E_{\text{obs}}(L;\lambda)$ to $H_{\text{obs}}(L;\lambda)$ is therefore dependent only on the internal fields. Without exact knowledge of the operator $\tau(\lambda)$, which in general requires knowledge of the conductivity structure of the Earth, one can not discover the internal electric field from measurements of the electric field at a single spatial location.

We remark that in the frequency domain the operator $\tau(\lambda)$ is simply a multiplier; it is an open question as to how this multiplier behaves over the range of practically measurable frequencies.

10 The Method in Practice

To implement the described procedure two issues must be addressed. The first is that the response function $W(L;\lambda)$ is only ever known at a finite number of frequency values, and the second is that the conductivity profile of a physical Earth is unlikely to belong to the class D^+. We have noted in Section 6 and Section 7 that the second problem is primarily a matter of correctly interpreting results in D^+. The ability to interpret solutions in D^+ allows for a solution to the first problem by fitting response functions in the class D^+ to finite data sets.

The form of the response function to be fit to the finite data set is crucial. In this section we outline the general procedure for Dirichlet boundary conditions and give an example. The same approach applies to Neumann boundary conditions, but the radiation boundary conditions require a different approach due to the overdetermination of the Weyl function; see Section 13.

We can use a partial fraction decomposition to express the response function in terms of its poles and residues:

$$W(L;\lambda) = \sum_{j=1}^{\infty} \frac{a_j}{\lambda - \lambda_j} + \frac{b_j}{\lambda - \overline{\lambda}_j}. \tag{50}$$

Since $\overline{\lambda_j} = -\lambda_j$ and $b_j = \overline{a}_j = -a_j$, we can rewrite (50) as

$$W(L;\lambda) = \sum_{j=1}^{\infty} \frac{\nu_j}{\lambda^2 - \lambda_j^2}, \tag{51}$$

where $\lambda_j^2 < 0$ and $\nu_j = 2\lambda_j a_j > 0$.

The procedure to solve the magnetotelluric problem with a finite number of data points for Dirichlet conditions is

(i) Use a truncation of (51) as an ansatz for a nonlinear least squares programming routine.

(ii) Apply the inversion routine of section 8 to the resulting rational function.

To demonstrate the procedure we first solve a simple example that illustrates the method, and then we show some results based upon synthetic frequency response data.

11 A Homogeneous Layer

Suppose the Earth is composed of an infinitely conductive halfspace with a uniformly conductive layer of thickness L and conductivity σ on top of the halfspace. Then the response function of the Earth is easily seen from (9) to be

$$\frac{E'(L;\lambda)}{E(L;\lambda)} = \sqrt{\sigma}\lambda \coth(\lambda\sqrt{\sigma}L). \tag{52}$$

To slightly simplify computations, we note that (41) contains the same information if we divide (52) by λ^2. Doing so gives

$$\frac{E'(L;\lambda)}{\lambda^2 E(L;\lambda)} = \frac{\sqrt{\sigma}\cosh(\lambda\sqrt{\sigma}L)}{\lambda\sinh(\lambda\sqrt{\sigma}L)}, \tag{53}$$

which has poles at $\lambda_n = \frac{i\pi n}{\sqrt{\sigma}L}, n \in \mathbb{Z}$. The residues are $\frac{1}{L}$. Combining terms we get

$$\frac{E'(L;\lambda)}{\lambda^2 E(L;\lambda)} = \frac{1}{L\lambda^2} + \sum_{i=1}^{\infty} \frac{2/L}{\lambda^2 + \pi^2 n^2/\sigma L^2} \tag{54}$$

Figure 1 shows some plots of the inversion procedure applied to truncations of (54) for homogeneous layer 1000m thick with conductivity 1 S/m. The solid line represents the model conductance, and the dots represent the conductance determined by the inversion procedure.

This is an idealized result of the procedure because the poles and residues used in the inversion are exactly equal to the poles and residues of the true response function. More realistic results of the procedure are the plots in Figure 2. To create these figures the exact response function (52) has been sampled at 58 frequencies that are practically measurable in exploratory magnetotelluric

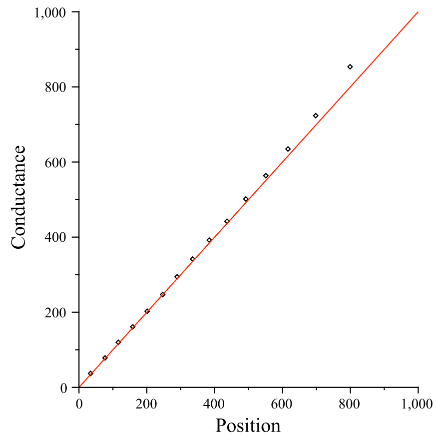

Figure 1: Plots of conductance versus position for exact spectral truncations of order 15 (left) and 25 (right).

surveying.[3] These data points were then interpolated with a spline curve, which was sampled at a larger number of chosen frequencies. A nonlinear least squares procedure was then used to fit a truncation of (51) to these datapoints.

12 Synthetic Example

As an example of a more physically realistic situation, we apply the full inversion procedure for Dirichlet boundary conditions to an Earth that is composed of a halfspace of conductivity 10 S/m, above which are three layers of specified conductivity. The result is shown in Figure 3; the solid curve indicates the model conductance.

13 Radiation Boundary Conditions

As previously mentioned the spectral properties of the general boundary condition are more complicated. There is no longer any guarantee that the spectrum is simple, or that it lies on the imaginary axis. It is in fact quite easy to find examples where the spectrum is degenerate. The one-parameter family of conductivity profiles in the class D^+ defined by

$$l_1 = \frac{4}{17\sigma_B},\ l_2 = \frac{25538}{24259\sigma_B},\ l_3 = \frac{6724}{1427\sigma_B},$$

$$\sigma_1 = \frac{289}{452}\sigma_B,\ \sigma_2 = \frac{2036329}{1519624}\sigma_B,\ \sigma_B = \sigma_B,$$

[3]The sampled frequencies lie in the range 5Hz-32kHz.

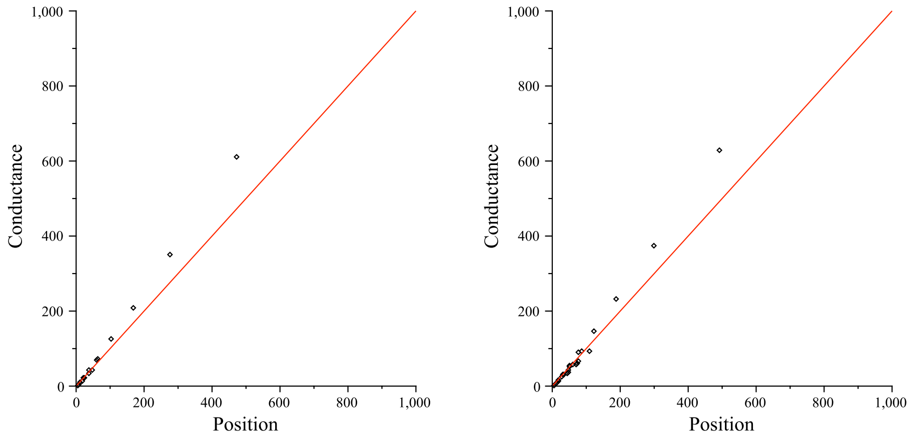

Figure 2: Plots of conductance versus position for truncations of order 15 (left) and 25 (right). The order-15 truncation was fit to 210 data points, and the order-25 truncation was fit to 240 data points.

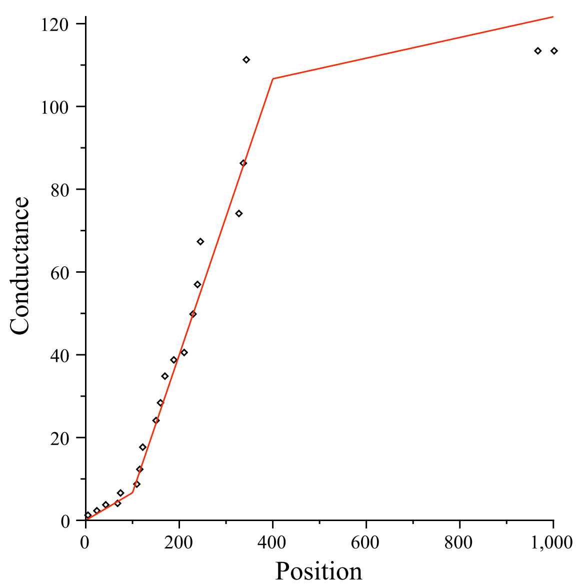

Figure 3: A 20-term truncation fit to 350 data points sampled from a spline curve generated by 58 data points.

always has the degenerate spectrum, $\{-1/4, -1-i, -1-i, -1+i, -1+i\}$, for any finite positive σ_B.

Numerical experiments suggests that the spectrum is typically simple. The spectrum of a conductivity profile with $l_1 = 1$, $l_2 = 2/5$, $l_3 = 1/5$, $l_4 = 3$, $\sigma_1 = 1/10$, $\sigma_2 = 5$, and $\sigma_3 = 1/2$ is shown for various values of $\sigma_B = 10^k$ in Figure 4.

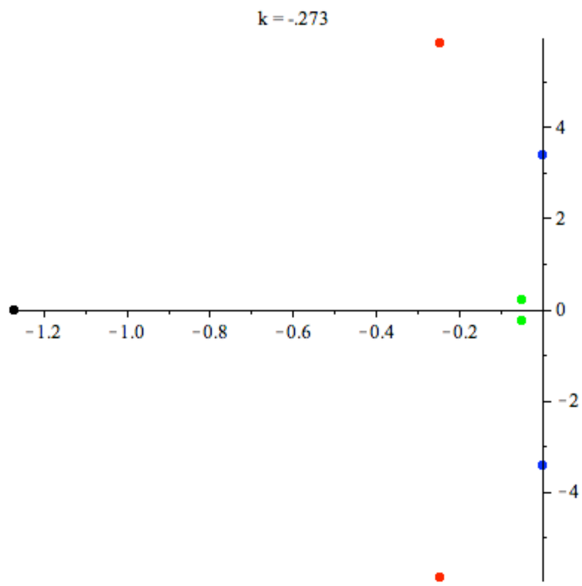

Figure 4: The spectrum for a conductivity profile $l_1 = 1$, $l_2 = 2/5$, $l_3 = 1/5$, $l_4 = 3$, $\sigma_1 = 1/10$, $\sigma_2 = 5$, and $\sigma_3 = 1/2$ for $\sigma_B = 10^k$. Click the figure to view an animated version. The colouration of the spectra is solely to highlight conjugate pairs.

This primarily horizontal movement of the individual eigenvalues appears to be typical, based on numerical experiments.

The primary difficulty in trying to solve the magnetotelluric problem for radiative boundary conditions lies in determining the correct ansatz for the least squares procedure. It is readily seen that, in general, a spectral truncation of the form (50) is not a response function for a discrete Earth. This is because, as shown in Section 9, the denominator (or numerator) alone determines the conductance. Spectral truncations of response functions can be constructed that do not correspond to conductances. It is unclear how to approach the radiative boundary condition using a rational ansatz.

14 Inversion via the Electric Field

Despite the remark in Section 9 regarding the experimental difficulty of determining the internal electric field, it is interesting to consider the solution of the magnetotelluric problem with radiative boundary conditions using the electric field alone. We assume that the value of the internal electric field is known at a finite set of frequencies $\{\omega_i\}_{i=1}^N$. Because the electric field $E(L;\lambda)$ is an entire function, it is known that the sequence of interpolating polynomials on a finite interval converges uniformly to $E(L;\lambda)$ [8]. One is therefore motivated to form an interpolating polynomial $q(\lambda)$ that approximates the electric field, determine the even and odd parts $q_e(\lambda)$ and $q_o(\lambda)$ of $q(\lambda)$, and then take the ratio of the even and odd parts to determine the conductance profile.

As noted in Table 2 the even and odd parts of the polynomial $q(\lambda)$ satisfy the same boundary conditions as a Dirichlet Earth, hence the method of Section 10 can be utilized by generating data from the ratio of $q_e(\lambda)$ to $q_o(\lambda)$.

The success of a such a procedure depends critically upon forming interpolating polynomials of sufficiently high degree. The precise notion of sufficiently high degree depends, in turn, on the frequency interval chosen. As a simple example, consider again a homogeneous layer of depth L with constant conductivity σ. The electric field at the surface is

$$E(L;\lambda) = \cosh(\lambda\sqrt{\sigma}L) + \frac{\sigma_B}{\sqrt{\sigma}}\sinh(\lambda\sqrt{\sigma}L). \tag{56}$$

Recalling that $\lambda = \sqrt{i\omega\mu_0}$ we can rewrite (56) as

$$\begin{aligned} E(L;\lambda) = &\left(\cosh(\sqrt{\frac{\omega\mu_0\sigma}{2}}L) + \frac{\sigma_B}{\sqrt{\sigma}}\sinh(\sqrt{\frac{\omega\mu_0\sigma}{2}}L)\right)\cos(\sqrt{\frac{\omega\mu_0\sigma}{2}}L) + \\ &i\left(\sinh(\sqrt{\frac{\omega\mu_0\sigma}{2}}L) + \frac{\sigma_B}{\sqrt{\sigma}}\cosh(\sqrt{\frac{\omega\mu_0\sigma}{2}}L)\right)\sin(\sqrt{\frac{\omega\mu_0\sigma}{2}}L). \end{aligned} \tag{57}$$

For sufficiently large ω $\cosh(\sqrt{\frac{\omega\mu_0\sigma}{2}}L) \approx \sinh(\sqrt{\frac{\omega\mu_0\sigma}{2}}L)$, so (57) can be rewritten as

$$\begin{aligned} E(L;\lambda) = &\left(\cosh(\sqrt{\frac{\omega\mu_0\sigma}{2}}L) + \frac{\sigma_B}{\sqrt{\sigma}}\cosh(\sqrt{\frac{\omega\mu_0\sigma}{2}}L)\right)\cos(\sqrt{\frac{\omega\mu_0\sigma}{2}}L) + \\ &i\left(\sinh(\sqrt{\frac{\omega\mu_0\sigma}{2}}L) + \frac{\sigma_B}{\sqrt{\sigma}}\sinh(\sqrt{\frac{\omega\mu_0\sigma}{2}}L)\right)\sin(\sqrt{\frac{\omega\mu_0\sigma}{2}}L). \end{aligned} \tag{58}$$

Rearrangement yields

$$E(L;\lambda) \approx (1 + \frac{\sigma_B}{\sqrt{\sigma}})\cosh(\lambda\sqrt{\sigma}L).$$

For large ω it is necessary to have high order polynomial interpolants to determine the even and odd parts of $E(L;\lambda)$. For typical values of σ ($\approx$ 1 S/m) and L ($\approx$ 1000 m) frequencies of 2 Hz or less are required for practical interpolation.

15 Conclusion

An algebraic method of inversion for the one-dimensional magnetotelluric inverse problem has been demonstrated. Examination of the radiation boundary conditions showed the existence of inversion methods based on measurements of only the electric or magnetic field, although these methods appear to be of solely theoretical interest. We have shown that the solutions in the class D^+ obtained by the inversion procedure can be viewed as approximations to the true physical conductance profile, which alleviates concerns over the class D^+ being unphysical.

16 Acknowledgments

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17 Appendix 1: Standard Inversion Expressions

As is well known, the Euclidean algorithm generates two sequences that can be used to form successive continued fraction approximations to a number x. These sequences are the partial numerators $\{a_i\}_i$, where $a_i = \lfloor x \rfloor$, and the remainders $\{b_i\}_i$, where $b_{i+1} = \{a_i\}$. The curly braces indicate that b_{i+1} is the fractional part of a_i. The continued fraction expansion for x is then

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + b_{n+1}}}}}. \tag{59}$$

The n^{th} approximate R_n to x is formed by setting $b_{n+1} = 0$:

$$R_n = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}. \tag{60}$$

Note that this method can easily be applied to rational functions $f(x)$ by using the Euclidean algorithm for polynomial division.

Proposition 5. *Given a function $f(x)$ and the sequence $\{a_i\}_i$ of partial denominators generated by the Euclidean algorithm, consider the recurrence relation*

$$u_j = a_j u_{j-1} + u_{j-2}, \quad j \geq 1 \tag{61}$$

Let P_j, Q_j be the unique solutions to (61) satisfying $P_{-1} = 1$, $P_0 = 0$, $Q_{-1} = 0$, and $Q_0 = 1$, respectively. Then for any $k \geq 1$

$$R_k = \frac{P_k}{Q_k} \tag{62}$$

Proof. The initial conditions imply $R_1 = 1/a_1$. Proceed by induction. Suppose (62) holds for $k = n$. Then set $a_n = a_n + 1/a_{n+1}$ in the expression for R_n to get R_{n+1}:

$$\begin{aligned} R_{n+1} &= \frac{(a_n + 1/a_{n+1})P_{n-1} + P_{n-2}}{(a_n + 1/a_{n+1})Q_{n-1} + Q_{n-2}} \\ &= \frac{a_{n+1}(a_n P_{n-1} + P_{n-2}) + P_{n-1}}{a_{n+1}(a_n Q_{n-1} + Q_{n-2}) + Q_{n-1}} \\ &= \frac{a_{n+1}P_n + P_{n-1}}{a_{n+1}Q_n + Q_{n-1}} \\ &= \frac{P_{n+1}}{Q_{n+1}} \end{aligned}$$

□

We can write (61) for P_n, P_{n-1}, Q_n, and Q_{n-1} more compactly in terms of 2×2 matrices

$$\begin{bmatrix} Q_n & P_n \\ Q_{n-1} & P_{n-1} \end{bmatrix} = \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}. \tag{63}$$

Given the known form of R_n, continued fraction forms of the ratios P_n/Q_n, P_{n-1}/Q_{n-1}, P_{n-1}/P_n, and Q_{n-1}/Q_n can be developed by reading off the partial denominators from (63) and the transpose of (63).

We can rewrite (45) as

$$T(\lambda^2) = \begin{bmatrix} \ell_{n+1} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda^2\sigma_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \ell_n & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} \lambda^2\sigma_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \ell_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{64}$$

by inserting the identity matrix into $T(\lambda^2)$ after each matrix containing an ℓ_i and using

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{65}$$

Equation (47) shows that we can express $T(\lambda^2)$ as

$$T(\lambda^2) = E_0 \begin{bmatrix} q_o(\lambda^2) & q_e(\lambda^2) \\ p_o(\lambda^2) & p_e(\lambda^2) \end{bmatrix}, \tag{66}$$

and therefore the matrix form of Proposition (5) gives continued fraction expansions for the discrete Earth data:

$$\frac{q_e}{\lambda q_o} = \cfrac{1}{\lambda \ell_1 + \cfrac{1}{\lambda \sigma_1 + \cfrac{1}{\ddots + \cfrac{1}{\lambda \ell_{n+1}}}}}, \qquad \frac{p_o}{\lambda q_o} = \cfrac{1}{\lambda \ell_{n+1} + \cfrac{1}{\lambda \sigma_n + \cfrac{1}{\ddots + \cfrac{1}{\lambda \ell_1}}}}, \tag{67}$$

$$\frac{p_e}{\lambda p_o} = \cfrac{1}{\lambda \ell_1 + \cfrac{1}{\lambda \sigma_1 + \cfrac{1}{\ddots + \cfrac{1}{\lambda \sigma_n}}}}, \qquad \frac{p_e}{\lambda q_e} = \cfrac{1}{\lambda \ell_{n+1} + \cfrac{1}{\lambda \sigma_n + \cfrac{1}{\ddots + \cfrac{1}{\lambda \sigma_1}}}}. \tag{68}$$

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