# A Gramian-Based Controller for Linear Periodic Systems

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**Abstract**

Many important real-world processes are best modelled by linear time-periodic systems. This paper proposes a new design method for the control of these systems. The method is based on the use of the reachability Gramian and a specific form for the feedback gain matrix to build a novel control law for the closed-loop system. The new controller is presented as full-state and allows the control engineer to assign all the invariants of the system; i.e., the full monodromy matrix. Calculation of the feedback matrix requires solving a matrix integral equation for the periodic Floquet factor of the new state-transition matrix of the closed-loop system. The effectiveness of the method is illustrated on a simple example.

**Keywords**

Floquet, periodic systems, periodic feedback, invariant factors, Gramian

## I. Introduction

Linear time-invariant (LTI) systems are the most common way of analyzing engineering processes. Consequently, they have been extensively studied, and many different strategies have been developed

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over the years for their control. Yet, modelling real-world processes often leads to a linear time-periodic (LTP) system. In mechanical engineering alone, many mechanical systems work under a periodic regime in steady-state conditions and can be reduced to a LTP formulation under "small perturbations." This is the case, for instance, of manipulators performing repetitive tasks [1], helicopters [2, 3, 4, 5], and satellites [6, 7, 8, 9].

Unfortunately, results established for LTI systems do not usually hold for time-varying systems. General time-varying systems must typically be treated on a case-by-case basis. Moreover, techniques devised for one type of time-varying system cannot be generalized for use with another. LTP systems are an exception in that they all exhibit similar behaviour, and thus form a unified class. Moreover, several aspects of Floquet-Lyapunov theory for LTP systems have connections with LTI systems, raising the prospect of being able to take advantage of this well-established body of knowledge.

### *A. Notation and Definitions*

Let $\mathbb{R}$ ($\mathbb{R}^n$) [$\mathbb{R}^{m\times n}$] denote the real field (space of real $n$-vectors) [set of real matrices with $m$ rows and $n$ columns], $\mathbb{C}$ ($\mathbb{C}^n$) [$\mathbb{C}^{m\times n}$] denote the complex field (space of complex $n$-vectors) [set of complex matrices with $m$ rows and $n$ columns], $\mathbb{Z}$ ($\mathbb{N}$) denote the sets $\{0,1,2,\ldots\}$ ($\{1,2,\ldots\}$), $\mathbf{I}$ denote the identity matrix of order $n$, and superscript $T$ $(-1)$ $[-T]$ denote matrix transpose (inverse) [inverse and transpose]. Consider the continuous-time system described by the differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \tag{1}$$

and its corresponding uncontrolled form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \tag{2}$$

where $\mathbf{A}(\cdot) \in \mathbb{R}^{n\times n}, \mathbf{B}(\cdot) \in \mathbb{R}^{n\times r}$ are piecewise continuous, $T$-periodic matrix functions. Denote by $\mathbf{\Phi}(\cdot, 0)$ the state-transition matrix (STM) of (2). The matrix $\mathbf{\Phi}(T, 0)$ is called the *monodromy matrix.*

### *B. Floquet Theory*

We give the main results and refer to [10, 11] for a complete treatment. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ define the following set of matrix functions:

$$\begin{aligned}\mathcal{L}_T^{\mathbb{K}} \;=\; & \left\{\mathbf{L}(\cdot) : \mathbb{R} \to \mathbb{K}^{n\times n} : \mathbf{L}(0) = \mathbf{I},\ \mathbf{L}(t+T) = \mathbf{L}(t),\right.\\ & \left.\det \mathbf{L}(t) \neq 0\ \forall t,\ \mathbf{L}(\cdot) \text{ absolutely continuous}\right\}.\end{aligned}$$

*Theorem 1* (Representation Theorem) The STM $\mathbf{\Phi}(\cdot, 0)$ of system (2) can be factored as

$$\mathbf{\Phi}(t, 0) = \mathbf{L}(t)\exp(t\mathbf{F}),\ \text{where } \mathbf{L}(\cdot) \in \mathcal{L}_T^{\mathbb{C}},\ \mathbf{F} \in \mathbb{C}^{n\times n}. \tag{3}$$

*Theorem 2* (Reducibility) The Lyapunov transformation

$$\mathbf{x}(t) = \mathbf{L}(t)\mathbf{z}(t)$$

transforms the original LTP system into a linear time-invariant (LTI) system

$$\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t),$$

where $\mathbf{L}(\cdot)$ and $\mathbf{F}$ are the same as those that appear in (3).

One disadvantage of Theorems 1-2 is that the *Floquet factors* $\mathbf{L}(t)$ and $\mathbf{F}$ may in general be complex even if $\mathbf{\Phi}(T,0)$ is real. It is well known (see e.g., [12]) that it is always possible to obtain real Floquet factors by treating (2) as having $2T$-periodic coefficients. However, in this case calculations must be made over two periods. Recently [13, 14] demonstrated how to obtain a real representation from computations performed solely from one period by generalizing a result from [15, 11]. The two main results are reproduced below.

*Theorem 3:* Consider the equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \tag{4}$$

and let $\mathbf{\Phi}(\cdot,0)$ be its (real) state-transition matrix. Let $\mathbf{Y} \in \mathbb{R}^{n\times n}$ such that

- $\mathbf{Y\Phi}(T,0)$ has a real logarithm;
- $\mathbf{\Phi}^k(T,0) = [\mathbf{\Phi}(T,0)\mathbf{Y}]^k$ for some positive integer $k$.

Then for any $\mathbf{F}_Y \in \mathbb{R}^{n\times n}$ satisfying

$$\exp(T\mathbf{F}_Y) = \mathbf{Y\Phi}(T,0), \tag{5}$$

the real factor

$$\mathbf{L}_{F_Y}(t) \stackrel{\triangle}{=} \mathbf{\Phi}(t,0)\exp(-t\mathbf{F}_Y)$$

is $kT$-periodic, continuous with a piecewise continuous derivative, and satisfies $\mathbf{L}_{F_Y}(T) = \mathbf{Y}^{-1}$.

Any $\mathbf{Y}$ satisfying the conditions of this theorem will be henceforth called a *Yakubovich matrix* [14]. It is always possible to find $\mathbf{Y}$ such that $\mathbf{L}_{F_Y}(\cdot)$ is $2T$-periodic.

*Note I.1:* In practice, it is often desirable to require as well that the eigenvalues of $\mathbf{Y\Phi}(T,0)$ and

$\mathbf{\Phi}(T,0)$ have the same moduli. This condition is not necessary to define a Yakubovich matrix. However, in order to apply the classical results from control theory, a specific relationship between the eigenvalues $\mu_i$ $(i = 1 \ldots n)$ of $\mathbf{\Phi}(T,0)$ and the eigenvalues $\lambda_i$ $(i = 1 \ldots n)$ of $\mathbf{F}$, where $\mathbf{F}$ is any complex matrix obtained from Theorem 1, namely

$$\mathbf{\Phi}(T_A, 0) = \exp(T_A \mathbf{F}).$$

Consider the polar representation of $\mu_i$:

$$\mu_i = \rho_i \exp(\iota \theta_i), \quad \iota^2 = -1,$$

$$\rho_i \stackrel{\Delta}{=} |\mu_i| \geq 0, \ -\pi \leq \theta_i < \pi.$$

Every $\lambda_i$ has the form

$$\lambda_i = \frac{\ln(\rho_i)}{T} + \iota \frac{\theta_i + 2k\pi}{T}, \ k \in \mathbb{Z}.$$

When introducing matrix $\mathbf{Y}$ in the definition of $\mathbf{F}_Y$ in Eq. (5), it is important to preserve the relation

$$\Re(\lambda_i) = \frac{\ln(\rho_i)}{T_A}$$

so that the $s$-plane strip image of the unit circle is not distorted, resulting in the $\lambda_i$ losing their quantitative and qualitative dynamical significance. These properties are important to the control engineer, and can be preserved by requiring that the eigenvalues of $\mathbf{\Phi}(T_A, 0)$ and $\mathbf{Y\Phi}(T_A, 0)$ have the same moduli.

*Theorem 4* (Converse) Let $\mathbf{L}(\cdot) \in \mathcal{L}^{\mathbb{R}}_{kT}$ and $\mathbf{F} \in \mathbb{R}^{n \times n}$. Then

$$\mathbf{\Phi}(t, 0) \stackrel{\Delta}{=} \mathbf{L}(t) \exp(t\mathbf{F})$$

is the STM of a system of the form (4), where $\mathbf{A}(\cdot)$ is piecewise continuous and $T$-periodic if and only if $\mathbf{Y} = \mathbf{L}^{-1}(T)$ is a Yakubovich matrix.

### C. Controllability

We first make a few definitions and state some relevant theory.

System (1) is *controllable on the time interval* $[t_1, t_2]$ if each initial condition at time $t_1$ can be driven to the origin at time $t_2$. System (1) is *controllable at time* $t_1$ if there exists a time $t_2 > t_1$ such that it is controllable on $[t_1, t_2]$. System (1) is *controllable* if it is controllable for all time.

Let $\mathbf{F}$ and $\mathbf{G}$ be $n \times n$ and $n \times r$ constant matrices. A pair $(\mathbf{F}, \mathbf{G})$ is controllable if System (1) with $\mathbf{A}(t) = \mathbf{F}$ and $\mathbf{B}(t) = \mathbf{G}$ is controllable. Now for integers $k = 0, 1, \ldots, n-1$ define the matrix

$$\mathbf{U}_k \triangleq \left[ \mathbf{G} \,\vdots\, \;\mathbf{FG} \,\vdots\, \;\cdots \,\vdots\, \;\mathbf{F}^k\mathbf{G} \right]. \tag{6}$$

The *controllability index* of the pair $\{\mathbf{F}, \mathbf{G}\}$ is the smallest positive integer $\mu$ $(1 \leq \mu \leq n)$ such that $\mathbf{U}_{\mu-1}$ has rank $n$. The pair is controllable if and only if such an integer $\mu$ exists. Finally, define the *reachability Gramian* on $[t_1, t_2]$ as

$$\mathbf{W}(t_2, t_1) \triangleq \int_{t_1}^{t_2} \mathbf{\Phi}(t_2, \tau)\mathbf{B}(\tau)\mathbf{B}^T(\tau)\mathbf{\Phi}^T(t_2, \tau)d\tau. \tag{7}$$

Classical results on the controllability of linear periodic systems [16, 17, 18] can be summarized as follows:

*Theorem 5:* Consider System (1) of order $n$, its corresponding monodromy matrix $\mathbf{\Phi}(T, 0)$, and its reachability Gramian $\mathbf{W}(T, 0)$ at $t = T$. Furthermore, let $\nu$ be the controllability index of the pair $\{\mathbf{\Phi}(T, 0), \mathbf{W}(T, 0)\}$.

The following statements are equivalent:

- System (1) is controllable;
- The pair $\{\mathbf{\Phi}(T,0), \mathbf{W}(T,0)\}$ is controllable;
- System (1) is controllable over $(0, \nu T)$;
- $\mathbf{W}(\nu T, 0)$ is positive definite.

The relationship between controllability and invariant assignment for LTP systems was provided by [16]:

*Theorem 6:* A LTP system is controllable if and only if there exists a $T$-periodic $r \times n$ matrix $\mathbf{K}(\cdot)$ which can assign the eigenvalues of the monodromy matrix in such a way that their product is positive and any complex eigenvalues appear as complex-conjugate pairs.

*Note I.2:* The restriction on the sign of the product of the eigenvalues is explained by Liouville's Theorem [19]:

$$\det \mathbf{\Phi}(t,0) = \exp\left(\int_0^t \mathrm{tr}\mathbf{A}(\sigma)d\sigma\right),$$

from which we can see that $\det \mathbf{\Phi}(t,0) > 0,\ \forall t$. This has to hold in particular at $t = T$ when $\det \mathbf{\Phi}(T,0)$ is the product of the eigenvalues of the monodromy matrix.

*Note I.3:* The results presented in this paper can be equivalently expressed in terms of the *controllability Gramian*

$$\mathbf{W}_C(t_2,t_1) \triangleq \int_{t_1}^{t_2} \mathbf{\Phi}(t_1,\tau)\mathbf{B}(\tau)\mathbf{B}^T(\tau)\mathbf{\Phi}^T(t_1,\tau)d\tau.$$

*Note I.4:* Recalling the duality between controllability and observability [e.g., 20, Theorem 5-10], in Theorem 5 all the results are preserved if 'controllable' is replaced by 'observable' and $\mathbf{W}(t_2,t_1)$ is replaced by the observability Gramian

$$\mathbf{W}_O(t_2,t_1) \triangleq \int_{t_1}^{t_2} \mathbf{\Phi}^T(\tau,t_1)\mathbf{C}^T(\tau)\mathbf{C}(\tau)\mathbf{\Phi}(\tau,t_1)dt.$$

## II. Review of Previous Work

Although assigning the invariants of LTI systems is widely described in classical control literature, this task has proved far more challenging in the case of LTP systems. The different attempts usually fall into two main categories, depending on whether or not the feedback is based on sampling the state.

### A. Discrete Feedback

Brunovský [16] proposed a feedback control consisting of impulsive feedbacks of the instantaneous state at $n$ suitably chosen times within a period. The limitations of this approach were discussed in [21]. Kabamba [18] introduced the concept of "sampled state periodic hold" and gave an explicit expression for a piecewise continuous periodic feedback. He showed that when the controllability index is less than $n$, the whole monodromy matrix can be assigned. Al-Rahmani and Franklin [22] adopted a similar framework and, using *generalized reachability Gramians*, designed a periodic piecewise constant feedback control to assign the Floquet multipliers (the eigenvalues of $\mathbf{\Phi}(T,0)$) arbitrarily, and under conditions similar to [18], the whole monodromy matrix.

### B. Continuous Feedback

The task of producing a continuous-state periodic feedback (CPF) control for system (1) has been arduous. Several approaches looked for the controller based on the transformed system of Theorem 2, viz.

$$\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t) + \mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{u}(t), \tag{8}$$

and assignment of the Floquet exponents (the eigenvalues of the matrix $\mathbf{F}$) by a feedback $\mathbf{u}(t) = -\mathbf{K}(t)\mathbf{z}(t)$.

Kern [23, 24] tried to solve this system of nonlinear differential equations by performing some changes of variables and imposing various assumptions on the form of $\mathbf{K}(\cdot)$. This approach only worked in some simple cases where the closed form of $\mathbf{\Phi}(\cdot, 0)$ or its Floquet factors are known, something that is not true in the general case for LTP systems.

Joseph et al. [25, 26] tried to overcome the periodicity of the input matrix in equation (8) by introducing an "auxiliary" LTI system which would asymptotically converge toward the original LTP system. Unfortunately the scheme was based on a misuse of the pseudo-inverse of matrix $\mathbf{B}(\cdot)$ and of some nonlinear results. Stability of the auxiliary system was proved [27] not to guarantee stability of the LTP system. The authors in [28] proposed to repair the feedback design, but their amended scheme still relied on knowing the closed form of $\mathbf{\Phi}(\cdot, 0)$ or its Floquet factors. This is of course unrealistic in practice.

Calico and Wiesel [29, 4] produced a modal control which could move Floquet exponents one at a time when matrix $\mathbf{F}$ was diagonalizable. However, placing complex-conjugate pairs was problematic, and the authors noted numerical instabilities in their scheme [5]. Despite its limitations, their approach offered a lot of insight into the problem and did provide solutions in simple cases.

On the other hand, Laptinsky [30], using results from [31], adopted an approach based on the closed-loop system

$$\dot{\mathbf{x}}(t) = [\mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}(t)]\,\mathbf{x}(t)$$

and used Theorem 1 to find $\mathbf{K}(\cdot)$: Having assigned a new constant Floquet factor $\mathbf{F}_K$ for the closed-loop system, he solved first for the corresponding Lyapunov transformation $\mathbf{L}_K(\cdot)$ and then for $\mathbf{K}(\cdot)$. Unfortunately the scheme is impractical in the most common case where the input matrix $\mathbf{B}(\cdot)$ is not invertible.

## III. New Problem Formulation

Willems, Kučera, and Brunovský [21] stressed the fundamental design restrictions of LTI feedback control systems: Eigenvalues can be assigned, but their algebraic and geometric multiplicity as well as their eigenvectors cannot. In the same paper, the authors conjectured that a periodic time-varying feedback would be able to relax these design restrictions. They provided an example of a completely unrestricted assignment of the invariant factors for periodic discrete-time systems, but only partial results were obtained in the continuous-time case. In the following, we present a novel continuous-time periodic controller which can assign the whole monodromy matrix.

The approach builds on the works of [23] and [30]. However, whereas their work was dependent on the invertibility of the input matrix $\mathbf{B}(\cdot)$, we base our design on the invertibility of the Gramian, something which is guaranteed from the controllability of the system. Properties of linear state-feedback strategies based on the Gramian have been previously investigated for LTI systems [32], and, more recently, for LTP systems [33]. This novel scheme can be put in parallel with the results obtained by [22] for discrete-time systems.

Recalling the result of Theorem 5, system (1) is controllable if and only if for some $\nu \in \mathbb{N}$

$$\det \mathbf{W}(\nu T, 0) \neq 0.$$

Since the rank of the Gramian is nondecreasing as the range of the integration is increased [34, §2.8]

$$\forall t > \nu T, \quad \det \mathbf{W}(t, 0) \neq 0. \tag{9}$$

Choosing a CPF control law of the form

$$\mathbf{u}(t) = -\mathbf{K}(t)\mathbf{x}(t), \quad \mathbf{K}(t + pT) = \mathbf{K}(t), \quad p \in \mathbb{N}, \quad p \geq \nu,$$

the closed-loop system becomes

$$\dot{\mathbf{x}}(t) = \mathbf{A}_K(t)\mathbf{x}(t), \quad \mathbf{A}_K(t) = \mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}(t). \tag{10}$$

Because this system is $T_{A_K}$-periodic ($T_{A_K} = pT$), we can apply Theorem 1 and thus its state-transition matrix $\mathbf{\Phi}_K(t, 0)$ can be written in the form

$$\forall t \in \mathbb{R}^+, \quad \mathbf{\Phi}_K(t, 0) \triangleq \mathbf{L}_K(t) \exp(t\mathbf{F}_K), \tag{11}$$

where $\mathbf{L}_K(\cdot) \in \mathcal{L}^{\mathbb{R}}_{kT_{A_K}} (k \in \mathbb{N})$ and $\mathbf{F}_K \in \mathbb{R}^{n \times n}$. Recalling the results of Theorem 3, the boundary conditions satisfied by $\mathbf{L}_K(\cdot)$ are

$$\mathbf{L}_K(0) = \mathbf{I}, \quad \mathbf{L}_K(T_{A_K}) = {\mathbf{Y}_K}^{-1}, \tag{12a}$$

for some Yakubovich matrix $\mathbf{Y}_K$ that determines the periodicity $T_{L_K}$ of $\mathbf{L}_K(\cdot)$. On the other hand, matrix $\mathbf{L}_K(\cdot)$ is known [10] to satisfy the equation

$$\forall t \in \mathbb{R}^+, \quad \dot{\mathbf{L}}_K(t) = \mathbf{A}_K(t)\mathbf{L}_K(t) - \mathbf{L}_K(t)\mathbf{F}_K. \tag{12b}$$

## IV. $T_{A_K}$-Periodic Feedback

The following lemma was first proposed without proof in [30].

*Lemma 1:* $\mathbf{L}_K(\cdot)$ is the solution of Eq. (12b) if and only if $\mathbf{L}_K(\cdot)$ satisfies

$$\forall t \in \mathbb{R}^+, \quad \mathbf{L}_K(t) = \mathbf{\Phi}(t,0)\left\{\mathbf{I} - \int_0^t \mathbf{\Phi}(0,\tau)\left[\mathbf{B}(\tau)\mathbf{K}(\tau)\mathbf{L}_K(\tau) + \mathbf{L}_K(\tau)\mathbf{F}_K\right] d\tau\right\}. \tag{13}$$

*Proof:* The proof is a straightforward application of the well known variation-of-constants formula. ■

Lemma 1 establishes the equivalence of the differential Eq. (12b) with an integral Eq. (13) which also contains the unknown gain $\mathbf{K}(t)$. In order to establish a valid CPF control law using Floquet-Lyapunov theory we need that $\mathbf{K}(t)$ be chosen such that the boundary conditions (12a) are also satisfied. To achieve this we proceed as follows.

Using the second boundary condition in Eq. (12a), Eq. (13) can be rewritten at $t = T_{A_K}$ as

$$\int_0^{T_{A_K}} \mathbf{\Phi}(0,\tau)\mathbf{B}(\tau)\mathbf{K}(\tau)\mathbf{L}_K(\tau)d\tau = -\left\{\int_0^{T_{A_K}} \mathbf{\Phi}(0,\tau)\mathbf{L}_K(\tau)d\tau\mathbf{F}_K + \mathbf{\Phi}(0,T_{A_K})\mathbf{Y}_K{}^{-1} - \mathbf{I}\right\}.$$

Multiplying both sides of the equation by $\mathbf{\Phi}(T_{A_K},0)$ yields

$$\begin{aligned}&\int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K},\tau)\mathbf{B}(\tau)\mathbf{K}(\tau)\mathbf{L}_K(\tau)d\tau \\ &= -\left\{\int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K},\tau)\mathbf{L}_K(\tau)d\tau\mathbf{F}_K + \mathbf{Y}_K{}^{-1} - \mathbf{\Phi}(T_{A_K},0)\right\}.\end{aligned} \tag{14}$$

We now have the following Lemma:

*Lemma 2:* Let system (10) be controllable. Then there exists a constant matrix $\mathbf{K}_c \in \mathbb{R}^{n\times n}$ such that $\forall t,\ 0 \leq t < T_{A_K}$,

$$\mathbf{K}(t) = \mathbf{B}^T(t)\mathbf{\Phi}^T(T_{A_K}, t)\mathbf{K}_c\mathbf{L}_K^{-1}(t), \tag{15a}$$

$$\text{and } \forall k \in \mathbb{N},\ \ \mathbf{K}(t + kT_{A_K}) = \mathbf{K}(t). \tag{15b}$$

*Proof:* The existence of $\mathbf{L}_K(.)$ is known from Eqs. (10) and (11) and, since $\mathbf{L}_K(.) \in \mathcal{L}^{\mathbb{R}}_{k_1 T_{A_K}}$ ($k_1 \in \mathbb{N}$), $\mathbf{L}_K(t)$ is invertible for all $t$. Using $\mathbf{K}(t)$ given by Eq. (15a), Eq. (14) takes the form

$$\begin{aligned}&\int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K}, \tau)\mathbf{B}(\tau)\mathbf{B}^T(\tau)\mathbf{\Phi}^T(T_{A_K}, \tau)d\tau\mathbf{K}_c \\ &= -\left\{\mathbf{Y}_K{}^{-1} - \mathbf{\Phi}(T_{A_K}, 0) + \int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K}, \tau)\mathbf{L}_K(\tau)d\tau\mathbf{F}_K\right\}.\end{aligned} \tag{16a}$$

Noting that

$$\mathbf{W}(T_{A_K}, 0) = \int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K}, \tau)\mathbf{B}(\tau)\mathbf{B}^T(\tau)\mathbf{\Phi}^T(T_{A_K}, \tau)d\tau,$$

and recalling that by construction

$$\det \mathbf{W}(T_{A_K}, 0) \neq 0,$$

we have that Eq. (16a) can be rewritten as

$$\mathbf{K}_c = -\mathbf{W}^{-1}(T_{A_K}, 0)\left\{\mathbf{Y}_K{}^{-1} - \mathbf{\Phi}(T_{A_K}, 0) + \int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K}, \tau)\mathbf{L}_K(\tau)d\tau\mathbf{F}_K\right\}, \tag{16b}$$

thus proving the existence of $\mathbf{K}_c$. ■

Substituting Eq. (16b) in Eq. (15a) yields $\forall t,\ 0 \le t < T_{A_K}$,

$$\mathbf{K}(t) = \Bigg[ -\mathbf{B}^T(t)\mathbf{\Phi}^T(T_{A_K},t)\mathbf{W}^{-1}(T_{A_K},0)\Bigg\{ {\mathbf{Y}_K}^{-1} - \mathbf{\Phi}(T_{A_K},0) + \int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K},\tau)\mathbf{L}_K(\tau)d\tau\mathbf{F}_K \Bigg\}\Bigg]\mathbf{L}_K^{-1}(t). \tag{17}$$

We note that the canonical Floquet theory requires the system matrix $\mathbf{A}(\cdot)$ to be piecewise continuous in order for the converse statements used in this Lemma to hold. Hence one can expect the feedback matrix $\mathbf{K}(t)$ to be at least piecewise continuous. Indeed it can be observed that a potential point of discontinuity exists at $t = T_{A_K}$ even if $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are periodic and $C^\infty$. The position of the discontinuity at $t = T_{A_K}$ is natural in the sense that $\mathbf{K}(t)$ is defined in terms of continuous and periodic quantities *except* possibly for $\mathbf{\Phi}(T_{A_K}, t)$, which although continuous is likely *not* periodic. However, plants are often $C^\infty$, and so it is of interest to determine conditions by which $\mathbf{K}(t)$ is also $C^\infty$. It is straightforward to show that a necessary and sufficient condition for $\mathbf{K}(t)$ to be $C^0$ is

$$\mathbf{B}^T(0)[\mathbf{K}_c\mathbf{Y}_K - \mathbf{\Phi}^T(T_{A_K},0)\mathbf{K}_c] = \mathbf{0};$$

i.e., $[\mathbf{K}_c\mathbf{Y}_K - \mathbf{\Phi}^T(T_{A_K},0)\mathbf{K}_c]$ lies in the nullspace of $\mathbf{B}^T(0)$. Clearly it is sufficient that

$$\mathbf{K}_c\mathbf{Y}_K - \mathbf{\Phi}^T(T_{A_K},0)\mathbf{K}_c = \mathbf{0}. \tag{18}$$

It is also straightforward to show that sufficient conditions for $\mathbf{K}(t)$ to be $C^1$ are (18) and

$$\mathbf{F}_K\mathbf{Y}_K = \mathbf{Y}_K\mathbf{F}_K. \tag{19}$$

Interestingly, the latter condition (19) is important for obtaining *real* Floquet factorizations on one period; see [15, 11, 14, 13] for details. For $\mathbf{K}(t)$ to be $C^\infty$, sufficient conditions in addition to (18)–(19)

above take the form

$$\mathbf{\Phi}^T(T_{A_K},0)\mathbf{f}(0) = \mathbf{f}(T_{A_K})\mathbf{Y}_K \text{ and } \mathbf{Y}_K\mathbf{f}(T_{A_K}) = \mathbf{f}(0)\mathbf{\Phi}^T(T_{A_K},0),$$

where $\mathbf{f}(t)$ is a matrix made up of time derivatives of $\mathbf{A}(t)$ and powers of $\mathbf{A}(t)$, $\mathbf{K}_c$, and $\mathbf{F}_K$.

## V. Derivation of an Integral Equation for $\mathbf{L}_K(\cdot)$

*Theorem 7:* With $\mathbf{L}_K(0) = \mathbf{I}$, $\mathbf{L}_K(T_{A_K}) = \mathbf{Y}^{-1}$, $\{\mathbf{L}_K(\cdot), \mathbf{F}_K\}$ is a pair of Floquet factors of closed-loop system (10) for $\mathbf{K}(\cdot)$ given in Eq. (17) if and only if

$$\begin{aligned}
&\forall t,\ 0 \le t < T_{A_K},\\
&\mathbf{L}_K(t) = \big[\mathbf{W}(t,0)\mathbf{\Phi}^T(T_{A_K},t)\mathbf{W}^{-1}(T_{A_K},0)\\
&\int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K},\tau)\mathbf{L}_K(\tau)d\tau - \int_0^t \mathbf{\Phi}(t,\tau)\mathbf{L}_K(\tau)d\tau\big]\mathbf{F}_K + \mathbf{L}_{K_0}(t),
\end{aligned} \tag{20}$$

where

$$\mathbf{L}_{K_0}(t) = \mathbf{\Phi}(t,0) + \mathbf{W}(t,0)\mathbf{\Phi}^T(T_{A_K},t)\mathbf{W}^{-1}(T_{A_K},0)\left\{\mathbf{Y}_K{}^{-1} - \mathbf{\Phi}(T_{A_K},0)\right\}$$

is the value of $\mathbf{L}_K(\cdot)$ when $\mathbf{F}_K = \mathbf{0}$.

*Proof:* [⇒] Lem. 1 showed that if $\mathbf{L}_K(\cdot)$ is the solution of boundary-value problem (12), then $\mathbf{L}_K(\cdot)$ satisfies Eq. (13). Starting from Eq. (13), we can write

$$\mathbf{L}_K(t) \triangleq \mathbf{L}_{K_1}(t) + \mathbf{L}_{K_2}(t) + \mathbf{L}_{K_3}(t),$$

where

$$\mathbf{L}_{K_1}(t) \triangleq \mathbf{\Phi}(t,0),$$
$$\mathbf{L}_{K_2}(t) \triangleq -\int_0^t \mathbf{\Phi}(t,\tau)\mathbf{B}(\tau)\mathbf{K}(\tau)\mathbf{L}_K(\tau)d\tau,$$

and

$$\mathbf{L}_{K_3}(t) \triangleq -\int_0^t \mathbf{\Phi}(t,\tau)\mathbf{L}_K(\tau)d\tau\mathbf{F}_K.$$

Upon consideration of $\mathbf{L}_{K_2}(t)$ using the expression for $\mathbf{K}(t)$ derived in Eq. (17), we obtain

$$\mathbf{L}_{K_2}(t) = -\int_0^t \mathbf{\Phi}(t,\tau)\mathbf{B}(\tau)\Bigg[-\mathbf{B}^T(\tau)\mathbf{\Phi}^T(t,\tau)\mathbf{\Phi}^T(T_{A_K},t)\mathbf{W}^{-1}(T_{A_K},0)$$
$$\left\{\mathbf{Y}_K{}^{-1} - \mathbf{\Phi}(T_{A_K},0) + \int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K},s)\mathbf{L}_K(s)ds\mathbf{F}_K\right\}\Bigg]\mathbf{L}_K^{-1}(\tau)\mathbf{L}_K(\tau)d\tau.$$

After simplification, this reduces to

$$\begin{aligned}\mathbf{L}_{K_2}(t) &\triangleq \mathbf{W}(t,0)\mathbf{\Phi}^T(T_{A_K},t)\mathbf{W}^{-1}(T_{A_K},0)\left\{\mathbf{Y}_K{}^{-1} - \mathbf{\Phi}(T_{A_K},0) + \int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K},s)\mathbf{L}_K(s)ds\mathbf{F}_K\right\}\\ &= \mathbf{L}_{K2a}(t) + \mathbf{L}_{K2b}(t),\end{aligned}$$

where

$$\mathbf{L}_{K2a}(t) \triangleq \mathbf{W}(t,0)\mathbf{\Phi}^T(T_{A_K},t)\mathbf{W}^{-1}(T_{A_K},0)\left\{\mathbf{Y}_K{}^{-1} - \mathbf{\Phi}(T_{A_K},0)\right\},$$

and

$$\mathbf{L}_{K2b}(t) \triangleq \mathbf{W}(t,0)\mathbf{\Phi}^T(T_{A_K},t)\mathbf{W}^{-1}(T_{A_K},0)\int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K},s)\mathbf{L}_K(s)\ ds\ \mathbf{F}_K.$$

We can now easily identify

$$\begin{aligned}\mathbf{L}_{K_0}(t) &= \mathbf{L}_{K_1}(t) + \mathbf{L}_{K_{2a}}(t) \\ \mathbf{L}_K(t) &= \mathbf{L}_{K_{2b}}(t) + \mathbf{L}_{K_3}(t) + \mathbf{L}_{K_0}(t).\end{aligned}$$

■

*Proof:* [$\Leftarrow$] $\mathbf{L}_K(\cdot)$ given in Eq. (20) corresponds to $\mathbf{L}_K(\cdot)$ in Eq. (13) for $\mathbf{K}(\cdot)$ given in Eq. (17). Lem. 1 already proved the equivalence between Eqs. (12b) and (13) for any $\mathbf{K}(\cdot)$. We need to show that the boundary conditions given in Eq. (12a) are satisfied for this choice of $\mathbf{K}(\cdot)$. At $t = 0$, recalling that $\mathbf{W}(0,0) = \mathbf{0}$, Eq. (20) simplifies to

$$\mathbf{L}_K(0) = \mathbf{L}_{K_0}(0) = \mathbf{\Phi}(0,0) = \mathbf{I}.$$

On the other hand, at $t = T_{A_K}$, since $\mathbf{\Phi}(T_{A_K}, T_{A_K}) = \mathbf{I}$, Eq. (20) becomes

$$\begin{aligned}\mathbf{L}_K(T_{A_K}) &= \mathbf{L}_{K_0}(T_{A_K}) \\ &= \mathbf{\Phi}(T_{A_K},0) + \left\{\mathbf{Y}_K{}^{-1} - \mathbf{\Phi}(T_{A_K},0)\right\} = \mathbf{Y}_K{}^{-1}.\end{aligned}$$

■

*Note V.1:* Eq. (20) can be rewritten as an integral equation by noting that

$$\mathbf{L}_K(t) = \int_0^{T_{A_K}} \mathbf{P}(t,\tau)\mathbf{L}_K(\tau)d\tau\mathbf{F}_K + \mathbf{L}_{K_0}(t),$$

where

$$\mathbf{P}(t,\tau) = \begin{cases} \mathbf{W}(t,0)\mathbf{\Phi}^T(T_{A_K},0)\mathbf{W}^{-1}(T_{A_K},0)\mathbf{\Phi}(T_{A_K},\tau) \\ -\mathbf{\Phi}(t,\tau), \quad 0 \le \tau \le t \le T_{A_K}, \\ \mathbf{W}(t,0)\mathbf{\Phi}^T(T_{A_K},0)\mathbf{W}^{-1}(T_{A_K},0)\mathbf{\Phi}(T_{A_K},\tau), \\ \qquad 0 \le t \le \tau \le T_{A_K}. \end{cases}$$

*Note V.2:* It is sufficient to determine $\mathbf{L}_K(\cdot)$ over $(0, T_{A_K})$ because only values within this interval are used for $\mathbf{K}(\cdot)$.

*Note V.3:* It is of course possible to use the controllability Gramian $\mathbf{W}_C$ instead of the reachability Gramian $\mathbf{W}(t_2,t_1)$. In this case, it can be verified that $\forall t, \quad 0 \le t < T_{A_K}, \quad \forall k \in \mathbb{N}$,

$$\begin{aligned}
\mathbf{K}(t) &= \mathbf{B}^T(t)\mathbf{\Phi}^T(0,t)\bar{\mathbf{K}}_c\mathbf{L}_K^{-1}(t), \quad \mathbf{K}(t+kT_{A_K}) = \mathbf{K}(t) \\
\bar{\mathbf{K}}_c &= -\mathbf{W}_C{}^{-1}(T_{A_K},0)\left\{\mathbf{\Phi}(0,T_{A_K})\mathbf{Y}_K^{-1} - \mathbf{I} + \int_0^{T_{A_K}} \mathbf{\Phi}(0,\tau)\mathbf{L}_K(\tau)d\tau\mathbf{F}_K\right\}, \\
\mathbf{L}_K(t) &= \mathbf{\Phi}(t,0)\left[\mathbf{W}_C(t,0)\mathbf{W}_C{}^{-1}(T_{A_K},0)\int_0^{T_{A_K}} \mathbf{\Phi}(0,\tau)\mathbf{L}_K(\tau)d\tau - \int_0^t \mathbf{\Phi}(0,\tau)\mathbf{L}_K(\tau)d\tau\right]\mathbf{F}_K \\
&\quad + \mathbf{L}_{K_0}(t), \\
\mathbf{L}_{K_0}(t) &= \mathbf{\Phi}(t,0)[\mathbf{I} + \mathbf{W}_C(t,0)\mathbf{W}_C{}^{-1}(T_{A_K},0)\left\{\mathbf{\Phi}(0,tT_{A_K})\mathbf{Y}_K{}^{-1} - \mathbf{I}\right\}].
\end{aligned}$$

*Note V.4:* The solution of the matrix integral equation (20) can be achieved via a spectral method (e.g., [35]); i.e., the unknown entries of the matrix $\mathbf{L}_K(t)$ are expanded as a linear combination of suitable basis functions (such as Chebyshev polynomials). Then using the operational matrices of integration and product, a linear system is constructed and solved for the unknown coefficients. The computation was implemented using the spectral software package COCA [36]; see [13] for full details and Matlab source code for such an example.

In summary, here are the steps in the design procedure:

1. Assign a new period $T_{A_K}$ for the closed-loop system such that

$$\det\left[\mathbf{W}(T_{A_K}, 0)\right] \neq 0.$$

2. Assign the desired monodromy matrix $\mathbf{\Phi}_K(T_{A_K}, 0)$ of the closed-loop system satisfying the relation (cf. Note I.2)

$$\det\left[\mathbf{\Phi}_K(T_{A_K}, 0)\right] > 0.$$

3. Choose a pair $\{\mathbf{Y}_K, \mathbf{F}_K\}$ such that

$$\mathbf{Y}_K \mathbf{\Phi}(T_{A_K}, 0) = \exp(T_{A_K} \mathbf{F}_K).$$

4. Solve Eq. (20) for $\mathbf{L}_K(\cdot)$.
5. Compute the constant matrix $\mathbf{K}_c$ from Eq. (16b).
6. Construct the feedback matrix $\mathbf{K}(t)$ from Eq. (17).

*Note V.5:* Should the condition $\det\left[\mathbf{\Phi}_K(T_{A_K}, 0)\right] > 0$ not be satisfied, a spurious solution $\mathbf{L}_K(\cdot)$ of the integral equation may still exist. However, such a solution will be singular for some $t_s$. The matrix $\mathbf{K}(\cdot)$ will therefore not be defined at $t_s$, and thus the closed-loop system cannot be synthesized.

## VI. Example

Let

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t)u(t), \tag{22}$$

where

$$\mathbf{A}(t) = 2\pi \begin{bmatrix} -1+\alpha\cos^2(2\pi t) & 1-\alpha\sin(4\pi t)/2 \\ -1-\alpha\sin(4\pi t)/2 & -1+\alpha\sin^2(2\pi t) \end{bmatrix},$$

$$\mathbf{b}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and $\alpha$ is a parameter. In all simulations, we will set $\alpha = 1.2$. This system was used in [27] to illustrate the shortcomings of a proposed control scheme.

A straightforward check on the rank of the reachability Gramian shows that system (22) is controllable at $t = 1/2$. The next step involves determining the period of the feedback and choosing an appropriate Yakubovich matrix $\mathbf{Y}_K$. For the purposes of this paper, we wish to avoid a lengthy theoretical discussion on the choice of $\mathbf{Y}_K$. Generally speaking, the construction of a Yakubovich matrix $\mathbf{Y}_K$ can be based on the stem function [37, p.411]

$$f(x) = \begin{cases} -1, & x < 0 \\ 1, & \text{otherwise.} \end{cases}$$

We note that the choice of $\mathbf{Y}_K$ gives an *a priori* knowledge of the periodicity of the factor $\mathbf{L}_K(t)$, something which could not be determined *a priori* from previous theory. We also note that depending on the choice of $\mathbf{Y}_K$, not all matrices $\mathbf{F}_K$ will necessarily contain the classical stability information of the original system; i.e., the periodic system is stable if and only if the eigenvalues of $\mathbf{F}_Y$ have negative real parts. Further details and examples are available in [13, 14].

For this example, it is possible to work on period $T = 1$ so that the choice $\mathbf{Y}_K = \mathbf{I}$ can be made. A pair of Floquet factors $\mathbf{L}(\cdot)$ and $\mathbf{F}$ of the state-transition matrix of (22) are known:

$$\mathbf{L}(t) = \begin{bmatrix} \cos(2\pi t) & \sin(2\pi t) \\ -\sin(2\pi t) & \cos(2\pi t) \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 2\pi(\alpha - 1) & 0 \\ 0 & -2\pi \end{bmatrix}.$$

It is clear that for $\alpha > 1$, one of the Floquet exponents lies in the right-hand plane. We know from Floquet-Lyapunov theory that this is a necessary and sufficient condition for instability of system (22). Indeed we can see the unstable behaviour in components $\mathbf{\Phi}_{[1,1]}(t,0)$ and $\mathbf{\Phi}_{[2,1]}(t,0)$ of the open-loop state transition matrix in Fig. 1. Using the method described in the previous section, we can now build a controller which will assign the monodromy matrix of the closed-loop system. Here we choose

$$\mathbf{\Phi}_K(T,0) = \begin{bmatrix} \mathrm{e}^{-2\pi T} & 0 \\ 0 & \mathrm{e}^{-2\pi T} \end{bmatrix},$$

which corresponds to

$$\mathbf{F}_K = \begin{bmatrix} -2\pi & 0 \\ 0 & -2\pi \end{bmatrix}, \quad \mathbf{Y}_K = \mathbf{I}.$$

Note that we are able to increase the multiplicity of the Floquet exponent $(-2\pi)$ of the closed-loop system. Note also that we do not influence the original stable Floquet exponent of the open-loop system. Thus by construction $\mathbf{\Phi}_{K[1,1]}(t,0) = \mathbf{\Phi}_{K[2,2]}(t,0) = \mathbf{\Phi}_{[2,2]}(t,0)$ and $\mathbf{\Phi}_{K[2,1]}(t,0) = \mathbf{\Phi}_{K[1,2]}(t,0) = \mathbf{\Phi}_{[1,2]}(t,0)$. The variations of the closed-loop state-transition matrix are also displayed in Fig. 1. The components of the gain matrix $\mathbf{K}(t)$ are displayed in Fig. 2, where we note the discontinuity in the gain matrix at $t = T$. The resulting feedback $u(t)$ is decomposed into two parts: `u1` that stabilizes the originally unstable mode, and u2, which is of course zero. These variations are also displayed in Fig. 2. The details of the implementation and calculations, including direct computation of $\mathbf{L}_K^{-1}(t)$ and computationally efficient control of subsystems containing only unstable modes, are described elsewhere.

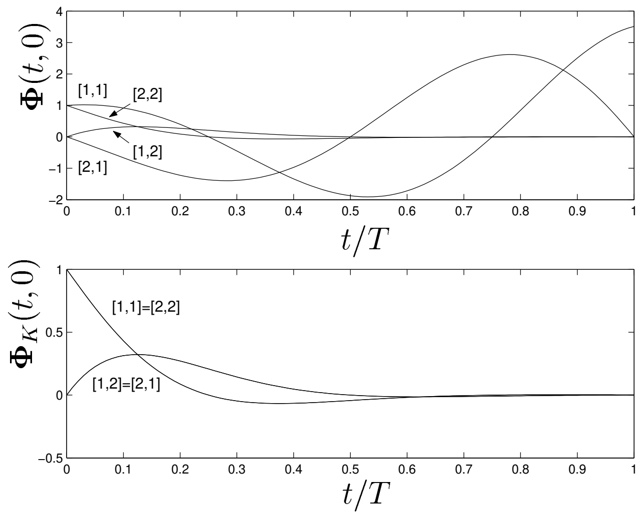


Fig. 1. Entries of the open-loop and closed-loop state-transition matrices.

## VII. Conclusion

In this paper we develop the synthesis of a novel controller for LTP systems. The design technique relies on assigning the constant matrix $\mathbf{F}$ of the Floquet pair of factors, and on solving for the periodic factor $\mathbf{L}(t)$. By doing so, this scheme intrinsically allows for the assignment of all the invariants of the system, i.e., the whole monodromy matrix, as opposed to the sole eigenvalue assignment that is well known for LTI systems. This provides a confirmation of the old conjecture that a continuous

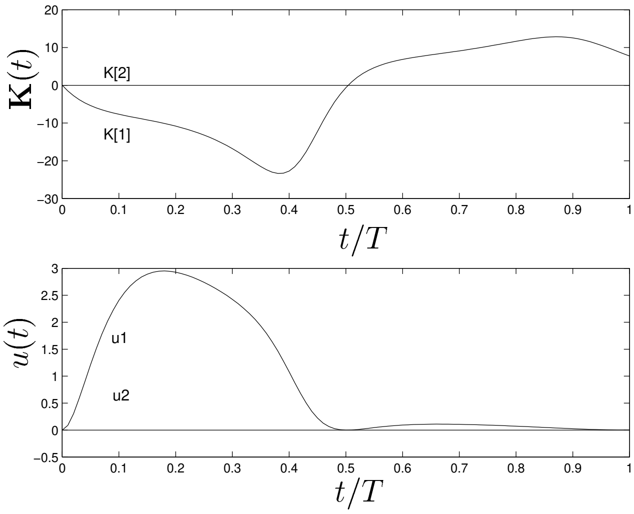


Fig. 2. Entries of the gain matrix $\mathbf{K}(t)$ and feedback $u(t)$.

periodically varying feedback can achieve results that are impossible with a constant gain matrix. The scheme presented here is the first ever to achieve in continuous time what so far only controllers based on the "sampled state periodic hold" strategy could achieve.

The method hinges on the specific form introduced for the time-varying feedback matrix presented in Eq. (17). This novel form leads to the introduction of the controllability index, which is the shortest number of periods over which the open-loop system is controllable, and the reachability Gramian, which is known to be invertible whenever the open-loop system is controllable. It is a significant improvement over existing work based on approximating the inverse of the input matrix by its generalized inverse, and provides a continuous-time companion to existing discrete schemes.

We illustrate our method on a simple periodic system. The implementation relies on a robust computational procedure for solving a matrix integral equation. We achieved this via a spectral method, with further refinement provided by a boundary-value problem formulation (see [13] for full details and a more difficult example). We are currently investigating computational methods to enhance the viability of our procedure.

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