CONTROL OF A SHAKING BELT CONVEYOR BY A NOVEL APPLICATION OF FLOQUET-LYAPUNOV THEORY

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ABSTRACT

One method for realizing the self-orientation of workpieces for grasping by a robot is by means of a shaking belt conveyor. Control of small perturbations around the steady state leads to the control of a linear time-periodic system. We propose and successfully implement an approach for control based on a novel application of Floquet-Lyapunov theory.

1 INTRODUCTION

A shaking belt conveyor was proposed in [1] to achieve the self-orientation of workpieces in flexible manufacturing systems. Workpieces, initially deposited onto the conveyor with a random orientation, would acquire a known orientation by the time they are to be grasped by a robotic manipulator (see Fig. 1) with the aid of a vision system. The task of the vision system would, therefore, consist of determining only the location of on landmark point on each workpiece, since the uncertainty regarding its orientation would have been eliminated. This would significantly alleviate the computational burden on the vision system.

This conveyor is to be driven by a multi-step cam mechanism whose synthesis was first reported in [1] and is still under development. The objective of this paper is to address the issues involved in the control of this system at operating speed.

2 PROBLEM FORMULATION

The transmission is driven by a DC motor, as shown in Fig. 2 and we assume that a steady-state has been reached.

We aim to design a controller that would reject any incoming perturbations, thus allowing the belt conveyor to be driven at a constant speed. We model the transmission taking into account the input shaft elasticity k_s, the viscous damping of the rotor c_1, and that of the bearings of the cam c_2 and the follower c_3 (see Fig. 2). Let ψ, θ and ϕ be the respective angular displacements of the rotor after speed reduction, of the cam and of the follower. The electro-mechanical equations of the system are then given by:

$$
\begin{aligned}
i + \frac{k_v}{R}\dot{\psi} &= i_s \\
\tau_m &= k_v i \\
I_1\ddot{\psi} - k_s(\theta - \psi) &= \tau_m - c_1\dot{\psi} \\
(I_2 + I_3\phi'^2(\theta))\ddot{\theta} & \\
+I_3\phi'(\theta)\phi''(\theta)\dot{\theta}^2 & \\
+k_s(\theta - \psi) &= -(c_2 + c_3\phi'^2(\theta))\dot{\theta},
\end{aligned}
$$

where $0 < r < 1$ is the reduction ratio and $I_1 = 1/r^2 I_{\text{rotor}}, I_2 = I_{\text{cam}}, I_3 = I_{\text{follower}} + I_{\text{load}}$.

The acceleration ratio over one complete revolution of the cam has been defined [1] as

$$
\phi''(\psi) = \begin{cases} \frac{1}{2}\alpha_M[1 - \cos(\frac{2\pi\psi}{\Delta\psi_1})], \\ \qquad 0 \le \psi \le \Delta\psi_1 \\ -\frac{1}{2}\alpha_m[1 - \cos(\frac{2\pi}{\Delta\psi_2}(\psi - \Delta\psi_1))], \\ \qquad \Delta\psi_1 \le \psi \le 2\pi \end{cases}
$$

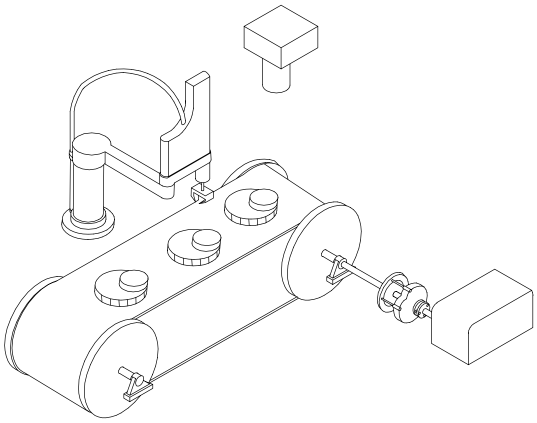

Figure 1: Shaking belt conveyor, robotic manipulator, and vision system.

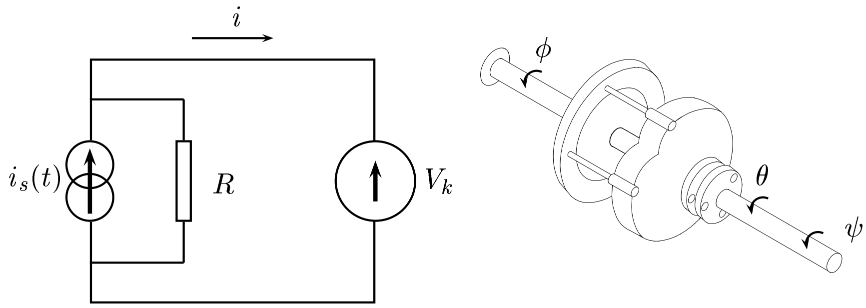

Figure 2: Iconic model of the electromechanical system.

where $\Delta\psi_1 + \Delta\psi_2 = 2\pi$. Moreover, the following condition [1] is imposed:

$$\frac{\alpha_M}{\alpha_m} = \frac{\Delta\psi_2}{\Delta\psi_1}$$

with α_M and α_m taking the forms

$$\alpha_M = \frac{4}{N\Delta\psi_1} - \frac{4a}{\Delta\psi_1},\ \alpha_m = \frac{4}{N\Delta\psi_2} - \frac{4a}{\Delta\psi_2},$$

where a is defined as $\phi'(0)$, and N is the number of steps of the mechanism.

After linearization about a constant speed ω_0, we eventually obtain:

$$\begin{aligned} I_1\delta\ddot{\psi} &= -k_s\delta\psi - (c_1 + {k_v}^2/R)\delta\dot{\psi} \\ &\quad + k_s\delta\theta + k_v\delta i_s \\ (I_2 + I_3\phi'^2)\delta\ddot{\theta} &= -(2\omega_0 I_3\phi'\phi'' + c_2 + c_3\phi'^2)\delta\dot{\theta} \\ &\quad - k_s\delta\theta + k_s\delta\psi. \end{aligned}$$

We choose as state vector

$$x = \begin{bmatrix} \delta\psi \\ \delta\dot{\psi} \\ \delta\theta \\ \delta\dot{\theta} \end{bmatrix},$$

while the current $\delta i_s(t)$ is the input variable $u(t)$. The system can then be in the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

where

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ A_{21} & A_{22} & -A_{21} & 0 \\ 0 & 0 & 0 & 1 \\ A_{41} & 0 & -A_{41} & A_{44} \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} 0 \\ k_v/I_1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\begin{aligned} A_{21} &= -k_s/I_1, \\ A_{22} &= \frac{-(c_1 + {k_v}^2/R)}{I_1}, \\ A_{41} &= \frac{k_s}{I_2 + I_3\phi'^2(t)}, \\ A_{44} &= \frac{-(2\omega I_3\phi'(t)\phi''(t) + c_2 + c_3\phi'^2(t))}{I_2 + I_3\phi'^2(t)}. \end{aligned}$$

We note that $\mathbf{A}(t)$ is now a T-periodic matrix. We display two of its entries, A_{22} and A_{44} in Fig. 2. Should both $\mathbf{A}(t)$ and $\mathbf{b}(t)$ have been constant matrices, then the use of well-known control strategies for linear time-invariant (LTI) systems would have been straightforward. Unfortunately, results deriving from LTI systems cannot usually be extended to the time-varying case. In fact, long-established and widely applied control schemes for LTI models, e.g. PID controllers, may even contribute to destabilizing a time-varying system.

3 LINEAR TIME-PERIODIC SYSTEMS AND FLOQUET-LYAPUNOV THEORY

Due to its periodicity, our system belongs to a very special class of time-varying systems. Whereas most time-varying systems are often dealt with on a case-by-case basis, linear time-periodic (LTP) systems have been the focus of many studies since the turn of the century. A detailed historical perspective of this topic can be found in [2]. This interest stems from the fact that many physical systems exhibit a periodic behaviour, and they can often be reduced to a set of linear differential equations with periodic coefficients.

One of the most commonly used mathematical method in the analysis of LTP systems is Floquet-Lyapunov theory, which consists of two main results [3]. The first main result states that given the linear time-periodic homogeneous system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t), \\ \mathbf{x}(0) &= \mathbf{x}_0, \\ \mathbf{A}(t+T) &= \mathbf{A}(t), \end{aligned}$$

the state transition matrix (STM) can be decomposed as a product of two factors

$$\Phi(t,0) = \mathbf{L}(t,0)e^{\mathbf{F}t}, \tag{2}$$

where $\mathbf{L}(.,0)$ is a matrix-valued function, periodic with period T, and $\mathbf{F}$ is a constant matrix defined by:

$$\mathbf{F} = \frac{1}{T}\ln(\Phi(T,0)). \tag{3}$$

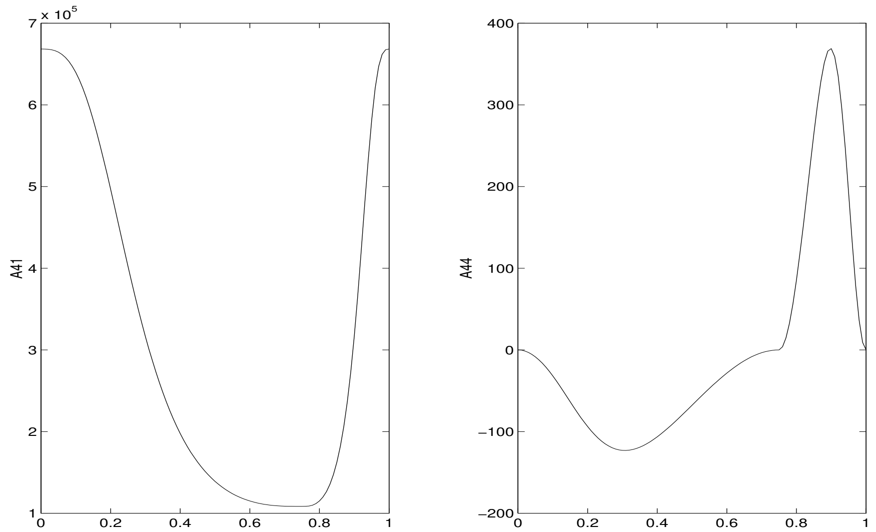

Figure 3: Selected components of A.

The eigenvalues of $\mathbf{F}$ are called the Floquet exponents and determine the stability of the homogeneous system[1].

However, in general, $\mathbf{F}$ is complex. We avoid the use of complex algebra by working on two periods rather than one. Redefining $\mathbf{L}$ and $\mathbf{F}$ as

$$\begin{aligned} \Phi_{2T}(t,0) &= \mathbf{L}_{2T}(t,0)e^{\mathbf{F}_{2T}t}, && (4)\\ \mathbf{F}_{2T} &= \frac{1}{2T}\ln(\Phi_{2T}(2T,0)), && (5)\\ \mathbf{L}_{2T}(t+2T,0) &= \mathbf{L}_{2T}(t,0), && (6)\\ \mathbf{L}_{2T}(0,0) &= \mathbf{1}, && (7) \end{aligned}$$

where $\mathbf{1}$ denotes the identity matrix, it is then known that $\mathbf{F}_{2T}$ will be real [4].

As mentioned above, study of the stability of the system requires only the knowledge of $\mathbf{F}$ — or, equivalently, of the monodromy matrix $\Phi(T,0)$. The monodromy matrix can be computed by a variety of ways, among which direct numerical integration has been the focus of most investigations so far [5, 6]. However, these studies have thus far only concentrated on an approximation of the monodromy matrix. Furthermore, even when restricted to the purpose of stability analysis, this approach is not always satisfying. For instance, Ewald [7] notes that when investigating helicopter stability, it can be difficult to obtain a good approximation of the Floquet multipliers by direct numerical integration.

The second main result of the Floquet-Lyapunov theory states that the equivalence transformation

$$\mathbf{x}(t) = \mathbf{L}(t,0)\mathbf{z}(t) \qquad (8)$$

will produce an equivalent LTI system, namely,

$$\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t). \qquad (9)$$

Note that if the system is not homogeneous but has an input $\mathbf{u}(t)$, application of the transformation (8) will result in a transformed system

$$\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t) + \mathbf{L}^{-1}(t,0)\mathbf{B}(t)\mathbf{u}(t), \qquad (10)$$

where it must be emphasised that the non-homogeneous term is still time-varying and periodic.

4 PREVIOUS WORK

Among the fields of mechanical engineering involving research on time-periodic systems, two of the most active throughout the last three decades have certainly been space mechanics and aeronautics. The issues related to the control of a spinning satellite [8, 9] and to the control of the blades of a helicopter rotor [10, 11] have provided the framework for many investigations. Early studies

[1]The system is stable if and only if all the Floquet exponents are in the left-hand plane.

were primarily focused on determining the stability of an existing system [12, 13, 14], and control schemes have only been considered since the late 1970's. A short review of these approaches is given by Pandiyan and Sinha [15]. Sinha et al. [16, 17] recently proposed a computation of the STM based on Chebyshev polynomials. Subsequently, this approach allowed them to derive control laws for the placement of the Floquet exponents [18] and the optimal control of an LTP system [19].

5 COMPUTATION OF THE L AND F MATRICES

It is known that $\mathbf{L}$ is periodic and satisfies the following first-order matrix differential equation [20]:

$$\dot{\mathbf{L}}(t,0) = \mathbf{A}(t)\mathbf{L}(t,0) - \mathbf{L}(t,0)\mathbf{F}, \tag{11a}$$

with the boundary conditions:

$$\mathbf{L}(0,0) = \mathbf{L}(2T,0) = \mathbf{1}. \tag{11b}$$

As $\mathbf{F}$ is constant, we can add the equation

$$\dot{\mathbf{F}} = 0 \tag{11c}$$

so that the problem is well posed.

Solving this system for $\mathbf{L}(.,0)$ and $\mathbf{F}$ is done by using the COLDAE software developed by Ascher and Spiteri [21], even though the full scope of the software goes far beyond the requirements for the solution of problem (11). As eq. (11a) is not linear, a numerical solution first requires providing an initial guess. In our approach, we have opted for producing this initial guess with a zero-order-hold-like algorithm proposed by Hsu [22, 23].

The main idea is to discretize the time interval $[0, 2T]$ into a sequence of subintervals and approximate the initial LTP system by an LTI system on each of these subintervals. Hsu showed in [22] that the monodromy matrix of the discretized system converges to that of the LTP system as the maximum subinterval length approaches zero.

We extend this approach to also obtain the initial guess for $\mathbf{L}$ once the initial guess for $\mathbf{F}$ has been made. On each subinterval, substituting $\Phi_k(t_k) = \mathbf{L}_k(t_k)e^{t_k\mathbf{F}}$ allows us to derive an approximation for the corresponding $\mathbf{L}(t_k)$.

We have not, so far, formally investigated the convergence of $\mathbf{L}_k$ towards $\mathbf{L}$, but, as far as the issue of finding an initial guess is concerned, it has always produced adequate results for our intended use. It should be noted that although good approximations for the eigenvalues of the monodromy matrix are obtained relatively easily, the same cannot be said for the individual elements of the matrices $\mathbf{L}$ and $\mathbf{F}$. Hence, this approximation does not constitute an efficient yet effective way to compute these matrices.

6 SIMULATION RESULTS

Rahmani and Franklin [24, 25] have proposed a novel control strategy for LTP systems based on the use of sampled data and constant output feedback. To the best of our knowledge, their schemes have not yet been put into practice or further investigated because they heavily depend on the knowledge of the STM at any instant (and the authors left aside the issue of how to compute the STM). We use our computation of the STM to implement the placement of the eigenvalues of $\mathbf{F}$ as described in [24]. The procedure is illustrated in Fig. 4 along with the results from simulation. These results are obtained for an initial state vector

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

7 CONCLUSIONS

In this paper, we have provided a framework for the control of a shaking belt conveyor by means of a linear time-periodic system. We have used our knowledge of Floquet-Lyapunov theory to compute the state-transition matrix at any time through its Floquet-Lyapunov factors $\mathbf{L}$ and $\mathbf{F}$. We then apply a strategy that is dependent on an accurate calculation of the state-transition matrix to control our system. We demonstrate the overall effectiveness of our approach by giving some typical simulation results.

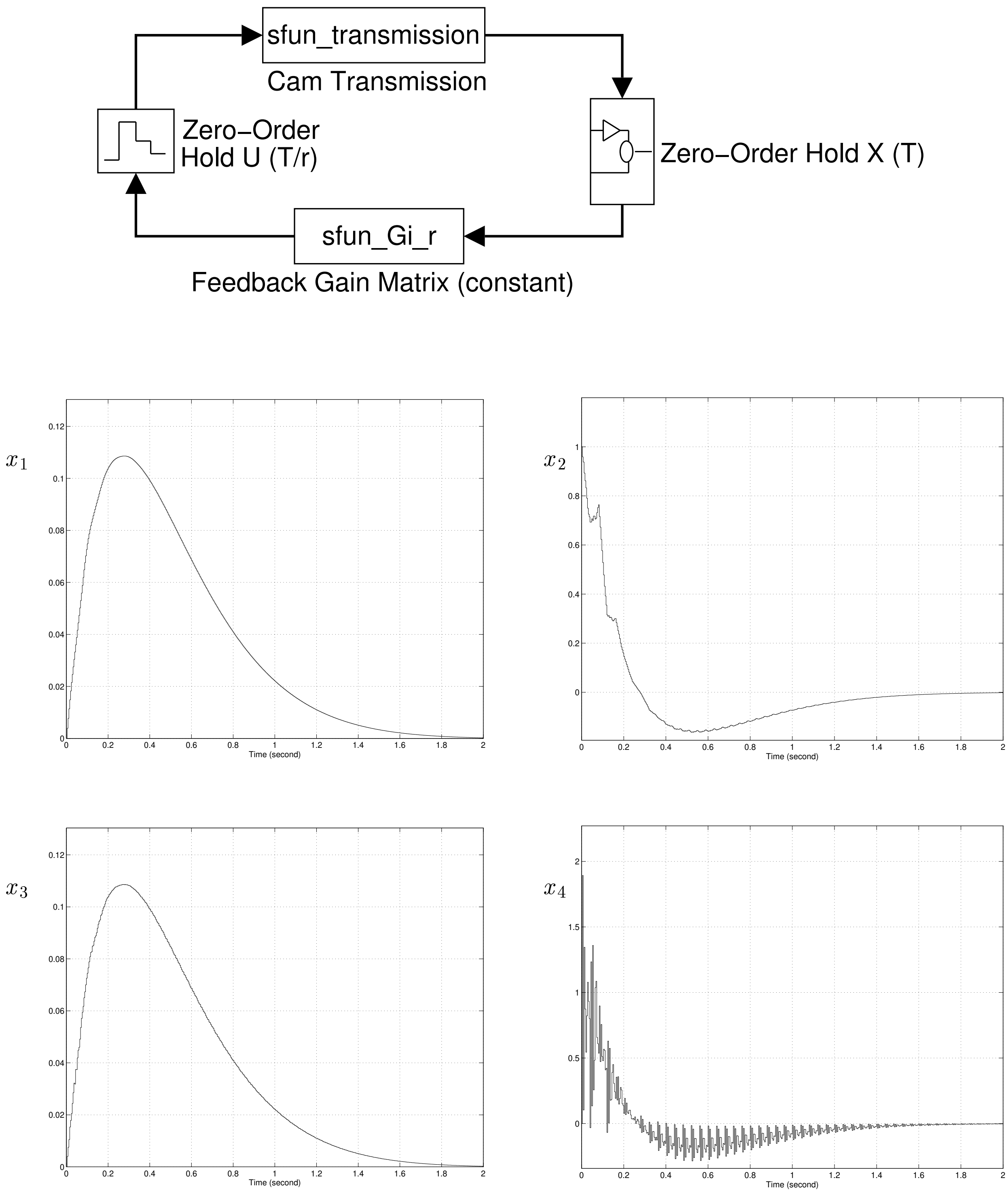

Figure 4: Simulation results.

8 ACKNOWLEDGEMENTS

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