Pitfalls of a Least-Squares-Equivalent Controller for Linear, Time-Periodic Systems

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Abstract

We review a technique for the design of controllers for linear, time-periodic systems. A major appeal of the technique, first proposed by Sinha and Joseph, is the use of Floquet-Lyapunov theory to transform the periodic system to a form where classical control strategies for time-invariant systems may be employed. However, it is normally impossible to find a completely time-invariant control system that is equivalent to the original time-varying system: Application of the Floquet-Lyapunov transformation in fact yields a time-varying control system that the technique makes equivalent to a time-invariant one in the least-squares sense, in order to subsequently synthesize the controller via pole placement using a constant feedback matrix.

However, classical control and Floquet-Lyapunov theory clearly show that it is erroneous to conclude that the behaviour of the least-squares-equivalent, time-invariant system always matches the behaviour of the original time-periodic system. Using an example found in the original paper, we provide a simple counter-example that illustrates the failure of the proposed strategy and analysis of the reasons for its failure.

1 Introduction

Linear, time-periodic (LTP) systems are quite common in scientific and engineering applications, e.g., linkages and cam mechanisms driven at a uniform speed, helicopter rotor

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blades in forward flight, and the study of bifurcation of periodic orbits. The control of these systems is also of great interest, and is made no less challenging by the fact that the (time-varying) eigenvalues of the periodic matrix in general play no part in determining the stability of the system (Wu 1974, Verhulst 1990). Consider the control system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), && \text{(1a)} \\ \mathbf{x}(0) &= \mathbf{x}_0, && \text{(1b)} \end{aligned}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{u} \in \mathbb{R}^m$, and $\mathbf{B} \in \mathbb{R}^{m\times n}$, and the matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are periodic with period T. Because the matrix $\mathbf{A}(t)$ of (1) is time periodic, the Floquet Representation Theorem (e.g., Brockett 1970) allows us to write the state-transition matrix $\mathbf{\Phi}(t)$ of this system in the form

$$\mathbf{\Phi}(t) = \mathbf{L}(t)e^{t\mathbf{R}},$$

where the matrix $\mathbf{L}(t)$ is T-periodic and the matrix $\mathbf{R} = (1/T)\ln \mathbf{\Phi}(T)$ is constant. The eigenvalues of $\mathbf{R}$ are called the *Floquet exponents* and fully determine the stability of the system (Coddington and Levinson 1955). By means of the Lyapunov Reducibility Theorem (Brockett 1970), the transformation $\mathbf{x}(t) = \mathbf{L}(t)\mathbf{z}(t)$ transforms system (1) into

$$\begin{aligned} \dot{\mathbf{z}}(t) &= \mathbf{R}\mathbf{z}(t) + \mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{u}(t), && \text{(2a)} \\ \mathbf{z}(0) &= \mathbf{x}_0. && \text{(2b)} \end{aligned}$$

It is apparent that system (2) is not completely time invariant. Hence, classical control methods to relocate the poles of $\mathbf{R}$ are not applicable.

In keeping with classical control theory, we seek a control law for the system of equations (1) of the form

$$\mathbf{u} = -\mathbf{K}(t)\mathbf{x}. \tag{3}$$

Indeed, it is well known (Brunovský 1970) that, if the pair $\{\mathbf{A}(t), \mathbf{B}(t)\}$ is controllable, there exists a periodic feedback $\mathbf{K}(t)$ to assign the Floquet exponents of $\mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}(t)$. The problem of assigning the Floquet invariants of a time-periodic system was discussed in detail by Willems, Kučera, and Brunovský (1984). Several approaches are available for discrete systems (e.g., Colaneri and Kučera 1997), but a general solution has yet to be found for the continuous case. There have been two main approaches to the pole-placement problem for continuous time LTP systems; a distinction can be made on the basis of whether the scheme focuses on placement of the eigenvalues of the state-transition matrix $\mathbf{\Phi}(T)$ versus those of the matrix $\mathbf{R}$. For instance, Yoshii and Hakomori (1989) proposed to assign the eigenvalues of the state-transition matrix of system (1) by constructing a feedback matrix as a sequence of periodic functions, while Fisher and Bryson Jr. (1995) used an analytical gain-scheduling approach. Al-Rahmani and Franklin (1989) gave a method for pole placement based on multi-rate control. Their work followed on Chammas and Leondes (1978a, 1978b, 1979), Kabamba (1986) and Juan and Kabamba (1989). On the other hand, Calico and Wiesel (1984), and Kern (1980, 1986) built schemes based on the transformed system (2).

In this paper, we provide an analysis of the method of pole placement given by Sinha and Joseph (1994) for general continuous-time LTP systems based on the control of a least-squares-equivalent auxiliary system.

2 The Auxiliary System

2.1 Procedure

Sinha and Joseph (1994) propose a new method, which was subsequently used by Pandiyan and Sinha (1995), based on an auxiliary linear, time-invariant (LTI) system, namely,

$$\begin{aligned} \dot{\tilde{\mathbf{z}}}(t) &= \mathbf{R}\tilde{\mathbf{z}}(t) + \tilde{\mathbf{B}}\mathbf{v}(t), && \text{(4a)} \\ \tilde{\mathbf{z}}(0) &= \mathbf{z}_0, && \text{(4b)} \end{aligned}$$

where $\tilde{\mathbf{B}}$ is any constant matrix such that the pair $(\mathbf{R}, \tilde{\mathbf{B}})$ is controllable. A control law for system (4) can be written as

$$\mathbf{v}(t) = -\bar{\mathbf{K}}\tilde{\mathbf{z}}(t),$$

where $\bar{\mathbf{K}}$ is a constant matrix chosen such that $(\mathbf{R} - \tilde{\mathbf{B}}\bar{\mathbf{K}})$ has a given set of stable eigenvalues.

Now, defining the error vector

$$\mathbf{e}(t) \equiv \mathbf{z}(t) - \tilde{\mathbf{z}}(t), \tag{5}$$

the *error dynamics* of systems (2) and (4) satisfy

$$\dot{\mathbf{e}}(t) = (\mathbf{R} - \tilde{\mathbf{B}}\bar{\mathbf{K}})\mathbf{e}(t) + \mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{u}(t) + \tilde{\mathbf{B}}\bar{\mathbf{K}}\mathbf{z}(t). \tag{6}$$

Because $(\mathbf{R} - \tilde{\mathbf{B}}\bar{\mathbf{K}})$ is, by construction, a time-invariant stability matrix, Sinha and Joseph (1994) conclude that the two systems, (2) and (4), can be made equivalent in the least-squares sense if the gain from the state feedback defined in equation (3) can be chosen such that

$$\mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{u}(t) = -\tilde{\mathbf{B}}\bar{\mathbf{K}}\mathbf{z}(t), \tag{7}$$

which leads to

$$\mathbf{u}(t) = -\mathbf{B}^{\dagger}(t)\mathbf{L}(t)\tilde{\mathbf{B}}\bar{\mathbf{K}}\mathbf{z}(t), \tag{8}$$

where $\mathbf{B}^{\dagger}$ is the applicable generalized inverse of $\mathbf{B}$. Recalling that $\mathbf{z}(t) = \mathbf{L}^{-1}(t)\mathbf{x}(t)$, the feedback law for system (1) yields:

$$\mathbf{u}(t) = -\mathbf{B}^{\dagger}(t)\mathbf{L}(t)\tilde{\mathbf{B}}\bar{\mathbf{K}}\mathbf{L}^{-1}(t)\mathbf{x}(t), \tag{9}$$

Note that, using this feedback law, systems (1) and (2), respectively, become:

$$\dot{\mathbf{x}}(t) = \left[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{B}^{\dagger}(t)\mathbf{L}(t)\tilde{\mathbf{B}}\bar{\mathbf{K}}\mathbf{L}^{-1}(t)\right]\mathbf{x}(t), \tag{10}$$

and

$$\dot{\mathbf{z}}(t) = \left[\mathbf{R} - \mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{B}^{\dagger}(t)\mathbf{L}(t)\tilde{\mathbf{B}}\bar{\mathbf{K}}\right]\mathbf{z}(t), \tag{11}$$

while the closed-loop auxiliary system is

$$\dot{\tilde{\mathbf{z}}}(t) = \left(\mathbf{R} - \tilde{\mathbf{B}}\bar{\mathbf{K}}\right)\tilde{\mathbf{z}}(t). \tag{12}$$

This scheme is simple to understand and apply, and it has some geometric appeal. Moreover, in cases where $\mathbf{B}(t)$ is invertible, it gives the desired results.

2.2 Analysis of Error Dynamics

Based upon equations (11) and (12), the error dynamics defined in (5) and (6) become

$$\dot{\mathbf{e}}(t) = (\mathbf{R} - \tilde{\mathbf{B}}\bar{\mathbf{K}})\mathbf{e}(t) + \left[\tilde{\mathbf{B}}\bar{\mathbf{K}} - \mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{B}^{\dagger}(t)\mathbf{L}(t)\tilde{\mathbf{B}}\bar{\mathbf{K}}\right]\mathbf{z}(t). \tag{13}$$

The strategy proposed in (Sinha and Joseph 1994) is based on the claim that, since $(\mathbf{R} - \tilde{\mathbf{B}}\bar{\mathbf{K}})$ is stable by construction, this offers a guarantee that the error dynamics are stable and, therefore, the transformed system (11) will behave like the auxiliary system (12). However, this is not entirely true. In general, equation (7) will not hold exactly when $\mathbf{B}(t)$ is not invertible. The consequence is that the term in $\mathbf{z}(t)$ in equation (13) will not vanish. Furthermore, this term cannot be considered simply as a non-homogeneous term because $\mathbf{e}(t)$ itself depends on $\mathbf{z}(t)$.

Looking at the error dynamics in the form of equation (13) can mislead the control-scheme designer to bounding this non-homogeneous term. Such procedures do exist in control; for instance, Bellman (1953) considers an equation of the form

$$\dot{\mathbf{z}}(t) = \mathbf{A}\mathbf{z}(t) + \mathbf{f}(\mathbf{z}), \quad \mathbf{z}(0) = \mathbf{c}, \tag{14}$$

where $\mathbf{A}$ is a constant matrix and $\mathbf{f}(\mathbf{z})$ satisfies the condition

$$||\mathbf{f}(\mathbf{z})||/||\mathbf{z}|| \to 0 \text{ as } ||\mathbf{z}|| \to 0. \tag{15}$$

Bellman shows that if,

- Every solution of $\dot{\mathbf{z}}(t) = \mathbf{A}\mathbf{z}(t)$ approaches zero as $t \to \infty$, and
- $\mathbf{f}(\mathbf{z})$ is continuous in some region about $\mathbf{z} = \mathbf{0}$,

then $\mathbf{z} = \mathbf{0}$ is a stable solution of (14), and every solution of (14) for which $||\mathbf{z}(0)||$ is sufficiently small approaches zero as $t \to \infty$. However, invoking Bellman's theorem does not generally apply in our case because condition (15) may not be satisfied.

2.3 Floquet Analysis

As all the components of equation (13) are periodic, the most appropriate tool for analysing its stability is Floquet-Lyapunov Theory. Combining equations (13) and (11), we obtain a linear, time-periodic system of equations for $\mathbf{e}(t)$ and $\mathbf{z}(t)$:

$$\begin{bmatrix} \dot{\mathbf{z}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{R} - \mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{B}^{\dagger}(t)\mathbf{L}(t)\tilde{\mathbf{B}}\bar{\mathbf{K}} & \mathbf{0}_{n\times n} \\ (\tilde{\mathbf{B}}\bar{\mathbf{K}} - \mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{B}^{\dagger}(t)\mathbf{L}(t)\tilde{\mathbf{B}}\bar{\mathbf{K}}) & \mathbf{R} - \tilde{\mathbf{B}}\bar{\mathbf{K}} \end{bmatrix} \begin{bmatrix} \mathbf{z}(t) \\ \mathbf{e}(t) \end{bmatrix}, \tag{16}$$

where $\mathbf{0}_{n\times n}$ is the $n \times n$ zero matrix. Therefore, an evaluation of the system stability can be obtained from a straightforward application of classical results.

First, if $\mathbf{B}(t)$ is invertible, then $\mathbf{B}(t)\mathbf{B}^{\dagger}(t) = \mathbf{B}(t)\mathbf{B}^{-1}(t) = \mathbf{1}$, and the system reduces to a linear time-invariant matrix, namely,

$$\begin{bmatrix} \dot{\mathbf{z}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{R} - \tilde{\mathbf{B}}\bar{\mathbf{K}} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{R} - \tilde{\mathbf{B}}\bar{\mathbf{K}} \end{bmatrix} \begin{bmatrix} \mathbf{z}(t) \\ \mathbf{e}(t) \end{bmatrix},$$

whose eigenvalues correspond to the eigenvalues of $(\mathbf{R} - \tilde{\mathbf{B}}\bar{\mathbf{K}})$. In this case, both $\mathbf{z}(t)$ and $\mathbf{e}(t)$ would indeed be stabilized through an appropriate choice of a stabilizing feedback gain matrix $\bar{\mathbf{K}}$, i.e., the algorithm from (Sinha and Joseph 1994) would be successful.

However, in the general case, equation (7) is not satisfied exactly, and system (16) is not time invariant; it is time periodic. As is well known from Floquet-Lyapunov Theory, the only way to draw a definitive conclusion on the stability of system (16) is to examine all of its Floquet exponents (Verhulst 1990). The block lower-triangular nature of the matrix of (16) implies that the Floquet exponents associated with $\mathbf{e}(t)$ correspond to the eigenvalues of $(\mathbf{R} - \tilde{\mathbf{B}}\bar{\mathbf{K}})$; however, nothing can be said a priori of the Floquet exponents associated with $\mathbf{z}(t)$. Therefore, any claim based solely on the eigenvalues of $(\mathbf{R} - \tilde{\mathbf{B}}\bar{\mathbf{K}})$ assuring the convergence of both $\mathbf{e}(t)$ and $\mathbf{z}(t)$ is not justified. Unfortunately, although the algorithm in (Sinha and Joseph 1994) is simple, it is not foolproof. We present below a simple counter-example, built by using one of the systems included in the original paper by Sinha and Joseph (1994), and illustrating that $\mathbf{e}(t)$ and $\mathbf{z}(t)$ have different Floquet exponents with different regions of stability.

3 Counter-Example

3.1 Feedback Synthesis

Let

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t)u(t), \tag{17}$$

where

$$\mathbf{A}(t) = 2\pi \begin{bmatrix} -1+\alpha\cos^2(2\pi t) & 1-\alpha\sin(2\pi t)\cos(2\pi t) \\ -1-\alpha\sin(2\pi t)\cos(2\pi t) & -1+\alpha\sin^2(2\pi t) \end{bmatrix}, \ \mathbf{b}(t) = \begin{bmatrix} 0 \\ \sin(2\pi t) \end{bmatrix},$$

and α is a parameter. In all simulations, we will set $\alpha = 1.2$, as in (Sinha and Joseph 1994).

It is known (Brunovský 1969, Bittanti, Guardabassi, Maffezzoni, and Silverman 1978) that a periodic system of order n is controllable if and only if it is controllable on $[0, nT]$. A straightforward check on the rank of the controllability Gramian,

$$\mathbf{W}(2T, 0) = \int_0^{2T} \mathbf{\Phi}(0,\tau)\mathbf{b}(\tau)\mathbf{b}^T(\tau)\mathbf{\Phi}^T(0,\tau)d\tau$$

shows that system (17) is controllable.

The Floquet-Lyapunov factors $\mathbf{L}(t)$ and $\mathbf{R}$ of the state-transition matrix of (17) are

$$\mathbf{L}(t) = \begin{bmatrix} \cos(2\pi t) & \sin(2\pi t) \\ -\sin(2\pi t) & \cos(2\pi t) \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} 2\pi(\alpha-1) & 0 \\ 0 & -2\pi \end{bmatrix}.$$

It is clear that for $\alpha = 1.2$, one of the Floquet exponents lies in the right-hand plane. We know from Floquet-Lyapunov theory that this is a necessary and sufficient condition for instability of system (17).

Using the transformation $\mathbf{x}(t) = \mathbf{L}(t)\mathbf{z}(t)$, system (17) becomes

$$\dot{\mathbf{z}}(t) = \mathbf{R}\mathbf{z}(t) + \mathbf{L}^{-1}(t)\mathbf{b}(t)u(t). \tag{18}$$

Following (Sinha and Joseph 1994), we consider an auxiliary system of the form:

$$\dot{\tilde{\mathbf{z}}}(t) = \mathbf{R}\mathbf{z}(t) + \tilde{\mathbf{b}}(t)v(t). \tag{19}$$

In this counter-example, we take

$$\tilde{\mathbf{b}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It can be readily be verified that the pair $(\mathbf{R}, \tilde{\mathbf{b}})$ is controllable, therefore satisfying the criterion for choosing the auxiliary system.

We now seek a feedback of the form $v(t) = -\bar{\mathbf{k}}^T\tilde{\mathbf{z}}(t)$. Applying this feedback, the auxiliary system (4) leads to:

$$\dot{\tilde{\mathbf{z}}}(t) = (\mathbf{R} - \tilde{\mathbf{b}}\bar{\mathbf{k}}^T)\tilde{\mathbf{z}}(t). \tag{20a}$$

On the other hand, the corresponding feedback law for the transformed system (18), based on least-squares equivalence and given in equation (8), yields:

$$\dot{\mathbf{z}}(t) = \left[\mathbf{R} - \mathbf{L}^{-1}(t)\mathbf{b}(t)\mathbf{b}^{\dagger}(t)\mathbf{L}(t)\tilde{\mathbf{b}}\bar{\mathbf{k}}^T\mathbf{L}^{-1}\right]\mathbf{z}(t). \tag{20b}$$

The next step is to choose a suitable constant feedback gain $\bar{\mathbf{k}}$ for the closed-loop auxiliary system $(\mathbf{R} - \tilde{\mathbf{b}}\bar{\mathbf{k}}^T)$, namely,

$$\mathbf{R} - \tilde{\mathbf{b}}\bar{\mathbf{k}}^T = \begin{bmatrix} 2\pi(\alpha-1) - \bar{k}_1 & -\bar{k}_2 \\ -\bar{k}_1 & -2\pi - \bar{k}_2 \end{bmatrix}.$$

Choosing $\bar{k}_2 = 0$, the eigenvalues $\{\lambda_i\}_{i=1}^2$ of the above matrix are readily found to be

$$\lambda_1 = 2\pi(\alpha-1) - \bar{k}_1,\ \ \lambda_2 = -2\pi.$$

Therefore, any value for $\bar{k}_1$, such that

$$\bar{k}_1 > 2\pi(\alpha-1) \tag{21}$$

guarantees the stability of $(\mathbf{R} - \tilde{\mathbf{b}}\bar{\mathbf{k}}^T)$, i.e., of the closed-loop auxiliary system described by equation (20a). The claim that this stability is extended to the transformed system (20b) is shown to be false by examining the closed-form solution of system (20b) for three values of $\bar{k}_1$.

Simplification of equation (20b) yields:

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} 2\pi(\alpha-1) + \bar{k}_1\sin(2\pi t)(\cos(2\pi t) - \sin(2\pi t)) & 0 \\ \bar{k}_1\cos(2\pi t)(\sin(2\pi\sin(2\pi t) - \cos(2\pi t))) & -2\pi \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}. \tag{22}$$

The first equation is readily integrated to yield

$$z_1(t) = z_1(0)\exp\left[\int_0^t 2\pi(\alpha-1) + \bar{k}_1\sin(2\pi\tau)(\cos(2\pi\tau) - \sin(2\pi\tau))d\tau\right],$$

which can be written as

$$z_1(t) = Q_1(t)\exp(\mu_1 t), \tag{23a}$$

with μ_1 constant and equal to

$$\mu_1 = 2\pi(\alpha-1) - \bar{k}_1/2, \tag{23b}$$

while $Q_1(t)$ is the periodic function given by

$$Q_1(t) = z_1(0)\exp\left[\bar{k}_1 \sin(2\pi t)(\sin(2\pi t) + \cos(2\pi t))/(4\pi)\right].$$

With $z_1(t)$ known, $z_2(t)$ satisfies:

$$\dot{z}_2(t) = -2\pi z_2(t) + z_1(t).$$

Solving the homogeneous equation leads immediately to

$$z_{2h}(t) = z_2(0)\exp(\mu_2 t),$$

where

$$\mu_2 = -2\pi.$$

After finding a particular solution, the general solution is

$$z_2(t) = Q_2(t)\exp(-2\pi t),$$

where $Q_2(t)$ is a periodic function, namely,

$$Q_2(t) = z_2(0) + z_1(0)\int_0^t \mathrm{e}^{(2\pi\alpha - \bar{k}_1/2)\tau}\mathrm{e}^{\left[\bar{k}_1 \sin(2\pi\tau)(\sin(2\pi\tau)+\cos(2\pi\tau))/(4\pi)\right]}\,d\tau.$$

3.2 Stability Analysis

It is easy to recognize μ_1 and μ_2 as the Floquet exponents of system (20b). Note that the exponent -2π has not changed, as could have been expected based on the form of equation (22). On the other hand, the Floquet exponent μ_1 associated with $z_1(t)$ and given in equation (23b) gives us the stability condition:

$$\bar{k}_1 > 4\pi(\alpha - 1).$$

This condition is more restrictive than the stability condition found for the auxiliary system set out in equation (21). Indeed, choosing to place the first Floquet exponent of the auxiliary system between 0 and $-4\pi(\alpha - 1)$ corresponds to

$$2\pi(\alpha - 1) < \bar{k}_1 < 4\pi(\alpha - 1).$$

$\bar{k}_1$	Auxiliary system (20a)		Transformed System (20b)	
	Poles: λ_1, λ_2		Floquet Exponents: μ_1, μ_2	
$3\pi(\alpha-1)$	$-\pi(\alpha-1)$	-2π	$\frac{\pi}{2}(\alpha-1)$	-2π
$4\pi(\alpha-1)$	$-2\pi(\alpha-1)$	-2π	0	-2π
$5\pi(\alpha-1)$	$-3\pi(\alpha-1)$	-2π	$-\frac{\pi}{2}(\alpha-1)$	-2π

Table 1: Floquet exponents of (6) depending on placement of unstable pole.

This will result in a scenario where although the auxiliary system will be stable, the transformed system and, hence, the original system will be unstable. For instance, consider $\bar{k}_1 = 3\pi(\alpha-1),\ 4\pi(\alpha-1),$ and $5\pi(\alpha-1)$. These three values satisfy condition (21) for the stability of the closed-loop auxiliary system (20a). Nevertheless, the closed-loop original system (10) and its transform (20b) are, respectively, unstable, oscillating, and stable. Indeed, the numerical values of the Floquet exponents, $\{\mu_i\}_{i=1}^2$ of system (20b) for each chosen $\bar{k}_1$ are provided in Table 1, showing that the first Floquet exponent is, respectively, positive, zero, and negative. Additionally, we display the system response obtained from the three different pole placements in Figure 1.

3.3 Remark

One might be tempted to correct this scheme by adopting a procedure where the choice for the constant feedback gain depends on the exponents of the closed-loop transformed system, rather than on the auxiliary system. We must stress again that the Floquet exponents in our counter-example were conveniently computed only due to the lower-triangular nature of the system matrix in equation (22), and the fact that we only needed to move a single exponent. This approach is similar to those in (Kern 1980, Calico and Wiesel 1984, Kern 1986), and has so far proved unsuccessful to provide a general solution to the assignment of Floquet exponents.

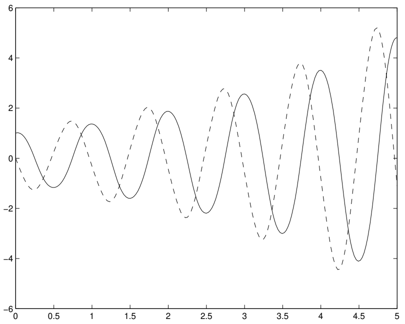
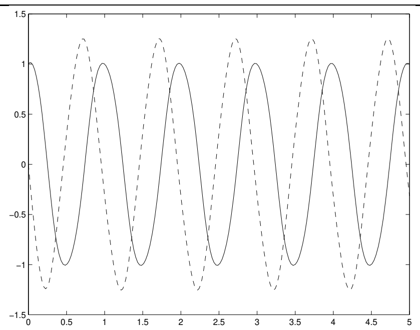
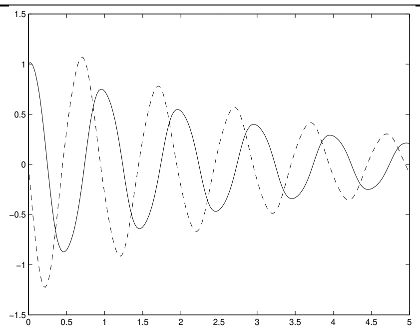
Pole placement of auxiliary system	Corresponding Floquet exponent in original system	Response of original system (x_1 —, x_2 - -)
$\lambda_1 = -\pi(\alpha - 1) < 0$ stable	$\mu_1 = \frac{\pi}{2}(\alpha - 1) > 0$ unstable	
$\lambda_1 = -2\pi(\alpha - 1) < 0$ stable	$\mu_1 = 0$ marginally stable	
$\lambda_1 = -3\pi(\alpha - 1) < 0$ stable	$\mu_1 = -\frac{\pi}{2}(\alpha - 1) < 0$ stable	

Figure 1: Illustration that stability of the auxiliary system does not guarantee stability of the original system

4 Conclusions

The novelty of the technique for the design of a controller presented by Sinha and Joseph (1994) hinges on the introduction of an 'auxiliary' least-squares-equivalent system to facilitate the pole-placement procedure for periodic systems. The scheme presented is easy to understand and implement, and in cases where the input matrix $\mathbf{B}(t)$ is invertible, it can be effective. However, using the conditions of stability for periodic systems provided by Floquet-Lyapunov theory, we show that stability of this 'auxiliary' system will not necessarily be conveyed to the original system when $\mathbf{B}(t)$ is not invertible. Moreover, attempts to repair this scheme lead back to already existing procedures.

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