The control of linear time-periodic systems using Floquet-Lyapunov theory

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In this paper we use Floquet-Lyapunov theory to derive the Floquet factors of the state-transition matrix of a given linear time-periodic system. We show how the periodicity of one of the factors can be determined a priori using a constant matrix, which we call the *Yakubovich matrix*, based upon the signs of the eigenvalues of the monodromy matrix. We then describe a method for the numerical computation of the Floquet factors, relying upon a boundary-value problem formulation and the Yakubovich matrix. Further, we show how the invertibility of the controllability Gramian and a specific form for the feedback gain matrix can be used to derive a control law for the closed-loop system. The controller can be full-state or observer-based. It also allows the engineer to assign all the invariants of the system; i.e., the full monodromy matrix. Deriving the feedback matrix requires solving a matrix integral equation for the periodic Floquet factor of the new state-transition matrix of the closed-loop system. This is achieved via a spectral method, with further refinement possible through a boundary-value problem formulation. The computational efficiency of the scheme may be further improved by performing the controller synthesis on the transformed system obtained from the Lyapunov reducibility theorem. The effectiveness of the method is illustrated with an application to a quick-return mechanism using a software toolbox developed for MATLAB™.

1 Introduction

The study of periodically varying systems dates back to the nineteenth century, an excellent detailed account being given by Richards (1983). Faraday (1831) is credited with the first recorded demonstration of parametric behavior using a vibrating membrane. With Mathieu (1868), research started to focus more specifically on periodic systems, and notably produced his eponymous equation. Floquet (1883) and Lyapunov (1992) independently established the fundamental form of solutions of linear time-periodic (LTP) systems. Their work is at the core of what is now known as *Floquet-Lyapunov theory.* Hill (1886) and Poincaré (1892), in turn, opened the field to celestial mechanics and were the first to discuss the motion of (natural) satellites.

Linear time-invariant (LTI) systems have been more extensively studied, with many different strategies developed for their control. Yet, modelling real-world processes often leads to a LTP system. In engineering alone, many mechanical systems work under a periodic regime in steady-state conditions and can be reduced to a LTP formulation under small perturbations. This is the case, for instance, of cam mechanisms (Spiteri *et al.* , 1998), manipulators for repetitive tasks (Streit *et al.* , 1985), helicopters (Calico & Wiesel, 1986; Friedmann & Silverthorn, 1974; Peters & Hohenemser, 1971; Webb *et al.* , 1991), and satellites (Calico Jr. & Yeakel, 1983; Cole & Calico, 1992; Kane & Barba, 1966; Mingori, 1969).

It is well known, however, that results established for LTI systems do not usually carry over to time-varying systems. LTP systems are an exception in that they exhibit similar behavior to LTI systems. This raises the prospect of enabling the control designer to apply the abundant knowledge of LTI systems to LTP systems.

2 Fundamental Concepts

For quick reference, we recall here a few concepts and results pertinent to LTP systems.

Definition 2.1 (Periodicity)
Let $\mathbb{R}^+$ and $\mathbb{R}^{+^*}$ denote the sets of real non-negative and real strictly positive numbers. A similar notation is applied for the set of integers $\mathbb{Z}$.

- *A function is* T-periodic *if $T \in \mathbb{R}^{+^*}$ and $\forall t,\ f(t+T) = f(t)$.*
- *A function is* T-antiperiodic *if $T \in \mathbb{R}^{+^*}$ and $\forall t,\ f(t+T) = -f(t)$.*
- *A function is* primarily T-periodic *or periodic with primary period T if $T \in \mathbb{R}^{+^*}$ is the smallest number such that $f(\cdot)$ is T-periodic.*
- *A function is* generally $2T$-periodic *if it is primarily $2T$-periodic and it is not T-antiperiodic.*

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Definition 2.2 (Linear Time-Periodic System) *A* linear time-periodic system *is a dynamical system represented by:*

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{1a}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t), \tag{1b}$$

where $\mathbf{x}(\cdot) \in \mathbb{R}^n$, $\mathbf{u}(\cdot) \in \mathbb{R}^r$, *and* $\mathbf{y}(\cdot) \in \mathbb{R}^m$; $\mathbf{A}(\cdot)$, $\mathbf{B}(\cdot)$, *and* $\mathbf{C}(\cdot)$ *are, respectively,* $n \times n$, $n \times r$, *and* $m \times n$ *matrices whose elements are known, piecewise continuous, real-valued, primarily* T_A*-periodic functions on* $\mathbb{R}^+$.

Definition 2.3 (Fundamental Matrix) *Any nonsingular solution* $\mathbf{X}(t)$ *of the homogeneous matrix differential system*

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t) \tag{2}$$

is called a fundamental matrix *of (2).*

Definition 2.4 (State-Transition Matrix) *There exists a unique fundamental matrix* $\mathbf{\Phi}(t, t_0)$ *of (2) such that* $\mathbf{\Phi}(t_0, t_0) = \mathbf{1}$, *the* $n \times n$ *identity matrix; this matrix is called the* state-transition matrix *(STM) of the system.*

Without loss of generality we shall assume henceforth that $t_0 = 0$.

Definition 2.5 (Controllability (Reachability) Gramian) *The* controllability Gramian $\mathbf{W}_C(t_2, t_1)$ *and the* reachability Gramian $\mathbf{W}_R(t_2, t_1)$ *on* $[t_1, t_2]$ *are*

$$\mathbf{W}_C(t_2, t_1) \triangleq \int_{t_1}^{t_2} \mathbf{\Phi}(t_1, \tau)\mathbf{B}(\tau)\mathbf{B}^T(\tau)\mathbf{\Phi}^T(t_1, \tau)d\tau, \tag{3a}$$

and

$$\mathbf{W}_R(t_2, t_1) \triangleq \int_{t_1}^{t_2} \mathbf{\Phi}(t_2, \tau)\mathbf{B}(\tau)\mathbf{B}^T(\tau)\mathbf{\Phi}^T(t_2, \tau)d\tau. \tag{3b}$$

In the results below, $\mathbf{W}(t_2, t_1)$ stands indistinctly for either $\mathbf{W}_R(t_2, t_1)$ or $\mathbf{W}_C(t_2, t_1)$.

Theorem 2.1 *(D'Angelo, 1970) System (1) is controllable on* $[t_1, t_2]$ *if and only if* $\mathbf{W}(t_2, t_1)$ *is nonsingular.*

Theorem 2.2 *(Kalman* et al. *, 1969; Weiss, 1968) The rank of* $\mathbf{W}(t_2, t_1)$ *is nondecreasing as the range of the integration is increased.*

In the special case where $\mathbf{A}(\cdot)$ and $\mathbf{B}(\cdot)$ are *constant*, controllability can also be evaluated as explained below.

Definition 2.6 (Controllability Index) *Given two* $n \times n$ *and* $n \times r$ *constant matrices* $\mathbf{M}$ *and* $\mathbf{N}$, *let*

$$\mathbf{U}_k = \left[\mathbf{N} \mid \ \mathbf{MN} \mid \ \cdots \mid \ \mathbf{M}^k\mathbf{N}\right].$$

The controllability index of the pair $\{\mathbf{M}, \mathbf{N}\}$ *is the smallest integer* k_0 *such that* $\mathbf{U}_{k_0-1}$ *has rank* n.

Theorem 2.3 *The pair* $\{\mathbf{M}, \mathbf{N}\}$ *is controllable if and only if* $k_0 \leq n$.

We now recall some useful results from linear algebra. Any solution $\mathbf{X}$ of the matrix equation $\exp(\mathbf{X}) = \mathbf{M}$, $\mathbf{M} \in \mathbb{C}^{n\times n}$, is called a *logarithm* of $\mathbf{M}$, denoted $\mathbf{X} = \log(\mathbf{M})$. We denote the set of all solutions by $\mathcal{L}og\{\mathbf{M}\} = \{\mathbf{X} : \exp(\mathbf{X}) = \mathbf{M}\}$ and the subset of all real solutions by $\mathcal{RL}og\{\mathbf{M}\}$.

Theorem 2.4 *(Culver, 1966) Let* $\mathbf{M}$ *be a real square matrix. There exists a real solution* $\mathbf{X}$ *to the equation* $\exp(\mathbf{X}) = \mathbf{M}$ *if and only if* $\mathbf{M}$ *is nonsingular and each Jordan block of* $\mathbf{M}$ *belonging to a negative eigenvalue occurs an even number of times.*

As a consequence, we have:

Corollary 2.5 *(Lukes, 1982) For each nonsingular* $\mathbf{M} \in \mathbb{R}^{n\times n}$ *there exists a real solution* $\mathbf{R} \in \mathbb{R}^{n\times n}$ *to the equation* $\mathbf{M}^2 = \exp(\mathbf{R})$.

Finally, we recall some results for spectral methods.

Definition 2.7 *Let matrices* $\mathbf{M}$ *and* $\mathbf{N}$ *be* $m_1 \times n_1$ *and* $m_2 \times n_2$ *respectively. The* Kronecker product *(or* tensor product*)* $\mathbf{M} \otimes \mathbf{N}$ *is the* $m_1 m_2 \times n_1 n_2$ *matrix defined as*

$$\mathbf{M} \otimes \mathbf{N} \triangleq \begin{bmatrix} M_{1,1}\mathbf{N} & M_{1,2}\mathbf{N} & \cdots & M_{1,n1}\mathbf{N} \\ \vdots & \vdots & & \vdots \\ M_{m_1,1}\mathbf{N} & M_{m_1,2}\mathbf{N} & \cdots & M_{m_1,n1}\mathbf{N} \end{bmatrix}.$$

Definition 2.8 *With each matrix* $\mathbf{M} = [M_{i,j}] \in \mathbb{R}^{m\times n}$, *we associate the vector* $\mathrm{vec}(\mathbf{M}) \in \mathbb{R}^{mn}$ *defined by*

$$\mathrm{vec}(\mathbf{M}) \triangleq [M_{1,1} \cdots M_{m,1} \ M_{1,2} \cdots M_{m,2} \cdots M_{1,n} \cdots M_{m,n}]^T.$$

Lemma 2.1 *(Horn & Johnson, 1994) Let* $\mathbf{M} \in \mathbb{R}^{m\times n}$, $\mathbf{N} \in \mathbb{R}^{p\times q}$, *and* $\mathbf{P} \in \mathbb{R}^{m\times q}$ *be given and let* $\mathbf{X} \in \mathbb{R}^{n\times p}$ *be unknown. The matrix equation*

$$\mathbf{MXN} = \mathbf{P}$$

is equivalent to the system of mq *equations in* np *unknowns given by*

$$\left[\mathbf{N}^T \otimes \mathbf{M}\right] \mathrm{vec}(\mathbf{X}) = \mathrm{vec}(\mathbf{P})$$

That is, $\mathrm{vec}(\mathbf{MXN}) = \left[\mathbf{N}^T \otimes \mathbf{M}\right] \mathrm{vec}(\mathbf{X})$.

2.1 Operational Matrices in Spectral Methods

Operational matrices are a useful tool in spectral methods (cf. e.g., Trefethen, 2001). The order-s approximation of a scalar function $f(\cdot)$ can be written as

$$f(t) \approx \boldsymbol{\sigma}^T(t)\mathbf{f}^s = \mathbf{f}^{sT}\boldsymbol{\sigma}(t),$$

where $\mathbf{f}^s \in \mathbb{R}^s$ is the vector of the *expansion coefficients* of $f(\cdot)$, and

$$\boldsymbol{\sigma}(t) = \begin{bmatrix}\sigma_0(t) & \sigma_1(t) & \cdots & \sigma_{s-1}(t)\end{bmatrix}^T$$

is the spectral basis chosen.

The *integration operation* can be approximated by a constant matrix $\mathbf{G}$ known as the *operational matrix of integration.* This matrix $\mathbf{G}$ depends upon the spectral basis only and is defined as the real constant matrix which satisfies

$$\int_0^t f(\tau)d\tau \approx \boldsymbol{\sigma}^T(t)\mathbf{G}^T\mathbf{f}^s. \tag{4}$$

The product of two scalar functions $f(\cdot)$ and $g(\cdot)$ can be approximated using an *operational matrix of product* $\mathbf{Q}(\cdot)$. This matrix $\mathbf{Q}(\cdot)$ also depends on the basis and its entries are functions of the approximation of either function. In fact, $\mathbf{Q}(\cdot)$ can be defined as the $s \times s$ real matrix which satisfies

$$f(t)g(t) \approx \mathbf{f}^{sT}\boldsymbol{\sigma}(t)\boldsymbol{\sigma}^T(t)\mathbf{g}^s = \boldsymbol{\sigma}^T(t)\mathbf{Q}(\mathbf{f}^s)\mathbf{g}^s. \tag{5}$$

3 Review of Floquet-Lyapunov Theory

In this section, we recall classical results from Floquet-Lyapunov theory. This allows us to motivate more recent theoretical results on real Floquet factors, our computational method for determining these factors, and our choice of control strategy.

3.1 Preliminaries

Although the results recalled here (e.g., Lukes, 1982; Yakubovich & Starzhinskii, 1975) apply to any fundamental matrix of solutions, we focus on the STM.

Definition 3.1 *We define two sets of matrices:*

- $\mathcal{A}_{T_A}$ *is the set of real, piecewise continuous, primarily T_A-periodic $n \times n$ matrices $\mathbf{A}(\cdot)$;*
- $\mathcal{L}_{T_L}^{\mathbb{K}}$ *is the set of $n \times n$ matrices $\mathbf{L}(\cdot)$ of $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, where the $\mathbf{L}(t)$ are T_L-periodic, nonsingular for all t, continuous with a piecewise continuous derivative, and such that $\mathbf{L}(0) = \mathbf{1}$.*

Theorem 3.1 *If $\mathbf{A}(\cdot) \in \mathcal{A}_{T_A}$, then the STM $\boldsymbol{\Phi}(\cdot, 0)$ of the system*

$$\forall t \in \mathbb{R}^+, \quad \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \tag{6}$$

satisfies

$$\forall t \in \mathbb{R}^+, \quad \boldsymbol{\Phi}(t + T_A, 0) = \boldsymbol{\Phi}(t, 0)\boldsymbol{\Phi}(T_A, 0). \tag{7}$$

Conversely, let $\mathbf{X}(\cdot)$, such that $\mathbf{X}(0) = \mathbf{1}$, be a nonsingular continuous $n \times n$ matrix function of time, having a piecewise continuous derivative. If T_A is the smallest number for which $\mathbf{X}(\cdot)$ satisfies Eq. (7), then $\mathbf{X}(\cdot)$ is the STM of some system of the form (6) where $\mathbf{A}(\cdot)$ is a periodic function of time whose primary period is T_A.

For a proof, see Chapter II, § 1.7 and § 2.1 of (Yakubovich & Starzhinskii, 1975).

Note 3.1 *The STM $\boldsymbol{\Phi}(\cdot, 0)$ of a LTP system is usually not periodic, but it is apparent that*

$$\forall (t, u) \in \mathbb{R}^{+*}, \quad \boldsymbol{\Phi}(t + T_A, u + T_A) = \boldsymbol{\Phi}(t, u).$$

Note 3.2 *By transitivity, Eq. (7) can be generalized to*

$$\forall t \in \mathbb{R}^+, \quad \boldsymbol{\Phi}(t + kT_A, 0) = \boldsymbol{\Phi}(t, 0)\boldsymbol{\Phi}^k(T_A, 0). \tag{8}$$

3.2 Floquet-Lyapunov Theory

Theorem 3.2 (Floquet Representation Theorem) *The STM $\boldsymbol{\Phi}(\cdot, 0)$ of Eq. (6) can be factored as*

$$\boldsymbol{\Phi}(t, 0) \triangleq \mathbf{L}(t)\exp(t\mathbf{F}), \tag{9}$$

where $\mathbf{L}(\cdot) \in \mathcal{L}_{T_A}^{\mathbb{C}}$, and $\mathbf{F} \in \mathbb{C}^{n \times n}$. Conversely, given $\mathbf{L}(\cdot) \in \mathcal{L}_{T_A}^{\mathbb{C}}$ and an arbitrary $\mathbf{F} \in \mathbb{C}^{n \times n}$, matrix $\boldsymbol{\Phi}(\cdot, 0)$ defined by Eq. (9) is the STM of some system (6) with $\mathbf{A}(\cdot) \in \mathcal{A}_{T_A}$.

At this point it is important to stress that the STM of a *real* matrix $\mathbf{A}(\cdot)$ will generally have *complex* Floquet factors $\mathbf{L}(\cdot)$ and $\mathbf{F}$. A well-known result follows:

Corollary 3.3 (Real Floquet Factors) *(e.g., Montagnier, 2002) If $\mathbf{A}(\cdot) \in \mathcal{A}_{T_A}$, then it is always possible to obtain a pair of real Floquet factors by regarding $\mathbf{A}(\cdot) \in \mathcal{A}_{T_R}$, where $T_R = 2T_A$.*

Note 3.3 *As we now apply Theorem 3.2 with $\mathbf{A}(\cdot) \in \mathcal{A}_{T_R}$, $\mathbf{L}(t, 0)$ can only be assumed to be T_R-periodic:*

$$\forall t \in \mathbb{R}^+, \ \mathbf{L}_{2T_A}(t + 2T_A) = \mathbf{L}_{2T_A}(t), \ \mathbf{L}(0) = \mathbf{1}.$$

In other words, the real T_A-periodic system of Eq. (6) always admits at least one pair of real Floquet factors $\{\mathbf{L}_{2T_A}(\cdot), \mathbf{F}\} \in \mathcal{L}_{2T_A}^{\mathbb{R}} \times \mathbb{R}^{n \times n}$.

Theorem 3.4 (Lyapunov Reducibility Theorem) *Let $\mathbf{L}(\cdot)$ and $\mathbf{F}$ be the same Floquet factors as in Theorem 3.2. The substitution*

$$\mathbf{x}(t) = \mathbf{L}(t)\mathbf{z}(t) \tag{10a}$$

transforms the LTP system (6) into the LTI system

$$\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t). \tag{10b}$$

Note 3.4 *In the case of a general nonhomogenous LTP system (1), the change of variable defined in Eq. (10a) will transform the system equations into*

$$\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t) + \mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{u}(t),$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{L}(t)\mathbf{z}(t).$$

3.3 Stability Analysis

Definition 3.2 (Monodromy Matrix) *The value of the STM of the homogeneous system (6) after one period, $\mathbf{\Phi}(T_A, 0)$, is called the* monodromy matrix. *The eigenvalues of the monodromy matrix are called the* Floquet multipliers *(a.k.a. Poincaré multipliers, characteristic multipliers, or characteristic roots) of the system.*

Definition 3.3 *The eigenvalues of $\mathbf{F}$ in Theorem 3.2 are called the* Floquet exponents *(a.k.a. Poincaré exponents or characteristic exponents) of the system.*

Theorem 3.5 *(e.g., Brockett, 1970) The Floquet multipliers of a LTP system are unique, whereas its Floquet exponents are not. Moreover, a LTP system is asymptotically stable if and only if all its Floquet multipliers lie within the unit circle; this is equivalent to its Floquet exponents having negative real parts.*

3.4 Dual Nature of LTP Systems

The results of the previous sections illustrate the dual nature of LTP systems: they can be interpreted either from the discrete- or the continuous-time point of view:

- At time $t = kT_A$, Eq. (8) becomes

  $$\mathbf{\Phi}(kT_A, 0) = \mathbf{\Phi}^k(T_A, 0),$$

  and any solution to (6) satisfies

  $$\begin{aligned}\mathbf{x}\left((k+1)T_A\right) &= \mathbf{\Phi}^{k+1}(T_A, 0)\mathbf{x}(0) \\ &= \mathbf{\Phi}(T_A, 0)\mathbf{\Phi}^k(T_A, 0)\mathbf{x}(0) = \mathbf{\Phi}(T_A, 0)\mathbf{x}(kT_A).\end{aligned}$$

  Because of this correspondence with a discrete time-invariant system, $\mathbf{\Phi}(T_A, 0)$ is sometimes called (D'Angelo, 1970) the *discrete transition matrix.* The system is thus described in the z-plane and the stability criterion of the Floquet multipliers being within the unit circle indeed corresponds to discrete LTI systems.

- Analogously, the transformed system of Eq. (10b) gives a description of the system in the s-plane and the stability criterion (exponents with negative real parts) is similar to continuous LTI system theory.

3.5 Controllability

The classical notion of controllability due to Kalman (1963) has been refined over the years for the specific framework of LTP systems (Bittanti *et al.* , 1978, 1984; Brunovský, 1969; Hewer, 1975; Kabamba, 1986; Kano & Nishimura, 1985; Silverman & Meadows, 1967).

In the general case, Brunovský in his seminal paper (Brunovský, 1969) proved that a LTP system of order n is controllable if and only if it is controllable over n periods. Bittanti *et al.* (1984) showed that this reduced to m periods, where m is the degree of the minimal polynomial of the monodromy matrix. Finally, Kabamba (1986) showed that the number of periods is at most equal to the controllability index of the pair $\{\mathbf{\Phi}(T_A, 0), \mathbf{W}(T_A, 0)\}$, which is known (Chen, 1984) to be less than m. A summary of the foregoing results is given below:

Theorem 3.6 *Consider system (1) of order n, its corresponding monodromy matrix $\mathbf{\Phi}(T_A, 0)$, and its Gramian $\mathbf{W}(T_A, 0)$ at $t = T_A$. Furthermore, let ν be the controllability index of the pair $\{\mathbf{\Phi}(T_A, 0), \mathbf{W}(T_A, 0)\}$. The statements below are equivalent:*

- *System (1) is controllable;*
- *The pair $\{\mathbf{\Phi}(T_A, 0), \mathbf{W}(T_A, 0)\}$ is controllable;*
- *System (1) is controllable over $(0, \nu T_A)$;*
- *$\mathbf{W}(\nu T_A, 0)$ is positive-definite.*

The relationship between controllability and invariant assignment for LTP systems was provided by Brunovský (1969):

Theorem 3.7 *A LTP system is controllable if and only if there exists a T_A-periodic $r \times n$ matrix $\mathbf{K}(\cdot)$ which can assign the eigenvalues of the monodromy matrix in such a way that they appear as complex conjugate pairs, and their product is positive (cf. Liouville's Theorem (Bellman, 1970a)).*

Note 3.5 *Recalling the duality between controllability and observability (e.g., Chen, 1984, Theorem 5-10), in Theorem 3.6 all the results are preserved if 'controllable' is replaced by 'observable' and $\mathbf{W}(t_2, t_1)$ is replaced by the observability Gramian*

$$\mathbf{W}_O(t_2, t_1) \triangleq \int_{t_1}^{t_2} \mathbf{\Phi}^T(t, t_1)\mathbf{C}^T(t)\mathbf{C}(t)\mathbf{\Phi}(t, t_1)dt.$$

4 Real Floquet Factors

As we saw in Section 3, the STM of a LTP system is unique, but its Floquet factors are not. In the case of real factors derived from Corollary 3.3, Note 3.3 pointed out that the period of $\mathbf{L}(\cdot)$ can then only be assumed to be $2T_A$. This leads to a loss of information on its primary periodicity, as illustrated via a simple example proposed by Wu (1978). Wu's example shows (Montagnier, 2002) that Corollary 3.3 is useful to prove the existence of real factorizations, but fails to provide all pertinent pieces of information relative to the periodicity of the Floquet factors. To keep consistent with the complex Floquet factors and the other general properties of LTP systems, a new result should be based on $\mathbf{\Phi}(T_A, 0)$, rather than on $\mathbf{\Phi}(2T_A, 0)$. This result should be able to indicate a priori the periodicity of $\mathbf{L}(\cdot)$. Moreover, we notice that to a primarily $2T_A$-periodic $\mathbf{L}_{F_1}(\cdot)$ corresponds a primarily T_A-periodic system, whereas one may have expected only a primarily $2T_A$-periodic $\mathbf{A}(\cdot)$. Therefore, a converse statement would also be welcome.

In this section, we review a result which achieves both objectives while highlighting the pitfalls of existing results. The details are available in (Montagnier *et al.* , 2003). Henceforth, the proofs of continuity do not vary from the standard theorems, and we focus on periodicity.

4.1 Previous Work

Yakubovich (1970) and Yakubovich & Starzhinskii (1975) were the first to propose a result (stated here with our current notation) which generalized Corollary 3.3.

Theorem 4.1 *In the equation*

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \tag{11a}$$

let $\mathbf{A}(\cdot)$ be a real matrix function, where $\mathbf{A}(\cdot)$ is integrable and piecewise continuous on $(0, T_A)$, and $\mathbf{A}(t + T_A) = \mathbf{A}(t)$. An arbitrary real matrix $\mathbf{X}(\cdot, 0)$ that is a fundamental solution of (11a) may be expressed as

$$\mathbf{X}(t, 0) = \mathbf{L}(t)\exp(t\mathbf{F}), \tag{11b}$$

where $\mathbf{F}$ is a real constant matrix, $\mathbf{L}(\cdot)$ is a real matrix function such that

$$\mathbf{L}(t + T_A) = \mathbf{L}(t)\mathbf{Y}, \tag{11c}$$

and $\mathbf{Y}$ is some real matrix such that

$$\mathbf{Y}^2 = \mathbf{1}, \tag{11d}$$

$$\mathbf{FY} = \mathbf{YF}. \tag{11e}$$

In particular,

$$\mathbf{L}(t + 2T_A) = \mathbf{L}(t) \quad \forall t.$$

The function $\mathbf{L}(\cdot)$ is continuous with an integrable piecewise-continuous derivative. Conversely, let $\mathbf{L}(\cdot)$, $\mathbf{F}$, and $\mathbf{Y}$ be arbitrary real matrices satisfying conditions (11c)–(11e), such that $\forall t \;\det[\mathbf{L}(t)] \neq 0$, and let $\mathbf{L}(\cdot)$ have an integrable piecewise-continuous derivative. Then (11b) is a fundamental matrix for some equation of the form (11a) with a real T_A-periodic matrix $\mathbf{A}(\cdot)$.

Unfortunately, this result does not cover all possible real Floquet factorizations. Indeed, if constant real matrices $\mathbf{K}$ and $\mathbf{M}$ commute, and if $\log(\mathbf{M})$ is a *primary matrix function* (PMF) of $\mathbf{M}$, then $\mathbf{K}$ and $\log(\mathbf{M})$ will also commute (Horn & Johnson, 1994). However, when $\log(\mathbf{M})$ is not a PMF, then the commutativity may not always be preserved.

4.2 Characterization

4.2.1 $\mathbf{F}_Y$ matrices

Let $\mathbf{Y} \in \mathbb{R}^{n\times n}$ such that $T_A\mathbf{F}_Y \in \mathcal{RL}og\{\mathbf{Y\Phi}(T_A, 0)\}$; we then have

$$\mathbf{Y\Phi}(T_A, 0) = \exp(T_A\mathbf{F}_Y). \tag{12}$$

We can now define matrix $\mathbf{L}_{F_Y}(\cdot)$ as

$$\forall t \in \mathbb{R}^+, \quad \mathbf{L}_{F_Y}(t) \stackrel{\Delta}{=} \mathbf{\Phi}(t, 0)\exp(-t\mathbf{F}_Y). \tag{13}$$

Note that $\mathbf{L}_{F_Y}(0) = \mathbf{1}$.

The classical results recalled in §3.3 and §3.4 rely on a specific relationship between the eigenvalues μ_i ($i = 1 \ldots n$) of $\mathbf{\Phi}(T_A, 0)$ and the eigenvalues λ_i ($i = 1 \ldots n$) of $\mathbf{F}$, where $\mathbf{F}$ is any complex matrix obtained from Theorem 3.2, namely

$$\mathbf{\Phi}(T_A, 0) = \exp(T_A\mathbf{F}).$$

Considering the polar representation of μ_i,

$$\mu_i = \rho_i \exp(\iota\theta_i), \quad \iota^2 = -1,$$
$$\rho_i \stackrel{\Delta}{=} |\mu_i| \geq 0, \; -\pi \leq \theta_i < \pi,$$

every λ_i has the form

$$\lambda_i = \frac{\ln(\rho_i)}{T_A} + \iota\frac{\theta_i + 2k\pi}{T_A}, \; k \in \mathbb{Z}.$$

When introducing matrix $\mathbf{Y}$ in the definition of $\mathbf{F}_Y$ in Eq. (12), it is important to preserve the relation

$$\mathrm{Re}(\lambda_i) = \frac{\ln(\rho_i)}{T_A}.$$

Otherwise, the s-plane strip image of the unit circle would be distorted and the resulting λ_i may lose quantitative and qualitative dynamical significance. All these properties are essential to the control engineer. Fortunately they can be preserved by requiring that the eigenvalues of $\mathbf{\Phi}(T_A, 0)$ and $\mathbf{Y\Phi}(T_A, 0)$ have the same moduli.

Theorem 4.2 *(Montagnier* et al. *, 2003) Let $\mathbf{\Phi}(\cdot,0)$ be the STM of (6), where $\mathbf{A}(\cdot) \in \mathcal{A}_{T_A}$. Further, let the real $n \times n$ matrices $\mathbf{Y}, \mathbf{F}_Y$, and $\mathbf{L}_{F_Y}(\cdot)$ be defined in Eqs. (12) and (13), and $k \in \mathbb{N}$. Then in the real factorization*

$$\mathbf{\Phi}(\cdot,0) = \mathbf{L}_{F_Y}(\cdot)e^{t\mathbf{F}_Y}$$

we have

$$\mathbf{L}_{\mathbf{F}_Y}(t+T) = \mathbf{L}_{F_Y}(t)e^{t\mathbf{F}_Y}\mathbf{L}_{F_Y}(T)e^{-t\mathbf{F}_Y} \ .$$

The choice of $\mathbf{Y}$ affects $\mathbf{L}_{F_Y}(t)$ as follows:

$$\mathbf{L}_{F_Y}(T_A) = \mathbf{Y}^{-1} \ .$$

$\mathbf{L}_{F_Y}(\cdot)$ is kT_A-periodic if and only if

$$\mathbf{\Phi}^k(T_A,0) = \left[\mathbf{\Phi}(T_A,0)\mathbf{Y}\right]^k, \tag{14a}$$

while $\mathbf{L}_{F_Y}(\cdot)$ is kT_A-antiperiodic if and only if

$$\mathbf{\Phi}^k(T_A,0) = -\left[\mathbf{\Phi}(T_A,0)\mathbf{Y}\right]^k. \tag{14b}$$

Equations (14a,b) are equivalent to

$$\mathbf{\Phi}^k(T_A,0) = [\mathbf{Y}\mathbf{\Phi}(T_A,0)]^k \tag{15a}$$

$$\mathbf{\Phi}^k(T_A,0) = -[\mathbf{Y}\mathbf{\Phi}(T_A,0)]^k \tag{15b}$$

respectively.

Any $\mathbf{Y}$ satisfying the condition of this theorem will be henceforth called a *Yakubovich matrix.*

Corollary 4.3 *(Montagnier* et al. *, 2003) There always exists a matrix $\mathbf{Y}$ such that $\mathbf{L}_{F_Y}$ is $2T_A$-periodic.*

Note 4.1 *In principle $\mathbf{Y}$ can be constructed using the Jordan decomposition of $\mathbf{\Phi}(T_A,0)$ (Montagnier* et al. *, 2003).*

5 Derivation of the Floquet Factors

In this section we review the existing methods for the derivation of the Floquet factors $\mathbf{L}(\cdot)$ and $\mathbf{F}$ of a given LTP system of the form (6). Bellman (1970b) best summarized the issues involved when he deemed the representation provided by Floquet-Lyapunov theory "an elegant and important one but not easy to use for quantitative purposes. There is no readily available way of determining [the Floquet factors]."

Existing methods are indirect in the sense that they consist of two steps: the initial computation of the STM followed by the derivation of its Floquet factors. They were until now the only approach available; we describe them in Subsection 5.1. We then propose a novel method which follows a direct approach (Spiteri *et al.* , 1998); i.e., no computation of the STM is performed.

5.1 Two-Step Methods

With the STM being known, a constant matrix $\mathbf{F}$ can be derived using the results of the previous chapter. $\mathbf{L}(\cdot)$ is then obtained from

$$\mathbf{L}(t) \triangleq \mathbf{\Phi}(t,0)\exp(-t\mathbf{F}).$$

$\mathbf{\Phi}(\cdot,0)$ is known in closed form only in specific cases (Erugin, 1966); for instance, when the product of $\mathbf{A}$ from (6) and $\exp[\int_0^t \mathbf{A}(\tau)\ d(\tau)]$ commute (Lukes, 1982), the STM takes on simple symbolic form. In the general case, the STM has to be computed numerically. Knowledge of $\mathbf{\Phi}(\cdot,0)$ is only required over the interval $(0,T_A)$, since other values may be obtained from Eq. (8).

A popular technique is based on the *direct numerical integration* of Eq. (6) for n different initial conditions

$$\mathbf{x}_k(0) = \mathbf{e}_k, \quad k \in \{1,\cdots,n\},$$

where $\mathbf{e}_k = [\delta_{ki}]$ is the k-th column of $\mathbf{1}$. Let $\mathbf{x}_i(t),\ i \in \{1,\cdots,n\}$ be the n independent solutions obtained for each initial condition. Then, the STM is

$$\mathbf{\Phi}(t,0) \triangleq [\mathbf{x}_1(t)|\mathbf{x}_2(t)|\cdots|\mathbf{x}_n(t)] \,.$$

Gaonkar *et al.* (1981) provide a comparison of the efficiency of different integration schemes.

An alternative to direct numerical integration schemes consists of computing the expansion of the STM on a spectral basis $\boldsymbol{\sigma}(\cdot)$ of order m,

$$\boldsymbol{\sigma}(\cdot) = \begin{bmatrix}\sigma_0(\cdot) & \sigma_1(\cdot) & \cdots & \sigma_{m-1}(\cdot)\end{bmatrix}^T,$$

which could be, say, a Fourier basis or a polynomial basis (e.g., Chang *et al.* , 1986; Wu, 1991). The expansion $\mathbf{C}_\Phi$ of the STM is obtained from the linear system

$$\mathbf{M}\mathbf{C}_\Phi = \mathbf{N},$$

where $\mathbf{M}$ and $\mathbf{N}$ are known constant matrices based on the spectral expansion of $\mathbf{A}(\cdot)$. This procedure was reviewed by Montagnier *et al.* (1998).

5.2 Direct Derivation

The two previous approaches are based on an initial-value problem (IVP) formulation. There is no guarantee that solutions $\mathbf{x}_i(t)$ will remain independent after numerical integration. On the other hand, it is well known (Ascher *et al.* , 1988; Keller, 1976) that the solution to a boundary-value problem (BVP) can be better conditioned than the solution to an IVP. This leads us to consider the following BVP formulation.

5.2.1 BVP Formulation

$\mathbf{L}(\cdot)$ can be shown to satisfy the first-order matrix differential equation

$$\forall t \in \mathbb{R}^+, \quad \dot{\mathbf{L}}(t) = \mathbf{A}(t)\mathbf{L}(t) - \mathbf{L}(t)\mathbf{F}. \tag{16a}$$

Moreover, we can add two boundary conditions (BCs) using Corollary 3.3:

$$\mathbf{L}(0) = \mathbf{L}(2T_A) = \mathbf{1}. \tag{16b}$$

The system as it stands has twice as many BCs as differential equations. However, noting that $\mathbf{F}$ is constant, we can add the equation

$$\dot{\mathbf{F}} = \mathbf{O}$$

with $\mathbf{O}$ denoting the $n \times n$ zero matrix, so that the problem is *well defined.*

Solving this system for $\mathbf{L}(\cdot)$ and $\mathbf{F}$ is done by using a standard BVP software package; e.g. `COLDAE`, developed by Ascher & Spiteri (1994), or `bvp4c` in `MATLAB`™ (Kierzenka, 1998; Kierzenka & Shampine, 2001).

Note 5.1 *As Eq. (16a) is nonlinear, a numerical solution requires an initial guess. We have opted to produce this initial guess with a zero-order-hold-like algorithm (Friedmann* et al. *, 1977; Hsu, 1974), whereby the period is discretized in a sequence of intervals and the initial LTP system approximated by a LTI system on each interval. Hsu (1974) showed that the monodromy matrix of the discretized system always converges to that of the LTP system as the interval length diminishes.*

Note 5.2 *A BC at $t = T_A$ rather than $t = 2T_A$ can be derived using a Yakubovich matrix $\mathbf{Y}$, as introduced in Theorem 4.2: Equation (16b) becomes*

$$\mathbf{L}(0) = \mathbf{1}, \quad \mathbf{L}(T_A) = \mathbf{Y}^{-1}.$$

Independent of its periodicity, values of $\mathbf{L}(\cdot)$ outside $(0, T_A)$ are obtained via $\forall (t,k) \in \mathbb{R}^+ \times \mathbb{Z}^{+*}$,

$$\mathbf{L}(t + kT_A) = \mathbf{L}(t)\exp(t\mathbf{F})\mathbf{\Phi}^{k-1}(T_A, 0)\times \mathbf{L}(T_A)\exp[-((k-1)T_A + t)\mathbf{F}].$$

Matrix $\mathbf{L}^{-1}(\cdot)$ plays a major role in the applications of the theory. Inverting $\mathbf{L}(\cdot)$ at each desired value may be computationally expensive and may lead to round-off error amplification. A better approach is to consider a BVP formulation for $\mathbf{L}^{-1}(\cdot)$ that mirrors Eq. (16b). Using the expression of $\dot{\mathbf{L}}(\cdot)$ given in Eq. (16a) and the well-known formula for the derivative of the inverse of a matrix, we can define a new BVP: $\forall t \in \mathbb{R}^+$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{L}^{-1}(t)\right] = -\mathbf{L}^{-1}(t)\mathbf{A}(t) + \mathbf{F}\mathbf{L}^{-1}(t), \tag{17a}$$

$$\dot{\mathbf{F}} = \mathbf{O}, \tag{17b}$$

$$\mathbf{L}^{-1}(0) = \mathbf{1}, \quad \mathbf{L}^{-1}(T_A) = \mathbf{Y}. \tag{17c}$$

6 Gramian-Based Controller

6.1 Review of Previous Work

When designing feedback controllers, periodic systems have often only been considered as general time-varying systems. The rationale behind those controllers was to take advantage of the intrinsic properties of each particular system (e.g., Fisher & Bryson Jr., 1995; Friedmann, 1983; Ilić-Spong & Mak, 1986; Ilić-Spong *et al.* , 1985; Kern, 1980), rather than trying to develop a general framework based upon Floquet-Lyapunov theory. The few instances of the latter approach have mainly focused on the classical control method of invariant assignment: It helps to build a better understanding of the relevant issues of Floquet-Lyapunov theory by relying upon stability concepts which are common to both time-periodic systems and time-invariant systems (cf. Sections 3.3 and 3.4), namely, that the eigenvalues of a LTI system are also its Floquet exponents.

Assigning the invariants of LTI systems is widely described in the literature; this was also achieved for discrete-time LTP systems (e.g., Willems *et al.* , 1984). However, the same task has proven far more challenging in the case of continuous-time LTP systems. The different attempts usually fall into two categories, depending on whether the (continuous-time) feedback is based on the discrete- or continuous-state model.

6.1.1 Discrete-State Feedback

Brunovský (1969) proposed a feedback control scheme consisting of impulsive feedback signals of the instantaneous state at n suitably chosen instants within a period. The limitations of this approach were discussed by Willems *et al.* (1984). Kabamba (1986) introduced the concept of "sampled state periodic hold" and gave an explicit expression for a piecewise continuous periodic feedback signal. He showed that, when the controllability index is less than n, the whole monodromy matrix can be assigned. Al-Rahmani & Franklin (1989) adopted a similar framework and, using *generalized reachability Gramians*, designed a periodic piecewise constant feedback control to assign the Floquet multipliers arbitrarily and, under conditions similar to those first derived by Kabamba (1986), the whole monodromy matrix.

6.1.2 Continuous-State Feedback

The task of producing a continuous-state periodic feedback (CPF) control has been arduous. Attempts can be separated into two categories, depending on whether

1. the controller is built based on the transformed sys-

tem of Note 3.4, viz.

$$\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t) + \mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{u}(t), \tag{18}$$

and the Floquet exponents are assigned by a feedback scheme $\mathbf{u}(t) = -\mathbf{K}(t)\mathbf{z}(t)$ (Calico & Wiesel, 1984, 1986; Joseph, 1993); or

2. one considers the closed-loop system

$$\dot{\mathbf{x}}(t) = [\mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}(t)]\,\mathbf{x}(t) \tag{19}$$

and uses Theorem 3.2 to find $\mathbf{K}(t)$ (Kern, 1980, 1986; Laptinsky, 1988).

Calico & Wiesel (1984, 1986) produced a "modal control" based on Eq. (18) that can move Floquet exponents one at a time. Placing complex-conjugate pairs is problematic; moreover, as noted later (Webb *et al.* , 1991), numerical instabilities occur. Despite its limitations, the foregoing approach offered deep insight into the problem and provided solutions to simple cases.

Joseph (1993) (see also Sinha & Joseph, 1994) tried to overcome the periodicity of the input matrix $\mathbf{L}^{-1}(\cdot)\mathbf{B}(\cdot)$ in Eq. (18) by introducing an "auxiliary" LTI system which would asymptotically converge toward the original LTP system. Unfortunately, this scheme was based on a misuse of the generalized inverse of $\mathbf{B}(\cdot)$ and of some results from nonlinear analysis. Montagnier *et al.* (2001) proved that stability of the auxiliary system did not guarantee stability of the LTP system. Lee & Balas (1999) proposed to repair the feedback design, but failed to notice that their new scheme was based on the knowledge of the closed form of the STM, which is not available in the general case for LTP systems.

Two other schemes are based on the system represented by Eq. (19), which is itself a Floquet system and has a pair of (initially unknown) Floquet factors $\mathbf{L}_K(\cdot)$ and $\mathbf{F}_K$. Kern (1980, 1986) established a differential system of nonlinear equations in $\mathbf{L}_K(\cdot)$ and $\mathbf{K}(\cdot)$. Having assigned the constant Floquet factor $\mathbf{F}_K$ for the closed-loop system, one solved first for the corresponding Lyapunov transformation $\mathbf{L}_K(\cdot)$ and then for $\mathbf{K}(\cdot)$. Laptinsky (1988), using results from Zoubov (1978), adopted a similar approach in which the feedback matrix $\mathbf{K}(\cdot)$ was assumed to be the sum of a periodic $n \times n$ matrix $\tilde{\mathbf{K}}(\cdot)$ and a constant $n \times n$ matrix $\bar{\mathbf{K}}$. Unfortunately, both schemes are impractical when $\mathbf{B}(\cdot)$ is not square and singular, which is the most common case.

6.2 Motivation

Willems *et al.* (1984) stressed the fundamental restrictions of LTI feedback control systems: Eigenvalues can be assigned, but neither their algebraic or geometric multiplicities nor their eigenvectors. In the same paper, the authors conjectured that a periodic time-varying feedback would be able to relax these design restrictions. They provided an example of a completely unrestricted assignment of the invariant factors (i.e., eigenvectors and eigenvalues) for discrete-time periodic systems, but only partial results were obtained in the continuous-time case.

We describe below a continuous-time, continuous-state periodic controller which can assign the whole monodromy matrix; i.e., its eigenvalues and eigenvectors (Montagnier, 2002). The technique can in principle be applied to any LTP system because it only relies upon properties derived from the Floquet-Lyapunov theory.

The approach builds on the works of Kern (1980) and Laptinsky (1988). However, whereas their work was dependent on the invertibility of the input matrix $\mathbf{B}(\cdot)$, we base our design on the invertibility of the Gramian, something which is guaranteed from the controllability of the system. Properties of linear state-feedback strategies based on the Gramian have been previously investigated for LTI systems (Pearson & Kwon, 1976), and, more recently, for LTP systems (Nicolao & Strada, 1997). This scheme can be put in parallel with the results obtained by Al-Rahmani & Franklin (1989) with discrete-state continuous-time controllers.

6.3 Novel Formulation

Consider the LTP system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t),$$

and let ν be the controllability index of the pair $\{\mathbf{\Phi}(T_A, 0), \mathbf{W}_R(T_A, 0)\}$. From Theorem 3.6, the system is controllable if and only if

$$\det[\mathbf{W}_R(\nu T_A, 0)] \neq 0.$$

If this is the case, then Theorem 2.2 leads to

$$\forall t > \nu T_A, \quad \det[\mathbf{W}_R(t, 0)] \neq 0. \tag{20}$$

Choosing a CPF control law of the form

$$\mathbf{u}(t) = -\mathbf{K}(t)\mathbf{x}(t), \quad \mathbf{K}(t + pT_A) = \mathbf{K}(t), \qquad p \in \mathbb{Z}^+, \quad p \geq \nu, \tag{21}$$

the closed-loop system becomes (cf. Fig. 1)

$$\dot{\mathbf{x}}(t) = \mathbf{A}_K(t)\mathbf{x}(t), \ \mathbf{A}_K(t) = \mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}(t). \tag{22}$$

Because this system is T_{A_K}-periodic ($T_{A_K} = pT_A$), we can apply Theorem 3.2 to conclude that the state-transition matrix $\mathbf{\Phi}_K(t, 0)$ is of the form

$$\forall t \in \mathbb{R}^+, \quad \mathbf{\Phi}_K(t, 0) \stackrel{\Delta}{=} \mathbf{L}_K(t)\exp(t\mathbf{F}_K), \tag{23}$$

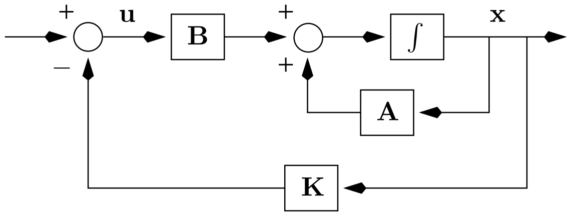


Figure 1: Closed-loop system

where $\mathbf{L}_K(\cdot) \in \mathcal{L}^{\mathbb{R}}_{kT_{A_K}}$ $(k \in \mathbb{Z}^{+*})$ and $\mathbf{F}_K \in \mathbb{R}^{n\times n}$. From Theorem 4.2, the BCs imposed on $\mathbf{L}_K(\cdot)$ are

$$\mathbf{L}_K(0) = \mathbf{1}, \quad \mathbf{L}_K(T_{A_K}) = \mathbf{Y}_K^{-1}, \tag{24a}$$

for some Yakubovich matrix $\mathbf{Y}_K$ that determines the periodicity T_{L_K} of $\mathbf{L}_K(\cdot)$. $\mathbf{L}_K(\cdot)$ also satisfies Eq. (16a):

$$\forall t \in \mathbb{R}^+,\ \dot{\mathbf{L}}_K(t) = \mathbf{A}_K(t)\mathbf{L}_K(t) - \mathbf{L}_K(t)\mathbf{F}_K. \tag{24b}$$

Note 6.1 *It is important to stress the correspondence with §5.2. When considering the open-loop problem, the system matrix $\mathbf{A}(\cdot)$ is usually known, but its STM and Floquet factors are not. Here, we have the converse: Given a desired STM, determine first the Floquet factors and then obtain the closed-loop system $\mathbf{A}_K(\cdot) = \mathbf{A}(\cdot) - \mathbf{B}(\cdot)\mathbf{K}(\cdot)$ and, thus, the feedback matrix $\mathbf{K}(\cdot)$.*

Note 6.2 *This approach is a direct application of the converse statements in Theorems 3.1 and 3.2 limited to the real field via Theorem 4.2. One should note that the canonical Floquet theory assumes the system matrix $\mathbf{A}(\cdot)$ to be piecewise continuous in order to obtain the converse statements. Therefore, one should expect the feedback matrix $\mathbf{K}(\cdot)$ to be piecewise continuous as well.*

6.4 T_{A_K}-Periodic Feedback

We recall a lemma first proposed by Laptinsky (1988):

Lemma 6.1 *$\mathbf{L}_K(\cdot)$ is the solution of Eq. (24b) if and only if $\mathbf{L}_K(\cdot)$ satisfies, $\forall t \in \mathbb{R}^+$,*

$$\mathbf{L}_K(t) = \mathbf{\Phi}(t,0)\times \left\{\mathbf{1} - \int_0^t \mathbf{\Phi}(0,\tau)\left[\mathbf{B}(\tau)\mathbf{K}(\tau)\mathbf{L}_K(\tau) + \mathbf{L}_K(\tau)\mathbf{F}_K\right]d\tau\right\} \tag{25}$$

Now, recalling the second BC in Eq. (24a), Eq. (25) can be rewritten at $t = T_{A_K}$ as

$$\int_0^{T_{A_K}} \mathbf{\Phi}(0,\tau)\mathbf{B}(\tau)\mathbf{K}(\tau)\mathbf{L}_K(\tau)d\tau = -\left\{\int_0^{T_{A_K}} \mathbf{\Phi}(0,\tau)\mathbf{L}_K(\tau)d\tau\mathbf{F}_K + \mathbf{\Phi}(0,T_{A_K})\mathbf{Y}_K^{-1} - \mathbf{1}\right\}.$$

Multiplying both sides of the above equation by $\mathbf{\Phi}(T_{A_K},0)$ leads to

$$\int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K},\tau)\mathbf{B}(\tau)\mathbf{K}(\tau)\mathbf{L}_K(\tau)d\tau = \mathbf{\Phi}(T_{A_K},0) - \mathbf{Y}_K^{-1} - \int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K},\tau)\mathbf{L}_K(\tau)d\tau\mathbf{F}_K. \tag{26}$$

Equation (26) leads to the result:

Lemma 6.2 *Suppose (22) is controllable. Then, a stabilizing T_{A_K}-periodic feedback $\mathbf{K}(t) \in \mathbb{R}^{r\times n}$ can be constructed of the form*

$$\mathbf{K}(t) = \mathbf{B}^T(t)\mathbf{\Phi}^T(T_{A_K},t)\mathbf{K}_c\mathbf{L}_K^{-1}(t) \tag{27}$$

where $\mathbf{K}_c \in \mathbb{R}^{n\times n}$ is constant and $\mathbf{L}_K(\cdot)$ is defined in (23). $\mathbf{K}_c$ is guaranteed to exist from controllability.

The proof is available in (Montagnier, 2002).

6.5 Derivation of an Integral Equation for $\mathbf{L}_K(\cdot)$

In this section we use Eq. (27) to obtain an integral equation for $\mathbf{L}_K(\cdot)$. We have

Theorem 6.1 *$\{\mathbf{L}_K(\cdot), \mathbf{F}_K\}$, with $\mathbf{L}_K(0) = \mathbf{1}$, $\mathbf{L}_K(T_{A_K}) = \mathbf{Y}_K^{-1}$, is a pair of Floquet factors of the closed-loop system (22) for $\mathbf{K}(\cdot)$ in (27) if and only if, $\forall t,\ 0 \le t < T_{A_K}$,*

$$\mathbf{L}_K(t) = \Big[\mathbf{W}_R(t,0)\mathbf{\Phi}^T(T_{A_K},t){\mathbf{W}_R}^{-1}(T_{A_K},0)\times \int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K},\tau)\mathbf{L}_K(\tau)d\tau - \int_0^t \mathbf{\Phi}(t,\tau)\mathbf{L}_K(\tau)d\tau\Big]\mathbf{F}_K + \mathbf{L}_{K_0}(t), \tag{28}$$

where

$$\mathbf{L}_{K_0}(t) = \mathbf{\Phi}(t,0) + \mathbf{W}_R(t,0)\mathbf{\Phi}^T(T_{A_K},t)\times {\mathbf{W}_R}^{-1}(T_{A_K},0)\left\{\mathbf{Y}_K^{-1} - \mathbf{\Phi}(T_{A_K},0)\right\}$$

is the value of $\mathbf{L}_K(\cdot)$ when $\mathbf{F}_K = \mathbf{O}$.

The proof is available in (Montagnier, 2002).

Note 6.3 *(28) can be written in integral form using*

$$\mathbf{L}_K(t) = \int_0^{T_{A_K}} \mathbf{P}(t,\tau)\mathbf{L}_K(\tau)d\tau\mathbf{F}_K + \mathbf{L}_{K_0}(t),$$

where

$$\mathbf{P}(t,\tau) = \begin{cases} \mathbf{R} - \mathbf{\Phi}(t,\tau), & 0 \le \tau \le t \le T_{A_K}, \\ \mathbf{R}, & 0 \le t \le \tau \le T_{A_K}, \end{cases}$$

with

$$\mathbf{R} \equiv \mathbf{W}_R(t,0)\mathbf{\Phi}^T(T_{A_K},0){\mathbf{W}_R}^{-1}(T_{A_K},0)\mathbf{\Phi}(T_{A_K},\tau).$$

Note 6.4 *It is of course possible to use* $\mathbf{W}_C(t_2,t_1)$ *instead of* $\mathbf{W}_R(t_2,t_1)$ *in the above discussion. In this case, it can be verified that,* $\forall t,\quad 0 \le t < T_{A_K}, \quad \forall k \in \mathbb{Z}^{+*}$

$$
\begin{aligned}
\mathbf{K}(t) &= \mathbf{B}^T(t)\mathbf{\Phi}^T(0,t)\bar{\mathbf{K}}_c\mathbf{L}_K^{-1}(t),\\
\mathbf{K}(t+kT_{A_K}) &= \mathbf{K}(t),\\
\bar{\mathbf{K}}_c &= -\mathbf{W}_C{}^{-1}(T_{A_K},0)\left\{\mathbf{\Phi}(0,T_{A_K})\mathbf{Y}_K^{-1} - \mathbf{1}\right.\\
&\quad \left. + \int_0^{T_{A_K}} \mathbf{\Phi}(0,\tau)\mathbf{L}_K(\tau)d\tau\mathbf{F}_K\right\},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{L}_K(t) &= \mathbf{\Phi}(t,0)\left[\mathbf{W}_C(t,0)\mathbf{W}_C{}^{-1}(T_{A_K},0)\times\right.\\
&\quad \int_0^{T_{A_K}} \mathbf{\Phi}(0,\tau)\mathbf{L}_K(\tau)d\tau\\
&\quad \left. - \int_0^t \mathbf{\Phi}(0,\tau)\mathbf{L}_K(\tau)d\tau\right]\mathbf{F}_K + \mathbf{L}_{K_0}(t),\\
\mathbf{L}_{K_0}(t) &= \mathbf{\Phi}(t,0)\left\{\mathbf{1} + \mathbf{W}_C(t,0)\mathbf{W}_C{}^{-1}(T_{A_K},0)\times\right.\\
&\quad \left.\left[\mathbf{\Phi}(0,T_{A_K})\mathbf{Y}_K^{-1} - \mathbf{1}\right]\right\}.
\end{aligned}
$$

6.6 Solving the Integral Equation

Although the solution of scalar integral equations has been investigated—see, e.g., (Delves & Mohamed, 1985) and references therein—to the best of our knowledge there are no standard techniques for solving matrix integral equations. In this section, we introduce an approach based on a spectral method.

Consider a spectral basis $\boldsymbol{\sigma}(\cdot)$ of order s defined over the interval $[0,1]$:

$$\boldsymbol{\sigma}(u) = \begin{bmatrix}\sigma_1(u) & \sigma_2(u) & \cdots & \sigma_s(u)\end{bmatrix}^T, \quad 0 \le u \le 1.$$

We introduce the change of variable

$$t = vT_{A_K},\ \tau = uT_{A_K},\ 0 \le t,\tau \le T_{A_K},\ 0 \le u,v \le 1,$$

in order to express Eq. (28) over $[0,1]$:

$$
\begin{aligned}
\mathbf{L}_K(vT_{A_K}) &= T_{A_K}\mathbf{B}_k\mathbf{F}_K + \mathbf{L}_{K_0}(vT_{A_K}), &\text{(29a)}\\
\widetilde{\mathbf{W}}(v) &\triangleq \mathbf{W}_R(vT_{A_K},0)\mathbf{\Phi}^T(T_{A_K},vT_{A_K})\times\\
&\quad \mathbf{W}_R{}^{-1}(T_{A_K},0)\mathbf{\Phi}(T_{A_K},0), &\text{(29b)}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{B}_k \equiv \widetilde{\mathbf{W}}(v)&\int_0^{T_{A_K}} \mathbf{\Phi}(0,uT_{A_K})\mathbf{L}_K(uT_{A_K})du\\
&- \int_0^v \mathbf{\Phi}(vT_{A_K},uT_{A_K})\mathbf{L}_K(uT_{A_K})du.
\end{aligned}
$$

Recalling Def. 2.8, let $\mathbf{l}(v)$ and $\mathbf{l}_{K_0}(v)$ be the n^2-dimensional vectors defined as:

$$
\begin{aligned}
\mathbf{l}(v) &\triangleq \mathrm{vec}\left(\mathbf{L}_K(vT_{A_K})\right),\\
\mathbf{l}_{K_0}(v) &\triangleq \mathrm{vec}\left(\mathbf{L}_{K_0}(vT_{A_K})\right).
\end{aligned}
$$

Using Lemma 2.1, Eq. (29a) becomes a n^2-dimensional vector equation, namely,

$$
\begin{aligned}
\mathbf{l}(v) &= T_{A_K}\left[\int_0^1 \mathbf{\Gamma}(v,u)\mathbf{l}(u)du\right.\\
&\quad \left. - \int_0^v \mathbf{\Theta}(v,u)\mathbf{l}(u)du\right] + \mathbf{l}_{K_0}(v), &\text{(30a)}\\
\mathbf{\Gamma}(v,u) &= \mathbf{F}_K^T \otimes \left(\widetilde{\mathbf{W}}(v)\mathbf{\Phi}(T_{A_K},uT_{A_K})\right), &\text{(30b)}\\
\mathbf{\Theta}(v,u) &= \mathbf{F}_K^T \otimes \mathbf{\Phi}(vT_{A_K},uT_{A_K}). &\text{(30c)}
\end{aligned}
$$

We now expand this equation on the spectral basis $\boldsymbol{\sigma}(\cdot)$.

Regarding the expansion of the $n^2 \times n^2$ matrix $\mathbf{\Theta}$ introduced in Eq. (30c), its (p,q) entry becomes

$$
\begin{aligned}
\Theta_{p,q}(v,u) &= F_{j_0,i_0}\Phi_{i_1,j_1}(vT_{A_K},uT_{A_K})\\
&= F_{j_0,i_0}\sum_{k=1}^n \Phi_{i_1,k}(vT_{A_K},0)\Phi_{k,j_1}(0,uT_{A_K})\\
&1 \le m,p,q \le n^2,\ 1 \le i_0,i_1,j_0,j_1 \le n,\\
&p = (i_1-1)n + i_0,\ q = (j_1-1)n + j_0.
\end{aligned}
$$

First, consider the expansions of $\Phi_{k,j_1}(0,uT_{A_K})$ and of $l_m(u)$, the m-th entry of $\mathbf{l}(u)$:

$$
\begin{aligned}
\Phi_{k,j_1}(0,uT_{A_K}) &\approx \boldsymbol{\sigma}^T(u)\boldsymbol{\phi}^s_{k,j_1}(0,\cdot), \boldsymbol{\phi}^s_{k,j_1}(0,\cdot) \in \mathbb{R}^s, &\text{(31a)}\\
l_m(u) &\approx \boldsymbol{\sigma}^T(u)\mathbf{l}^s_m, \quad \mathbf{l}^s_m \in \mathbb{R}^s, &\text{(31b)}
\end{aligned}
$$

which lead to the expansion of $\Theta_{p,q}(v,u)l_m(u)$ with respect to u :

$$
\begin{aligned}
&\Theta_{p,q}(v,u)l_m(u)\\
&\approx F_{j_0,i_0}\left[\boldsymbol{\sigma}^T(u)\sum_{k=1}^n \Phi_{i_1,k}(vT_{A_K},0)\boldsymbol{\phi}^s_{k,j_1}(0,\cdot)\right]\left[\boldsymbol{\sigma}^T(u)\mathbf{l}^s_m\right]\\
&= \boldsymbol{\sigma}^T(u)F_{j_0,i_0}\sum_{k=1}^n \Phi_{i_1,k}(vT_{A_K},0)\mathbf{Q}\left(\boldsymbol{\phi}^s_{k,j_1}(0,\cdot)\right)\mathbf{l}^s_m,
\end{aligned}
$$

where $\mathbf{Q}(\cdot)$ is the product matrix defined in Eq. (5). Using the integration matrix $\mathbf{G}$ defined in Eq. (4), the integral is approximated by

$$
\begin{aligned}
&\int_0^v \Theta_{p,q}(v,u)l_m(u)du\\
&\approx \boldsymbol{\sigma}^T(v)F_{j_0,i_0}\mathbf{G}^T\sum_{k=1}^n \Phi_{i_1,k}(vT_{A_K},0)\mathbf{Q}\left(\boldsymbol{\phi}^s_{k,j_1}(0,\cdot)\right)\mathbf{l}^s_m,
\end{aligned}
$$

which is a function of v. The expansion of $\Phi_{i_1,k}(vT_{A_K},0)$:

$$\Phi_{i_1,k}(vT_{A_K},0) \approx \boldsymbol{\sigma}^T(v)\boldsymbol{\phi}^s_{i_1,k}(\cdot,0), \quad \boldsymbol{\phi}^s_{i_1,k}(\cdot,0) \in \mathbb{R}^s,$$

leads to

$$
\begin{aligned}
&\int_0^v \Theta_{p,q}(v,u)l_m(u)du \approx F_{j_0,i_0}\sum_{k=1}^n \left[\boldsymbol{\sigma}^T(v)\boldsymbol{\phi}^s_{i_1,k}(\cdot,0)\right]\times\\
&\boldsymbol{\sigma}^T(v)\mathbf{G}^T\mathbf{Q}\left(\boldsymbol{\phi}^s_{k,j_1}(0,\cdot)\right)\mathbf{l}^s_m = \boldsymbol{\sigma}^T(v)\mathbf{Q}_{\Theta_{p,q}}\mathbf{l}^s_m,
\end{aligned}
$$

where matrix $\mathbf{Q}_{\Theta_{p,q}} \in \mathbb{R}^{s\times s}$ is such that

$$\mathbf{Q}_{\Theta_{p,q}} = F_{j_0,i_0}\sum_{k=1}^{n}\mathbf{Q}\left(\boldsymbol{\phi}^s_{i_1,k}(\cdot,0)\right)\mathbf{G}^T\mathbf{Q}\left(\boldsymbol{\phi}^s_{k,j_1}(0,\cdot)\right).$$

The expansion of the vector integral is of the form

$$\int_0^v \boldsymbol{\Theta}(v,u)\mathbf{l}(u)du \approx \left[\mathbf{1}_{n^2}\otimes\boldsymbol{\sigma}^T(v)\right]\mathbf{C}_\Theta\mathbf{l}^s, \qquad (32a)$$

where $\mathbf{C}_\Theta \in \mathbb{R}^{n^2s\times n^2s}$ is a block matrix whose (p,q) block is $\mathbf{Q}_{\Theta_{p,q}}$:

$$\mathbf{C}_\Theta = \begin{bmatrix} \mathbf{Q}_{\Theta_{1,1}} & \mathbf{Q}_{\Theta_{1,2}} & \cdots & \mathbf{Q}_{\Theta_{1,n^2}} \\ \vdots & \vdots & & \vdots \\ \mathbf{Q}_{\Theta_{n^2,1}} & \mathbf{Q}_{\Theta_{n^2,2}} & \cdots & \mathbf{Q}_{\Theta_{n^2,n^2}} \end{bmatrix}. \qquad (32b)$$

Similarly, expanding Eq. (30b), we obtain the (p,q) entry of the $n^2\times n^2$ matrix $\boldsymbol{\Gamma}$:

$$\begin{aligned}\Gamma_{p,q}(v,u) &= F_{j_0,i_0}\sum_{k=1}^{n}\widetilde{W}_{i_1,k}(v)\Phi_{k,j_1}(0,uT_{A_K})\\ &1\le m,p,q\le n^2,\quad 1\le i_0,i_1,j_0,j_1\le n,\\ &p=(i_1-1)n+i_0,\quad q=(j_1-1)n+j_0.\end{aligned}$$

Recalling Eq. (31), we obtain the s-order expansion with respect to u :

$$\begin{aligned}&\Gamma_{p,q}(v,u)l_m(u)\\ &\approx F_{j_0,i_0}\boldsymbol{\sigma}^T(u)\left[\sum_{k=1}^{n}\widetilde{W}_{i_1,k}(v)\boldsymbol{\phi}^s_{k,j_1}(0,\cdot)\right]\times\left[\boldsymbol{\sigma}^T(u)\mathbf{l}^s_m\right]\\ &=\boldsymbol{\sigma}^T(u)F_{j_0,i_0}\sum_{k=1}^{n}\widetilde{W}_{i_1,k}(v)\mathbf{Q}\left(\boldsymbol{\phi}^s_{k,j_1}(0,\cdot)\right)\mathbf{l}^s_m. \qquad (33)\end{aligned}$$

The integral of Eq. (33) with respect to u thus becomes

$$\begin{aligned}&\int_0^1\Gamma_{p,q}(v,u)l_m(u)du\\ &\approx F_{j_0,i_0}\sum_{k=1}^{n}\widetilde{W}_{i_1,k}(v)\boldsymbol{\sigma}^T(1)\mathbf{G}^T\mathbf{Q}\left(\boldsymbol{\phi}^s_{k,j_1}(0,\cdot)\right)\mathbf{l}^s_m,\end{aligned}$$

which is a scalar function of v. Now, considering the expansion of $\widetilde{W}_{i_1,k}(v)$ with respect to v,

$$\widetilde{W}_{i_1,k}(v)\approx\boldsymbol{\sigma}^T(v)\widetilde{\mathbf{w}}^s_{i_1,k},$$

we obtain:

$$\int_0^1\Gamma_{p,q}(v,u)l_m(u)du\approx\boldsymbol{\sigma}^T(v)\mathbf{Q}_{\Gamma_{p,q}}\mathbf{l}^s_m,$$

$$\mathbf{Q}_{\Gamma_{p,q}} = F_{j_0,i_0}\sum_{k=1}^{n}\widetilde{\mathbf{w}}^s_{i_1,k}\boldsymbol{\sigma}^T(1)\mathbf{G}^T\mathbf{Q}\left(\boldsymbol{\phi}^s_{k,j_1}(0,\cdot)\right).$$

Finally, the expansion of the vector integral is

$$\int_0^1\boldsymbol{\Gamma}(v,u)\mathbf{l}(u)du\approx\left[\mathbf{1}_{n^2}\otimes\boldsymbol{\sigma}^T(v)\right]\mathbf{C}_\Gamma\mathbf{l}^s, \qquad (34)$$

where $\mathbf{C}_\Gamma\in\mathbb{R}^{n^2s\times n^2s}$ is a block matrix whose (p,q) block is $\mathbf{Q}_{\Gamma_{p,q}}$.

Taking into account the expansion of $\mathbf{l}_{K_0}(\cdot)$,

$$\mathbf{l}_{K_0}(v)\approx\left\{\mathbf{1}_{n^2}\otimes\boldsymbol{\sigma}^T(v)\right\}\mathbf{l}^s_{K_0},\quad \mathbf{l}^s_{K_0}\in\mathbb{R}^{n^2r},$$

and using Eqs. (32) and (34), Eq. (29a) becomes

$$\begin{aligned}\left[\mathbf{1}_{n^2}\otimes\boldsymbol{\sigma}^T(v)\right]\mathbf{l}^s\approx\left[\mathbf{1}_{n^2}\otimes\boldsymbol{\sigma}^T(v)\right]&\left[T_{A_K}\left(\mathbf{C}_\Gamma-\mathbf{C}_\Theta\right)\mathbf{l}^s\right.\\ &\left.+\mathbf{l}^s_{K_0}\right].\end{aligned}$$

The expansion coefficients of $\mathbf{L}_K(vT_{A_K})$ contained in $\mathbf{l}^s$ are the solution of the $n^2s\times n^2s$ linear system

$$\left[\mathbf{1}_{n^2s}-T_{A_K}\left(\mathbf{C}_\Gamma-\mathbf{C}_\Theta\right)\right]\mathbf{l}^s=\mathbf{l}^s_{K_0},$$

which can be solved using standard techniques (Golub & Van Loan, 1996).

6.7 Design Rationale

The synthesis of a controller for a LTP system described in the previous sections can be performed in five steps:

Algorithm 6.1 Feedback Synthesis

1. *Assign a new period T_{A_K} for the closed loop-system such that*
$$\det\left[\mathbf{W}_R(T_{A_K},0)\right]\neq 0. \qquad (35a)$$
2. *Assign the desired monodromy matrix $\boldsymbol{\Phi}_K(T_{A_K},0)$ of the closed-loop system satisfying the relation (cf. Theorem 3.7)*
$$\det\left[\boldsymbol{\Phi}_K(T_{A_K},0)\right]>0. \qquad (35b)$$
3. *Choose a pair $\{\mathbf{Y}_K,\mathbf{F}_K\}$ such that*
$$\mathbf{Y}_K\boldsymbol{\Phi}(T_{A_K},0)=\exp(T_{A_K}\mathbf{F}_K). \qquad (35c)$$
4. *Solve Eq. (29a) for $\mathbf{L}_K(\cdot)$.*
5. *Derive the feedback matrix $\mathbf{K}(t)$ from Eq. (27).*

Note 6.5 *Should the condition* $\det\left[\boldsymbol{\Phi}_K(T_{A_K},0)\right]>0$ *not be satisfied, a spurious solution $\mathbf{L}_K(\cdot)$ of the integral equation will still exist, but will be singular for some t_s. The matrix $\mathbf{K}(\cdot)$ will therefore not be defined at t_s, and thus the closed-loop system cannot be synthesized with this form of feedback.*

6.7.1 Computing $\mathbf{L}_K^{-1}(\cdot)$

The spectral solution of §6.6 can be used as an initial guess to a BVP approach similar to the open-loop case discussed in §5.2. Noting that $\mathbf{K}(\cdot)$ makes only use of $\mathbf{L}_K(\cdot)^{-1}$, it is computationally more efficient to obtain the inverse directly: using Eqs. (22) and (27), Eq. (17) becomes, $\forall t \in \mathbb{R}^+$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\left[\mathbf{L}_K^{-1}(t)\right] &= -\mathbf{L}_K^{-1}(t)\left[\mathbf{A}(t) + \mathbf{F}_K\mathbf{L}_K^{-1}(t)\right.\\ &\left.\quad -\mathbf{B}(t)\mathbf{B}^T(t)\mathbf{\Phi}^T(T_{A_K}, t)\mathbf{K}_c\mathbf{L}_K^{-1}(t)\right]\\ \mathbf{L}_K^{-1}(0) = \mathbf{1}, \quad &\mathbf{L}_K^{-1}(T_{A_K}) = \mathbf{Y}_K, \quad \dot{\mathbf{K}}_c = 0.\end{aligned}$$

Note that $\mathbf{F}_K$ has been assigned and we are now solving for the constant matrix $\mathbf{K}_c$ in the BVP.

6.7.2 Feedback Synthesis on the Transformed System

The synthesis of the controller can alternatively be performed on the transformed system

$$\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t) + \mathbf{L}^{-1}\mathbf{B}(t)\mathbf{u}(t),$$

where $\{\mathbf{L}(\cdot), \mathbf{F}\}$ is a pair of Floquet factors of the STM $\mathbf{\Phi}(\cdot, 0)$. Using Algorithm 6.1, we can obtain a gain matrix $\mathbf{K}_z(\cdot)$ which can assign the monodromy matrix of

$$\dot{\mathbf{z}}(t) = \left[\mathbf{F} - \mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{K}_z(t)\right]\mathbf{z}(t).$$

Using the change of variable $\mathbf{x}(t) = \mathbf{L}(t)\mathbf{z}(t)$, the original system becomes

$$\dot{\mathbf{x}}(t) = \left[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}(t)\right]\mathbf{x}(t), \quad \mathbf{K}(t) = \mathbf{K}_z(t)\mathbf{L}^{-1}(t).$$

While there is no change in the assignment of the invariants of the resulting closed-loop system, there is a computational advantage to this technique 1) when finding the spectral solution for $\mathbf{L}_K(\cdot)$ and 2) when assigning only the unstable multipliers of the closed-loop system.

Finding the spectral expansion of $\mathbf{L}_K(\cdot)$ requires expanding $\mathbf{\Phi}(\cdot, 0)$ and $\mathbf{\Phi}(0, \cdot)$. These expansions can be obtained more efficiently for the STM $\mathbf{\Psi}(t, 0) \stackrel{\Delta}{=} \exp(t\mathbf{F})$ of the transformed system, since $\mathbf{F}$ is constant. Note that

$$\mathbf{F} = \left\{\mathbf{1} \otimes \boldsymbol{\sigma}^T(t)\right\}\mathbf{C}_F, \quad \mathbf{C}_F \stackrel{\Delta}{=} \begin{bmatrix} \mathbf{f}_{1,1}^s & \mathbf{f}_{1,2}^s & \cdots & \mathbf{f}_{1,n}^s \\ \vdots & \vdots & & \vdots \\ \mathbf{f}_{n,1}^s & \mathbf{f}_{n,2}^s & \cdots & \mathbf{f}_{n,n}^s \end{bmatrix}$$

where vector $\mathbf{f}_{i,j}^s$ of the expansion coefficients can usually be obtained as explained below, without any expansions.

Integrating the differential equation

$$\dot{\boldsymbol{\psi}}_i(t) = \mathbf{F}\boldsymbol{\psi}_i(t), \tag{36a}$$

where $\boldsymbol{\psi}_i(\cdot)$ is the i-th column vector of $\mathbf{\Psi}(\cdot, 0)$, yields

$$\boldsymbol{\psi}_i(t) - \mathbf{e}_i = \int_0^t \mathbf{F}\boldsymbol{\psi}_i(u, 0)du, \tag{36b}$$

$\mathbf{e}_i$ being the ith column of the $n \times n$ identity matrix. Using the operational matrix $\mathbf{G}$ of integration

$$\int_0^t \mathbf{F}\boldsymbol{\psi}_i(u, 0)du \approx \boldsymbol{\sigma}^T(t)\mathbf{F} \otimes \mathbf{G}^T\boldsymbol{\psi}_i^s, \quad \mathbf{e}_i \approx \boldsymbol{\sigma}^T(t)\mathbf{e}_i^s. \tag{36c}$$

The spectral approximation of Eq. (36b) then becomes

$$\boldsymbol{\sigma}^T(t)\left[\mathbf{1}_{ns} - \mathbf{F} \otimes \mathbf{G}^T\right]\boldsymbol{\psi}_i^s = \boldsymbol{\sigma}^T(t)\mathbf{e}_i^s \tag{36d}$$

and the expansion of the i-th column of $\mathbf{\Psi}(\cdot, 0)$ is the solution of a linear system of order ns, namely,

$$\left[\mathbf{1}_{ns} - \mathbf{F} \otimes \mathbf{G}^T\right]\boldsymbol{\psi}_i^s = \mathbf{e}_i^s. \tag{36e}$$

We thus derived the expansion of the exponential without having to evaluate the matrix exponential itself, thereby adding one more to the nineteen ways reviewed by Moler & Van Loan (1978). Similarly, by replacing $\mathbf{C}_F$ by $-\mathbf{C}_F$ in Eq. (36e), we obtain the expansion of the inverse of $\mathbf{\Psi}(0, \cdot)$.

If one aims at assigning only the unstable multipliers of the closed-loop system, the whole approach can be scaled down by constructing a lower-order controller. Consider the real Jordan decomposition of matrix $\mathbf{F}$

$$\mathbf{F} \stackrel{\Delta}{=} \mathbf{W}_J \begin{bmatrix} \mathbf{J}_1 & \mathbf{O} \\ \mathbf{O}^T & \mathbf{J}_2 \end{bmatrix} \mathbf{W}_J^{-1},$$

where $\mathbf{O}$ is the $p \times (n-p)$ zero matrix, while the $p \times p$ block $\mathbf{J}_1$ contains all the eigenvalues of $\mathbf{F}$ with positive real part. The change of variable $\mathbf{x}(t) = \mathbf{L}(t)\mathbf{W}_J\mathbf{w}(t)$ leads to the new transformed system

$$\dot{\mathbf{w}}(t) = \begin{bmatrix} \mathbf{J}_1 & \mathbf{O} \\ \mathbf{O}^T & \mathbf{J}_2 \end{bmatrix}\mathbf{w}(t) + \begin{bmatrix} \mathbf{D}_1(t) \\ \mathbf{D}_2(t) \end{bmatrix}\mathbf{u}(t), \tag{37}$$

where the $p \times r$ matrix $\mathbf{D}_1(\cdot)$ contains the first p rows of $\mathbf{W}_J^{-1}\mathbf{L}^{-1}(\cdot)\mathbf{B}(\cdot)$, and the $(n-p) \times r$ matrix $\mathbf{D}_2(\cdot)$ its last $(n-p)$ rows. We can now synthesize a $r \times p$ controller $\mathbf{K}_1(\cdot)$ for the p-order system

$$\dot{\mathbf{w}}_1(t) = \mathbf{J}_1\mathbf{w}_1(t) + \mathbf{D}_1(t)\mathbf{u}(t), \tag{38a}$$

such that

$$\dot{\mathbf{w}}_1(t) = \left[\mathbf{J}_1 - \mathbf{D}_1(t)\mathbf{K}_1(t)\right]\mathbf{w}_1(t) \tag{38b}$$

has a prescribed monodromy matrix. Choosing the feedback law

$$\mathbf{u}(t) = -\mathbf{K}_w(t)\mathbf{w}(t), \tag{39a}$$

with

$$\mathbf{K}_w(t) = \begin{bmatrix}\mathbf{K}_1(t) & \mathbf{O}_{r\times(n-p)}\end{bmatrix} \tag{39b}$$

and $\mathbf{O}_{r\times(n-p)}$ defined as the $r \times (n-p)$ zero matrix, system (37) becomes

$$\dot{\mathbf{w}}(t) = \begin{bmatrix}\mathbf{J}_1 - \mathbf{D}_1(t)\mathbf{K}_1(t) & \mathbf{O}_{p\times(n-p)} \\ -\mathbf{D}_2(t)\mathbf{K}_1(t) & \mathbf{J}_2\end{bmatrix}\mathbf{w}(t) \tag{39c}$$

Due to the block-triangular nature of system (39c), its STM has a specific form (O'Malley, 1997):

$$\boldsymbol{\Psi}(t,0) \triangleq \begin{bmatrix}\boldsymbol{\Psi}_{11}(t,0) & \mathbf{O}_{p\times(n-p)} \\ \boldsymbol{\Psi}_{21}(t,0) & \boldsymbol{\Psi}_{22}(t,0)\end{bmatrix}.$$

Note that the eigenvalues of the monodromy matrix are those of $\boldsymbol{\Psi}_{11}(T_{A_K},0)$ and $\boldsymbol{\Psi}_{22}(T_{A_K},0)$; we assigned the unstable Floquet exponents without modifying the stable Floquet exponents of $\mathbf{J}_2$.

Finally, the controller for the original closed-loop system has the form

$$\dot{\mathbf{x}}(t) = [\mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}(t)]\,\mathbf{x}(t), \tag{40a}$$

$$\mathbf{K}(t) = \mathbf{K}_w(t)\mathbf{W}_J^{-1}\mathbf{L}^{-1}(t). \tag{40b}$$

Working on this unstable subsystem may offer a substantial computational efficiency: 1) A controller based on the original system would require the expansions of the n^2 entries of $\boldsymbol{\Phi}(\cdot,0)$, its inverse, and the nr elements of $\mathbf{B}(\cdot)$; 2) Recalling Eq.(36e) to obtain the coefficients of $\exp(t\mathbf{J}_1)$, the only expansions required are for the pr entries of $\mathbf{D}(\cdot)$; and 3) There are additional computational savings when manipulating matrices of order sp^2 instead of sn^2, cf. Eqs. (32a) and (34).

6.7.3 State Reconstruction

The discussion has so far assumed a full-state feedback. We can take advantage of the duality of the optimal regulator and optimal observer problem (Kwakernaak & Sivan, 1972) by considering the dual of system (1):

$$\dot{\mathbf{x}}(t) = \mathbf{A}^T(t)\mathbf{x}(t) + \mathbf{C}^T(t)\mathbf{u}(t), \tag{41a}$$

$$\mathbf{y}(t) = \mathbf{B}^T(t)\mathbf{x}(t). \tag{41b}$$

From the duality between controllability and observability (Chen, 1984), system (1) is observable if and only if system (41) is controllable. In this case, we can design a controller $\mathbf{K}_d(\cdot)$ for the dual system following Algorithm 6.1. Defining the observer equation as

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}_O(t)\hat{\mathbf{x}}(t) + \mathbf{K}_O(t)\mathbf{y}(t) + \mathbf{B}(t)\mathbf{u}(t),$$

where $\hat{\mathbf{x}}(\cdot) \in \mathbb{R}^m$ is the observer state vector, and

$$\mathbf{A}_O(t) \triangleq \mathbf{A}(t) - \mathbf{K}_O(t)\mathbf{C}(t).$$

Kwakernaak & Sivan (1972) showed that the observer gain matrix is $\mathbf{K}_O(\cdot) = \mathbf{K}_d^T(\cdot)$, while the observer-based closed-loop system of Fig. 2 becomes

$$\begin{bmatrix}\dot{\mathbf{x}}(t) \\ \dot{\hat{\mathbf{x}}}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{A}(t) & -\mathbf{B}(t)\mathbf{K}(t) \\ \mathbf{K}_O(t)\mathbf{C}(t) & \mathbf{A}_O(t) - \mathbf{B}(t)\mathbf{K}(t)\end{bmatrix}\begin{bmatrix}\mathbf{x}(t) \\ \hat{\mathbf{x}}(t)\end{bmatrix}.$$

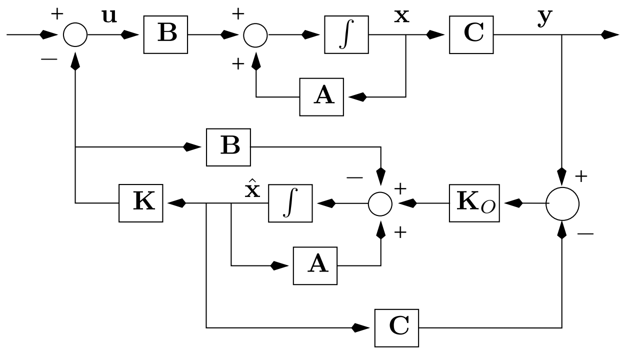


Figure 2: Observer-based controller

7 Application: Control of a Quick-Return Mechanism

7.1 Introduction

A quick-return mechanism is a mechanical transmission that produces a slow feed motion under a load in one direction, followed by a fast return stroke under no load in the opposite direction. Quick-return mechanisms are quite common in manufacturing processes, e.g. in pick-and-place operations, metal-cutting, and metal-forming.

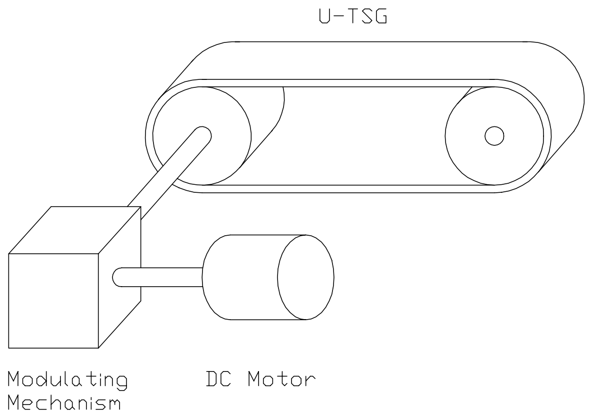


Figure 3: Layout of the quick-return mechanism

Figure 3 depicts a quick-return mechanism intended to transport and glue veneer strips. The uniform speed of the motor is rendered periodic by the modulating mechanism that produces the required motion, which is then transmitted to the uniform-transmission stroke generator (U-TSG) to move the workpiece. The U-TSG can be a ballscrew, a belt-pulley, or a rack-and-pinion mechanism. The stroke of the mechanism is 3.0 m and the period required for each cycle is about 2.0 s.

The system, illustrated in Fig. 4, is driven by a DC motor operating at its nominal speed of 3000 rpm; the shaft between the speed amplifier and the belt-pulley mechanism has a stiffness k_s. The values of the parameters given below are taken from our design studies and those of Asselin & Doléac (1997). These studies investigated only the mechanical aspects of the system; based upon these we now look at the open-loop and closed-loop stability of the system: We assume that a steady-state has been reached, our aim being to design a controller which would reject any incoming disturbances. The electromechanical system at hand was designed with the parameters listed below:

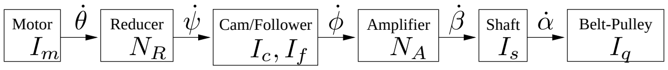


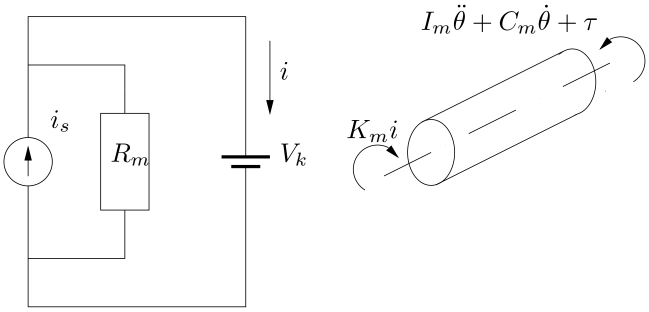
Figure 4: Modules of the quick-return mechanism

Belt-pulley mechanism

- pulley diameter: $D_p = 100$ mm
- diameter of the flexible shaft: $D_s = 7$ mm
- stroke length: $l = 3000$ mm
- length of the flexible shaft: $l_s = 35.0$ mm
- mass of the transported object: $m_p = 0.72$ kg
- mass of the fixtures: $m_f = 1.0$ kg
- amplification ratio: $N_A = 16$
- reduction ratio: $N_R = 1/100$
- pulley thickness: $t_p = 12.0$ mm

Motor

- operating speed: $\omega_0 = 3000$ rpm
- motor resistance: $R_m = 9.65\ \Omega$
- motor electro-mechanical constant: $K_m = 0.484$ V s rad^{-1}
- motor viscous damping: $C_m = 5.25\ 10^{-5}$ N m s rad^{-1}
- non-dimensional open-loop period: $T_A = 146.0$
- constant in Eq. (47c): $T_s = 0.136$ s
- cycle time: $T_c = 2.0$ s



(a) Electrical model

(b) Mechanical model

Figure 5: Model of the motor

7.2 Mathematical Model

The electro-mechanical model of the motor dynamics is derived from Fig. 5, namely,

$$R_m\,[i_s(t) - i(t)] = K_m\dot{\theta}(t), \tag{42a}$$

$$I_m\ddot{\theta}(t) + C_m\dot{\theta}(t) = K_m i(t) - \tau(t), \tag{42b}$$

where the inductance of the armature has been neglected and τ is the torque applied on the reducer. The above equations can be recast in the form

$$K_m i_s(t) - \left[C_m + \frac{K_m{}^2}{R_m}\right]\dot{\theta}(t) - I_m\ddot{\theta}(t) = \tau(t).$$

7.2.1 Mechanical System Model

Let ψ and ϕ be the angular displacement of the cam and the follower of the system, as shown in Fig. 4. The speed reduction is $N_R = 1/100$ and the speed amplification is $N_A = 16$, providing the relations:

$$\dot{\psi}(t) = N_R\dot{\theta}(t), \quad \dot{\beta}(t) = N_A\dot{\phi}(t). \tag{43}$$

From the chain rule, we have

$$\dot{\phi}(t) = \phi'(\theta)\dot{\theta}(t), \quad \text{where } \phi'(\theta) \triangleq \frac{\partial\phi}{\partial\theta}\,. \tag{44}$$

A cam was synthesized to produce the quick-return motion illustrated in Figs. 6 and 7, so that each operation takes exactly $T_c = 2.0$ s.

7.2.2 Lagrange Equations

The kinetic energy E_k of the system, excluding the motor, can be written as

$$E_k \triangleq \frac{1}{2}I_c\dot{\psi}^2 + \frac{1}{2}I_f\dot{\phi}^2 + \frac{1}{2}I_s\dot{\beta}^2 + \frac{1}{2}I_q\dot{\alpha}^2,$$

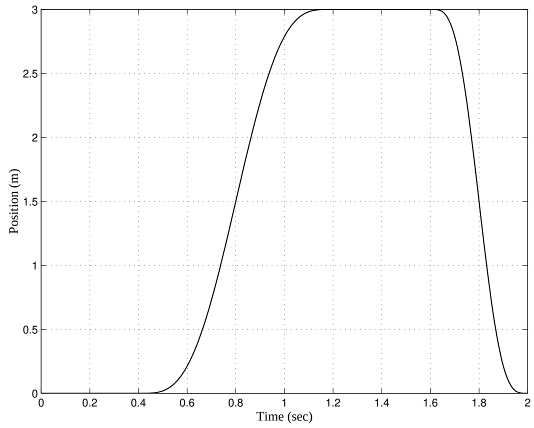


Figure 6: Quick-return motion

where I_c, I_f, I_s, and I_q are the moments of inertia of the cam, the follower, the shaft and the belt-pulley mechanism, respectively. Recalling (43), this simplifies to

$$\begin{aligned} E_k &= \frac{1}{2} I_e(\theta)\dot{\theta}^2 + \frac{1}{2} I_q \dot{\alpha}^2, \\ I_e(\theta) &\triangleq I_c {N_R}^2 + \left[I_f + I_s {N_A}^2\right] {\phi'}^2(\theta). \end{aligned}$$

On the other hand, the potential energy of the same system comes solely from the elastic shaft:

$$E_p \triangleq \frac{1}{2} k_s (\beta - \alpha)^2,$$

or, taking into account Eqs. (43) and (44),

$$E_p = \frac{1}{2} k_s \left[N_A \phi'(\theta)\theta - \alpha\right]^2.$$

Choosing θ and α as generalized coordinates, the generalized forces are

$$Q_\theta = \tau(t), \quad Q_\alpha = 0.$$

Define

$$\begin{aligned} C_e(\theta) &\triangleq \frac{1}{2}\frac{dI_e}{d\theta} = \left[I_f + I_s {N_A}^2\right] \phi'(\theta)\phi''(\theta), \\ {C_e}'(\theta) &\triangleq \frac{dC_e}{d\theta} = \left[I_f + I_s {N_A}^2\right] \left[{\phi''}^2(\theta) + \phi'(\theta)\phi^3(\theta)\right], \\ K(\theta) &\triangleq k_s N_A \left[\phi''(\theta)\theta + \phi'(\theta)\right]. \end{aligned}$$

Using the Lagrangian formulation we obtain the system of nonlinear governing equations

$$I_e(\theta)\ddot{\theta} + C_e(\theta)\dot{\theta}^2 + K(\theta)\left[N_A \phi'(\theta)\theta - \alpha\right] = \tau(\theta), \quad (45a)$$

$$I_q \ddot{\alpha} + k_s \left[\alpha - N_A \phi'(\theta)\theta\right] = 0. \quad (45b)$$

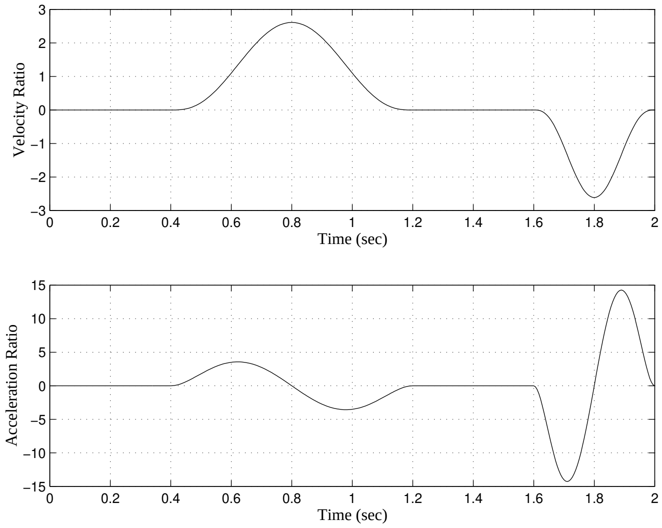


Figure 7: Velocity ratio $\phi'(\theta)$ and acceleration ratio $\phi''(\theta)$ of the quick-return motion

7.3 Linearization

We first linearize the system at the operating point (θ_0, α_0) :

$$\begin{aligned} \theta &= \theta_0 + \delta\theta, & \theta_0 &= \omega_0 t, \\ \alpha &= \alpha_0 + \delta\alpha, & \alpha_0 &= N_A \phi'(\theta_0)\omega_0 t. \end{aligned}$$

System (45) then becomes, after simplification:

$$\begin{aligned} &\left[I_e(\theta_0) + I_m\right]\delta\ddot{\theta} + \left[2C_e(\theta_0)\omega_0 + C_m + \frac{{K_m}^2}{R_m}\right]\delta\dot{\theta} \\ &\quad + K(\theta_0)\left(N_A\left\{\phi'(\theta_0)\theta_0 + \left[\phi'(\theta_0) + \phi''(\theta_0)\theta_0\right]\delta\theta\right\}\right. \\ &\quad\quad \left. - \left[\alpha_0 + \delta\alpha\right]\right) + K'(\theta_0)\left[N_A \phi'(\theta_0)\theta_0\delta\theta - \alpha_0\delta\right] \\ &\quad = \tau(\theta_0) + K_m \delta i_s - \left[C_e(\theta_0) + {C_e}'(\theta_0)\delta\theta\right]\omega_0^2, \quad (46a) \end{aligned}$$

$$\begin{aligned} &I_q \delta\ddot{\alpha} + k_s\left\{\alpha_0 + \delta\alpha\right. \\ &\quad\quad \left. - N_A\left[\phi'(\theta_0)\theta_0 + \phi''(\theta_0)\theta_0\right]\delta\theta\right\} = 0. \quad (46b) \end{aligned}$$

Using the equilibrium conditions, system (46) yields

$$\begin{aligned} \left[I_e(\Omega_s\bar{t}) + I_m\right]\delta\ddot{\theta}(\bar{t}) &= -{T_s}^2\left\{K(\Omega_s\bar{t})N_A\left[\phi'(\Omega_s\bar{t})\right.\right. \\ &\quad \left. + \phi''(\Omega_s\bar{t})\Omega_s\bar{t}\right] + K'(\Omega_s\bar{t})N_A\phi'(\Omega_s\bar{t}) \\ &\quad \left. - \alpha_0(T_s\bar{t}) + {C_e}'(\Omega_s\bar{t})\omega_0^2\right\}\delta\theta(\bar{t}) \\ &\quad - T_s\left[2C_e(\Omega_s\bar{t})\omega_0 + C_m + \frac{{K_m}^2}{R_m}\right]\delta\dot{\theta}(\bar{t}) \\ &\quad + {T_s}^2 K(\Omega_s\bar{t})\delta\alpha(\bar{t}) + K_m\delta i_s(\bar{t}), \quad (47a) \end{aligned}$$

$$\begin{aligned} I_q\delta\ddot{\alpha}(\bar{t}) &= {T_s}^2 k_s N_A\left[\phi'(\Omega_s\bar{t}) + \phi''(\Omega_s\bar{t})\Omega_s\bar{t}\right]\delta\theta(\bar{t}) \\ &\quad - {T_s}^2 k_s \delta\alpha(\bar{t}), \quad (47b) \end{aligned}$$

where we have introduced non-dimensional time $\bar{t}$ through the change of variable

$$t = T_s \bar{t}, \quad T_s = \sqrt{\frac{I_q}{k_s}}, \quad \Omega_s = \omega_0 T_s. \tag{47c}$$

7.3.1 State-Space Representation

Choosing the state vector

$$\mathbf{x}(t) \triangleq \begin{bmatrix} \delta\theta & \delta\dot{\theta} & \delta\alpha & \delta\dot{\alpha} \end{bmatrix}^T,$$

and the current δi_s as the input variable $u(\bar{t})$, system (47) can be rewritten in the form

$$\dot{\mathbf{x}}(\bar{t}) = \mathbf{A}(t)\mathbf{x}(\bar{t}) + \mathbf{b}(\bar{t})u(\bar{t}),$$

where

$$\mathbf{A}(\bar{t}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ A_{21}(\bar{t}) & A_{22}(\bar{t}) & A_{23}(\bar{t}) & 0 \\ 0 & 0 & 0 & 1 \\ A_{41}(\bar{t}) & 0 & A_{43} & 0 \end{bmatrix},$$

$$\begin{aligned}
A_{21}(\bar{t}) &= -{T_s}^2 \left\{ K(\Omega_s \bar{t}) N_A \left[\phi'(\Omega_s \bar{t}) + \phi''(\Omega_s \bar{t}) \Omega_s \bar{t} \right] \right. \\
&\quad \left. + K'(\Omega_s \bar{t}) N_A \phi'(\Omega_s \bar{t}) \Omega_s \bar{t} - \alpha_0 + {C_e}'(\Omega_s \bar{t}) \omega_0^2 \right\} \\
&\quad / \left[I_e(\Omega_s \bar{t}) + I_m \right], \qquad (48) \\
A_{22}(\bar{t}) &= N/D, \\
N &= -T_s \left[2C_e(\Omega_s \bar{t})\omega_0 + C_m + \frac{{K_m}^2}{R_m} \right], \\
D &= \left[I_e(\Omega_s \bar{t}) + I_m \right], \\
A_{23}(\bar{t}) &= {T_s}^2 K(\Omega_s u) / \left[I_e(\Omega_s \bar{t}) + I_m \right], \\
A_{41}(\bar{t}) &= {T_s}^2 k_s N_A \left[\phi'(\Omega_s \bar{t}) + \phi''(\Omega_s \bar{t}) \Omega_s \bar{t} \right] / I_q, \\
A_{43} &= -{T_s}^2 k_s / I_q, \\
\mathbf{b}(\bar{t}) &= \begin{bmatrix} 0 & b_{21}(\bar{t}) & 0 & 0 \end{bmatrix}^T, \\
b_{21}(\bar{t}) &= K_m / \left[I_e(\Omega_s \bar{t}) + I_m \right].
\end{aligned}$$

Note that $\mathbf{A}(\cdot)$ and $\mathbf{b}(\cdot)$ are periodic with primary period $T_A = T_c/T_s$.

7.4 System Dynamics

7.4.1 Stability

Using the method described in §5.2, we obtain a real pair of Floquet factors $\{\mathbf{L}(\cdot), \mathbf{F}\}$, from which we derive the STM of the open-loop system. With $\iota = \sqrt{-1}$, the Floquet exponents λ_i (the eigenvalues of $\mathbf{F}$) are

$$\begin{aligned}
\lambda_1 &= -6.9137 \times 10^{-7}, \\
\lambda_2 &= \bar{\lambda}_3 = -5.5222 \times 10^{-8} + 2.0258\iota, \\
\lambda_4 &= -3.6806,
\end{aligned}$$

corresponding to the Floquet multipliers

$$\begin{aligned}
\mu_1 &= 1.0000, \\
\mu_2 &= \bar{\mu}_3 = -4.3946 \times 10^{-1} + 8.9826 \times 10^{-1}\iota, \\
\mu_4 &= 2.5207 \times 10^{-2}.
\end{aligned}$$

Note that all these eigenvalues are dimensionless, since matrix $\mathbf{A}(\cdot)$ is. The potential pitfalls of the open-loop mechanical-system performance are best illustrated with a simple example. Consider an initial perturbation of one degree on the variable $\delta\alpha$; i.e. the initial state vector is

$$\mathbf{x}_0 = \begin{bmatrix} 0 & \pi/180 & 0 & 0 \end{bmatrix}^T. \tag{49}$$

Figure 8 displays the resulting oscillations of the system, shown here in radians, for $\delta\alpha$ and $\delta\theta$, and in rad/s for $\delta\dot{\alpha}$ and $\delta\dot{\theta}$. Thus, the system requires a closed-loop controller in order to eliminate incoming perturbations.

7.4.2 Controller Synthesis

We follow the design rationale described in § 6.7. Starting with the transformed system

$$\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t) + \mathbf{L}^{-1}(t)\mathbf{b}(t)u(t),$$

we consider the real Jordan form of $\mathbf{F}$:

$$\mathbf{F} \triangleq \mathbf{W}_J \begin{bmatrix} \mathbf{J} & \mathbf{0}_3 \\ \mathbf{0}_3^T & \lambda_4 \end{bmatrix} \mathbf{W}_J^{-1}, \quad \mathbf{J} \triangleq \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \mathrm{Re}(\lambda_2) & \mathrm{Im}(\lambda_2) \\ 0 & -\mathrm{Im}(\lambda_2) & \mathrm{Re}(\lambda_2) \end{bmatrix}$$

where $\mathbf{0}_3$ denotes the three-dimensional zero vector. We focus now on the 3×3 subsystem including the three Floquet exponents λ_1, λ_2, and $\lambda_3 = \bar{\lambda}_2$:

$$\dot{\mathbf{w}}(t) = \mathbf{J}\mathbf{w}(t) + \mathbf{d}(t)u(t),$$

where $\mathbf{d}(\cdot)$ contains the first three rows of $\mathbf{W}_J^{-1}\mathbf{L}^{-1}(\cdot)\mathbf{b}(\cdot)$. In this example $\mathbf{W}_R(\cdot, 0)$ is invertible after three periods, and thus, we build a controller $\mathbf{k}_3(\cdot)$ of period $3T_A$ that assigns the monodromy matrix of the subsystem to $\exp(3T_A\mathbf{F}_{K_3})$, the constant matrix $\mathbf{F}_{K_3}$ being chosen as

$$\mathbf{F}_{K_3} = \begin{bmatrix} -0.8 & 0 & 0 \\ 0 & -0.8 & 0.8 \\ 0 & -0.8 & -0.8 \end{bmatrix}. \tag{50}$$

Note that any $\mathbf{F}_{K_3}$ with eigenvalues in the right-hand side plane would produce a stable system. The values in Eq. (50) are an example chosen so that quantities satisfy the small-perturbation hypotheses.

Finally we synthesize the controller for the original system by building the feedback matrix:

$$\begin{aligned}
&\forall t \in [0, 3T_A[, \quad \forall k \in \mathbb{Z}^+, \quad \mathbf{k}^T(t) = \mathbf{k}_w^T(t)\mathbf{W}_J^{-1}\mathbf{L}^{-1}(t), \\
&\mathbf{k}(t + 3kT_A) = \mathbf{k}(t), \quad \mathbf{k}_w(t) = \begin{bmatrix} \mathbf{k}_3^T(t) & 0 \end{bmatrix}^T.
\end{aligned}$$

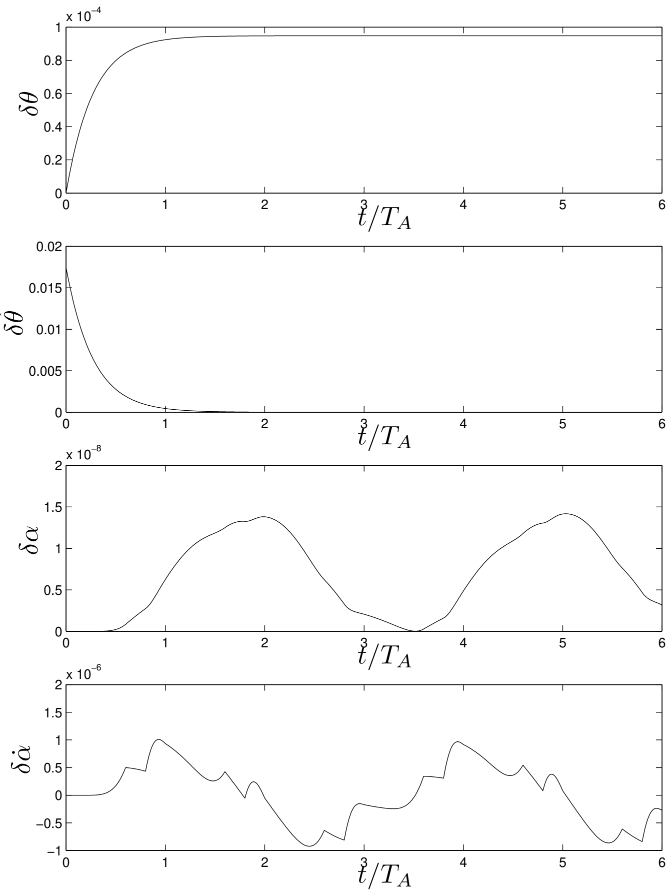


Figure 8: Effect of an initial perturbation on the open-loop system

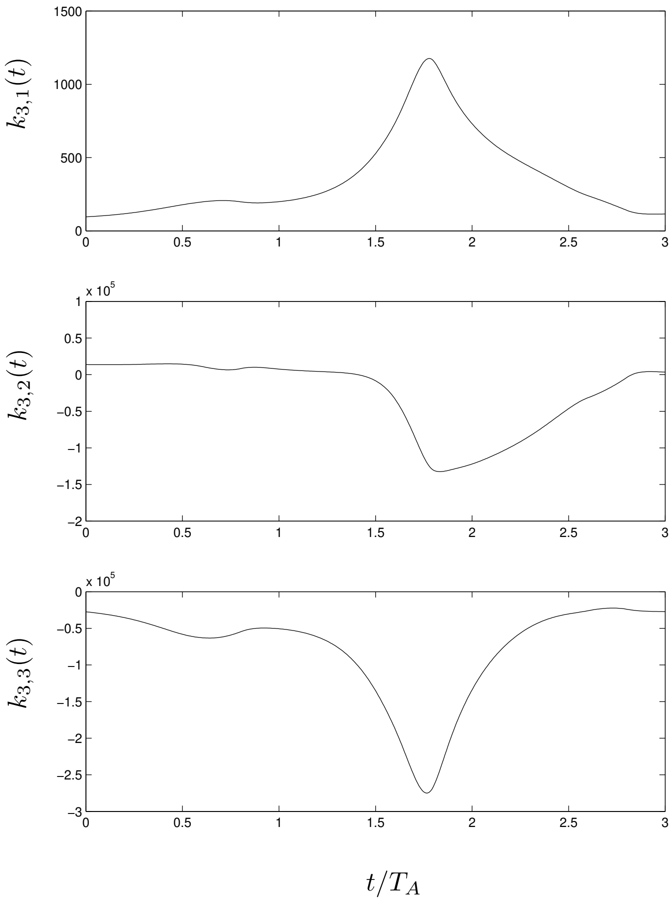


Figure 9: Entries of matrix $\mathbf{k}_3(t)$

The variations of entries of the gain 3×1 matrices $\mathbf{k}_3(\cdot)$ and $\mathbf{k}(\cdot)$ are displayed in Figs. 9 and 10. Figures 11 and 12 display the variations of the input $u(\cdot) = -\mathbf{k}^T(\cdot)\mathbf{x}(\cdot)$ and the state of the closed-loop system for the initial perturbation (49). The oscillations decrease by one order of magnitude thanks to the controller.

8 Conclusions

We introduced a novel technique for the design of feedback controllers for linear dynamical systems with periodically varying coefficients. This approach can be implemented using either full-state feedback or observer-based controllers. It also provides a solution to the long-standing problem of assigning all the invariants of a system. The technique hinges on the Floquet-Lyapunov decomposition of the state-transition matrix, and exploits recent theoretical results related to the Floquet factors.

The controller relies on the specific form for the time-varying feedback matrix introduced in Subsection 6.4. This form led to the introduction of the controllability Gramian that is known to be invertible whenever the open-loop system is controllable. This form brings about a significant improvement compared to existing work based on approximating the inverse of the input matrix with its generalized inverse. Obtaining the gain matrix first required solving a matrix integral equation; this was accomplished via a spectral method.

We also provided a design rationale for the synthesis of the periodic controller. We described how the spectral solution could be used as an initial guess to a boundary-value problem to further refine the solution when necessary. We showed that computational benefits are gained by performing the design on the transformed system obtained by the Reducibility Theorem. Though mainly introduced as a full-state feedback controller, we showed how the approach can be seamlessly transformed into an observer-based controller.

Finally, we applied the control scheme to a quick-return mechanism. After having determined the stabil-

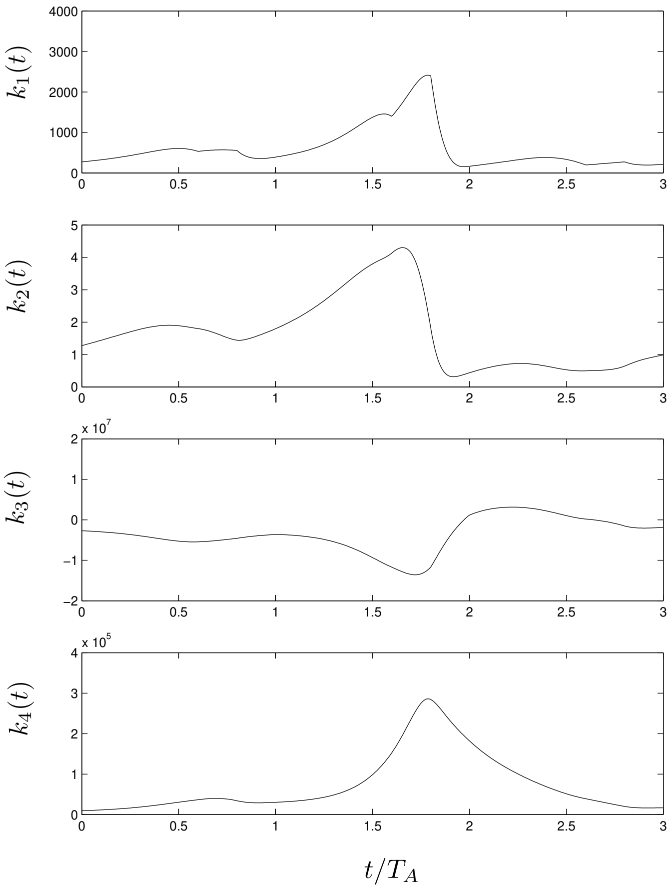


Figure 10: Entries of matrix $\mathbf{k}(t)$

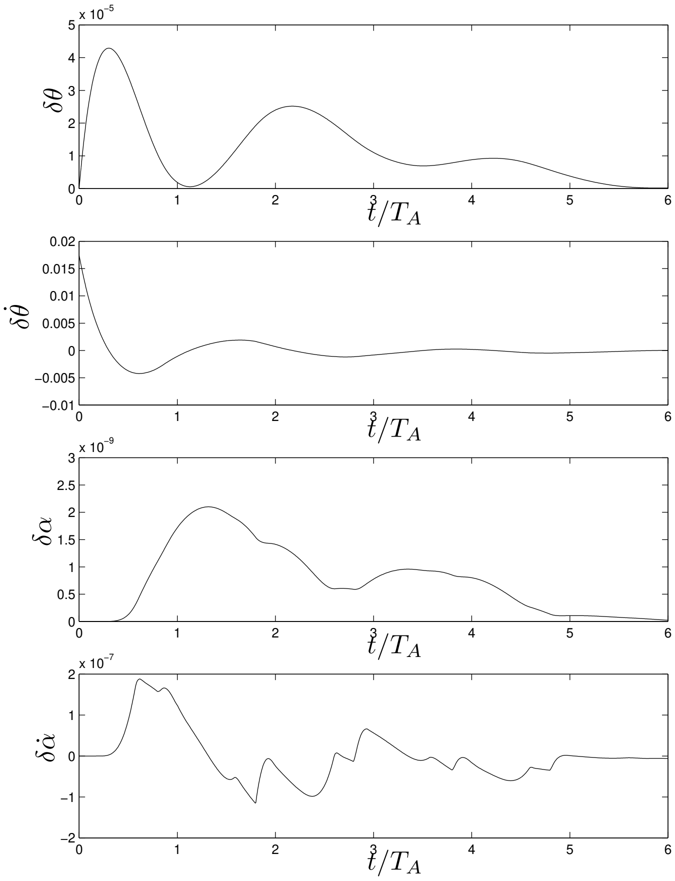


Figure 11: Effect of an initial perturbation on the closed-loop system

ity of the system, a closed-loop controller was built using the software toolbox we developed for the MATLAB$^{\text{TM}}$ environment. The source code for this toolbox is available in (Montagnier, 2002).

We are currently investigating how to take advantage of the extra degree of freedom of the periodic feedback when applied to the control of LTI systems; see e.g., (Willems *et al.* , 1984). We hope this will enable a better understanding of how the dynamics of the output is affected by the proposed controller. We are also interested in investigating the computational consequences brought about by the lack of uniqueness of the real Floquet decomposition.

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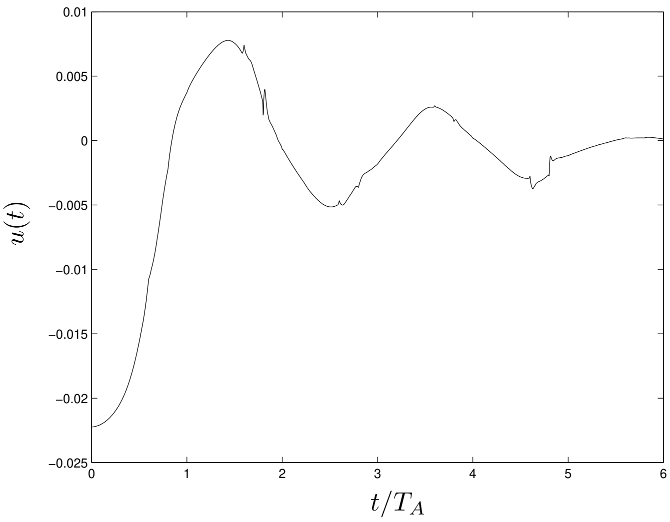

Figure 12: Variations of the input $u(t)$ for $\mathbf{x}(0) = \mathbf{x}_0$

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