The Real Story behind Floquet-Lyapunov Factorizations

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Abstract: Floquet-Lyapunov theory plays a ubiquitous role in the analysis and control of time-periodic systems. A real linear system with T-periodic coefficients will generally lead to a complex Floquet factorization. However, it is well known that by regarding the system as having $2T$-periodic coefficients, it is possible to obtain real factors. In this paper, we show that there exists a simple way to define real factors using known results for the real matrix logarithm and the knowledge of the state transition matrix of the system on a single period. The proposed procedure also determines whether the real Lyapunov transformation is T-periodic, T-antiperiodic, or generally $2T$-periodic.

Key-Words: Floquet theory, time-periodic systems, Lyapunov transformation, real matrix logarithm

1 Introduction

The Floquet-Lyapunov theorem (see, e.g., [2], [3], [4]) is a well-known and celebrated result in the field of linear, time-periodic (LTP) systems. The theorem consists of two main parts: the *Floquet representation theorem* and the *Lyapunov reducibility theorem*. In the following, we specialize the discussion of these results in terms of the *state transition matrix*; namely, the fundamental solution matrix $\mathbf{\Phi}(t,0)$ that satisfies $\mathbf{\Phi}(0,0) = \mathbf{I}$.

The Floquet representation theorem says that the state transition matrix $\mathbf{\Phi}(t,0)$ of the LTP system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \tag{1}$$

where $\mathbf{A}(t) = \mathbf{A}(t+T)$ can be written in the form

$$\mathbf{\Phi}(t,0) = \mathbf{L}(t,0)e^{t\mathbf{F}}. \tag{2}$$

In the classical theory, the factors $\mathbf{L}(t,0)$ and $\mathbf{F}$ are $n \times n$ matrices such that $\mathbf{L}(t,0) = \mathbf{L}(t+T,0)$, $\mathbf{L}(0,0) = \mathbf{I}_n$, where $\mathbf{I}_n$ is the $n \times n$ identity matrix, and $\mathbf{F} = \frac{1}{T}\ln\mathbf{\Phi}(T,0)$ is constant.

On the other hand, the Lyapunov reducibility theorem says that the time-dependent change of variables $\mathbf{x}(t) = \mathbf{L}(t,0)\mathbf{z}(t)$ transforms the original LTP system into a linear, time-invariant system $\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t)$, where $\mathbf{L}(t,0)$ and $\mathbf{F}$ are the same as in equation (2).

In general, the state transition matrix of a real system will have complex factors $\mathbf{L}(t,0)$ and $\mathbf{F}$. It is well known that it is always possible to obtain real factorizations within the classical theory by consid-

$\mathbf{A}(.)$	=	system matrix
$\mathbf{F}$	=	Floquet matrix
$\mathbf{I}_n$	=	$n \times n$ identity matrix
$\mathbf{J}$	=	Jordan canonical form of $\mathbf{\Phi}(T)$
$\mathbf{L}(.,0)$	=	Lyapunov transformation
$\mathbf{S}$	=	$n \times n$ diagonal matrix
$\mathbf{U}_n$	=	$n \times n$ super-diagonal matrix
$\mathbf{W}$	=	generalized eigenvectors of $\mathbf{\Phi}(T)$
$\mathbf{Y}$	=	$n \times n$ Yakubovich matrix
r_i	=	algebraic multiplicity of μ_i
s_i	=	geometric multiplicity of μ_i
$\mathbf{\Phi}(.,0)$	=	state transition matrix
ι	=	$\sqrt{-1}$
λ_i	=	eigenvalues of $\mathbf{F}$
μ_i	=	eigenvalues of $\mathbf{\Phi}(T)$

Table 1: Notation

ering the state transition matrix over *two* periods,

$$\begin{aligned} \mathbf{F}_{2T} &= \frac{1}{2T} \ln \mathbf{\Phi}(2T,0), \\ \mathbf{L}_{2T}(t+2T,0) &= \mathbf{L}_{2T}(t,0), \quad \mathbf{L}(0,0) = \mathbf{I}_n. \end{aligned} \tag{3}$$

This result offers sufficient conditions only. The issue of finding necessary and sufficient conditions to obtain a real factorization on one period was first raised in a short note by [8]. His result is summarized in the following theorem:

Theorem 1 *In equation (1), let $\mathbf{A}(t)$ be a real matrix function, where $\mathbf{A}(t+T) = \mathbf{A}(t)$. An arbitrary real matrix $\mathbf{X}(t,0)$ that is a fundamental solution of system (1) can be represented in the form (2) where $\mathbf{F}$ is real matrix, $\mathbf{L}(t,0)$ is a real matrix function,* $\det \mathbf{L}(t,0) \neq 0$, *and $\mathbf{L}(t,0)$ is such that*

$$\mathbf{L}(t+T,0) = \mathbf{L}(t,0)\mathbf{Y}, \tag{4}$$

and $\mathbf{Y}$ is a real matrix satisfying

$$\mathbf{Y}^2 = \mathbf{I}_n, \quad \mathbf{FY} = \mathbf{YF}. \tag{5}$$

Moreover, the converse statement is true as well: Let $\mathbf{L}(t,0), \mathbf{F}, \mathbf{Y}$ be arbitrary real matrices satisfying the above conditions, and let $\mathbf{X}(t,0)$ be described by (2). Then, $\mathbf{X}(t,0)$ is a real matrix of fundamental solutions of some equation (1) with a T-periodic real matrix function $\mathbf{A}(t)$.

On the other hand, the existence, uniqueness, and construction of real factorizations using information on only one period are issues that have not been investigated. These issues are treated in this report.

2 The Matrix Logarithm

Because $\mathbf{L}(t,0)$ satisfies $\mathbf{L}(t,0) = \mathbf{\Phi}(t,0)e^{-t\mathbf{F}}$, where $\mathbf{\Phi}(t,0)$ is real for a real system (1), conditions on the real nature of the factors can be focused on conditions for obtaining a real $\mathbf{F}$. This leads us to examine which conditions are necessary and sufficient for the logarithm of a real matrix to be real.

2.1 The Real Matrix Logarithm

We have the following results for the existence and uniqueness of a real logarithm of a real matrix [5]:

Theorem 2 *Suppose $\mathbf{M} \in \mathbb{R}^{n\times n}$. There is a real $\mathbf{X} \in \mathbb{R}^{n\times n}$ such that $e^{\mathbf{X}} = \mathbf{M}$ if and only if $\mathbf{M}$ is nonsingular and has an even number of Jordan blocks of each size for every negative eigenvalue.*
If $\mathbf{X}$ exists, it is uniquely defined if and only if it satisfies the above conditions and all the eigenvalues of $\mathbf{M}$ are real, positive, and have geometric multiplicity 1. Otherwise, there exist at least a countably infinite number of matrices $\mathbf{X}$ such that $e^{\mathbf{X}} = \mathbf{M}$.

Remark 1: For real systems (1), the state transition matrix $\mathbf{\Phi}(t,0)$ is real and non-singular. Therefore, its logarithm always exists. This theorem provides necessary and sufficient conditions for the logarithm to be real. Hence, we can focus our discussion on the existence of a real $\mathbf{F}$ in terms of the nature of the eigenvalues of the monodromy matrix $\mathbf{\Phi}(T,0)$ (also called the *Floquet multipliers*), and the structure of their associated Jordan blocks.

Remark 2: The classical result (3) is a straightforward consequence of Theorem 2: Because $\mathbf{\Phi}(T,0)$ is real, its eigenvalues must either be real or occur in complex-conjugate pairs. Because $\mathbf{\Phi}(2T,0) = \mathbf{\Phi}^2(T,0)$, any negative eigenvalues of $\mathbf{\Phi}(2T,0)$ must have come from purely imaginary eigenvalues of $\mathbf{\Phi}(T,0)$. Because these eigenvalues occur in complex-conjugate pairs, the Jordan blocks of $\mathbf{\Phi}(2T,0)$ corresponding to negative eigenvalues must occur in pairs.

when Floquet multipliers μ_i satisfy		real factorizations	for log branch $k \neq 0$, choose $\mathbf{P}$
$\mu_i \in \mathbb{R} - \{0\},\ r_i = s_i$			$\mathbf{P} = \mathbf{W}^{-1}\mathbf{T}$
$\forall i,\ r_i = 1$	with $\lvert\mu_i\rvert \neq \lvert\mu_j\rvert$ for $i \neq j$	1	$\mathbf{T} = \begin{bmatrix} -\iota & -\iota \\ 1 & -1 \end{bmatrix}$
$\forall i,\ r_i = 1$	with $\lvert\mu_i\rvert = \lvert\mu_j\rvert$ for some $i \neq j$	∞	
$\exists i,\ r_i > 1$	i.e., at least one paired	∞	
$\mu_i \in \mathbb{R} - \{0\},\ r_i > s_i$			$\mathbf{P} = \mathbf{\Pi}\,\mathrm{diag}\,(\mathbf{T})\,\mathbf{\Pi}^{-1}$
$\forall i,\ s_i = 1$	i.e., unpaired Jordan blocks	1	with $\mathbf{\Pi}\ \ln(\mathbf{J}_\Phi)\,\mathbf{\Pi}^{-1} =$
$\exists i,\ s_i > 1$	i.e., ≥ 1 paired Jordan blocks	∞	$\lambda \mathbf{I}_{2r_i} - \sum_{j=1}^{r_i-1} \frac{1}{j} \left(\frac{-1}{\mu_i}\right)^j \mathbf{U}_{2r_i}^{2j}$
$\mu_i \notin \mathbb{R}$ in conjugate pairs			
$r_i = s_i$	simple or multiple	1	$\mathbf{P} = \mathbf{I}_{r_i}$
$r_i > s_i$		∞	$\mathbf{P} = \mathbf{\Pi}\,\mathrm{diag}\,(\mathbf{T})\,\mathbf{\Pi}^{-1}$,

Table 2: Multiplicity of Real Factorizations

2.2 Characterization

The matrix logarithm can be constructed by means of the following procedure [6]:
1. Choose a Jordan form $\mathbf{J}$ for the non-singular matrix $\mathbf{\Phi}(T,0)$;
2. Choose a matrix $\mathbf{W}$ that puts $\mathbf{\Phi}(T,0)$ into the Jordan form $\mathbf{J}$ (i.e., $\mathbf{\Phi}(T,0) = \mathbf{W}\mathbf{J}\mathbf{W}^{-1}$);
3. Choose a logarithm $\ln(\mu_i)$ for each elementary Jordan block $\mathbf{J}_i = \mu_i \mathbf{I}_{r_i} + \mathbf{U}_{r_i}$ of $\mathbf{J}$;
4. Define

$$\begin{aligned} \ln\left(\mathbf{\Phi}(T,0)\right) &= \mathbf{W}\left(\sum_{i=1}^{m} \left[\ln(\mu_i)\mathbf{I}_{r_i} \right.\right. \\ &\left.\left. - \sum_{j=1}^{r_i-1} \frac{1}{j}\left(\frac{-\mathbf{U}_{r_i}}{\mu_i}\right)^j\right]\right)\mathbf{W}^{-1} \quad (6) \end{aligned}$$

It is shown in [6] that this approach can produce all the logarithms of matrix $\mathbf{\Phi}(T,0)$ when $\mathbf{W}$ is extended to $\mathbf{WP}$, where $\mathbf{P}$ is any matrix that commutes with $\mathbf{J}$.

3 Summary of Results

3.1 Generalized Definition of F

The classical definition $\mathbf{F} = \frac{1}{T}\ln(\mathbf{\Phi}(T,0))$ is not conducive to generating real factors. However, the use of the construction on two periods may result in a loss of information about $\mathbf{L}(.,0)$ by treating it as a $2T$-periodic function.

In his study of classical periodic differential equations, [1] investigates when solutions of Mathieu's equations are periodic or anti-periodic. As far as we know, no one has attempted to extend these results to general LTP systems, with an obvious application to the Floquet representation theorem.

Using Yakubovich's result, we adopt a generalized definition for $\mathbf{F}$:

$$\mathbf{F} = \frac{1}{T}\ln\left(\mathbf{Y}\mathbf{\Phi}(T)\right),$$

where $\mathbf{Y}$ is constant matrix, which we call the Yakubovich matrix. $\mathbf{Y}$ satisfies Theorem 1 and can be chosen according to $\mathbf{\Phi}(T)$ to give a real factorization. This leads to a novel Real Floquet Representation Theorem that supersedes the result on two periods. The results reported are as follows:

3.2 Existence and Uniqueness

It can be verified that $\ln(\mathbf{Y}\mathbf{\Phi}(T))$ is always real for the choice $\mathbf{Y} = \mathbf{WSW}$, where $s_{ii} = -1$ if μ_i is real and negative, and $s_{ii} = 1$ otherwise; $i = 1, 2, \ldots, n$.

When there are an infinite number of real representations, they can be constructed by taking $\lambda_k = \lambda_0 + \iota k\pi/T,\ k \in \mathbb{Z}$ in equation (6), with blocks of $\mathbf{P}$ chosen as shown in Table 2.

3.3 Periodicity

We see that $\mathbf{Y}$ indicates precisely the periodicity of the corresponding real $\mathbf{L}(t,0)$:

• If $\mathbf{Y} = \mathbf{I}_n$, $\mathbf{L}(t,0) = \mathbf{L}_T(t,0)$ is T-periodic;
• If $\mathbf{Y} = -\mathbf{I}_n$, $\mathbf{L}(t,0) = \mathbf{L}_{T^*}(t,0)$ is T-antiperiodic;
• Otherwise, $\mathbf{L}(t,0) = \mathbf{L}_{2T}(t,0)$ is generally $2T$-periodic.

We also note that when real factorizations are not unique, $\mathbf{Y}_k = (-1)^k\mathbf{Y}$ and the above is still true for the corresponding real transformation matrix $\mathbf{L}_k$.

4 Illustration

The following system first appeared in [7].

Let $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$, with $\mathbf{A}(t) = \frac{1}{\epsilon}\mathbf{U}(t)^T\mathbf{Z}\mathbf{U}(t)$, where $\mathbf{U}(t) = \begin{bmatrix} \cos\alpha t & -\sin\alpha t \\ \sin\alpha t & \cos\alpha t \end{bmatrix}$, $\mathbf{Z} = \begin{bmatrix} -1 & \eta \\ 0 & -1 \end{bmatrix}$, and $\alpha,\ \epsilon,\ \eta > 0$. The period of $\mathbf{A}(t)$ is $T = \frac{\pi}{\alpha}$.

One real pair of Floquet factors is given in [7] as

$$\mathbf{\Phi}(t) = \mathbf{L}_{2T}(t)e^{t\mathbf{F}_{2T}}, \tag{7}$$

$$\text{where} \quad \mathbf{L}_{2T}(t) = \begin{bmatrix} \cos\alpha t & \sin\alpha t \\ -\sin\alpha t & \cos\alpha t \end{bmatrix}, \tag{8}$$

$$\text{and} \quad \mathbf{F}_{2T} = \begin{bmatrix} -1/\epsilon & \eta/\epsilon - \alpha \\ \alpha & -1/\epsilon \end{bmatrix}. \tag{9}$$

The monodromy matrix is thus

$$\mathbf{\Phi}\left(\frac{\pi}{\alpha}\right) = \mathbf{L}_{2T}\left(\frac{\pi}{\alpha}\right)e^{\frac{\pi}{\alpha}\mathbf{F}_{2T}} = -e^{\begin{bmatrix} -\frac{\pi}{\alpha}\frac{1}{\epsilon} & \frac{\pi}{\alpha}\frac{\eta}{\epsilon} - \pi \\ \pi & -\frac{\pi}{\alpha}\frac{1}{\epsilon} \end{bmatrix}}, \tag{10}$$

and its eigenvalues satisfy

$$\mu_\pm = -e^{-\frac{\pi}{\alpha\epsilon}\beta_\pm}, \quad \beta_\pm = 1 \pm \sqrt{\alpha\epsilon(\eta - \alpha\epsilon)}. \tag{11}$$

4.1 High-Order Unpaired Jordan Blocks

The case $\eta = \alpha\epsilon$ leads to negative and repeated eigenvalues for the monodromy matrix:

$$\mathbf{\Phi}\left(\frac{\pi}{\alpha}\right) = -\begin{bmatrix} e^{-\pi/\eta} & 0 \\ \pi e^{-\pi/\eta} & e^{-\pi/\eta} \end{bmatrix} = \mathbf{W}_\Phi\mathbf{J}_\Phi(\mu)\mathbf{W}_\Phi^{-1},$$

where

$$\mathbf{W}_\Phi = \begin{bmatrix} 0 & 1 \\ -\pi e^{-\pi/\eta} & 0 \end{bmatrix}$$

is the matrix of generalized eigenvectors and

$$\mathbf{J}_\Phi(\mu) = \begin{bmatrix} -e^{-\pi/\eta} & 1 \\ 0 & -e^{-\pi/\eta} \end{bmatrix}.$$

The matrix $\mathbf{J}_\Phi(\mu)$ consists of only one Jordan block associated with a negative real repeated eigenvalue (multiplier). Hence, no real, T-periodic factor $\mathbf{L}_T(t)$ is possible. Only an antiperiodic factor $\mathbf{L}_{T^*}(.)$ exists, as given in equation (8), and this is the unique real decomposition. Figure 1 shows the variations of $\mathbf{A}(t)$ and $\mathbf{L}_{T^*}(t)$ when $\eta = \alpha = 2\pi$ and $\epsilon = 1$.

Let $\mu = -e^{-\pi/\eta}$, $\lambda = \frac{1}{T}\ln(-\mu) = -\alpha/\eta$. Then,

$$\frac{1}{T}\ln\left(\mathbf{J}_\Phi(\mu)\right) = \begin{bmatrix} \lambda & \frac{1}{T\mu} \\ 0 & \lambda \end{bmatrix},$$

$$\mathbf{F}_0 = \mathbf{W}_\Phi\mathbf{J}_{F_0}\mathbf{W}_\Phi^{-1} = \begin{bmatrix} -\alpha/\eta & 0 \\ \alpha & -\alpha/\eta \end{bmatrix},$$

as expected.

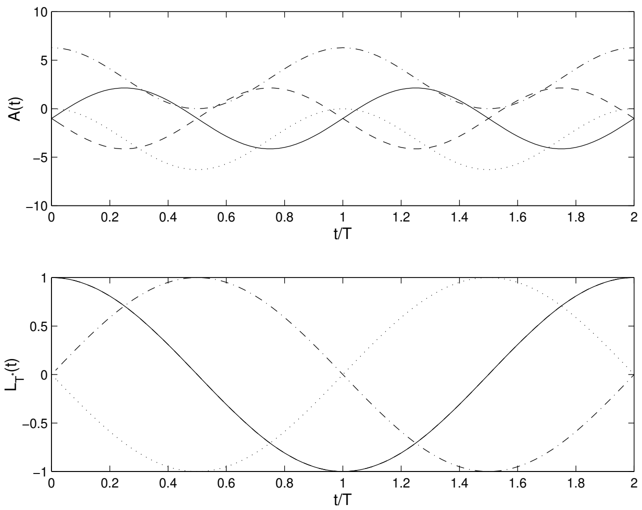

Figure 1: Periodic quantities when $\eta = \alpha\epsilon$. Top: T-periodic $A(t)$; Bottom: T-antiperiodic $L_{T^*}(t)$ (Note: $L_{11}(t) = L_{22}(t)$)

4.2 Complex-Conjugate Multipliers

In this example, we will consider the case where the monodromy matrix has two complex-conjugate eigenvalues, i.e., $\eta < \alpha\epsilon$. Regardless of the sign of the real parts of the multipliers, it is possible to have real, T-periodic or T-antiperiodic factors.

4.2.1 Derivation of $\mathbf{F}_{T_0}$

Using equation (10), we obtain the following monodromy matrix for $\eta = \pi,\ \alpha = 2\pi,$ and $\epsilon = 1$,

$$\mathbf{\Phi}(T) = -\begin{bmatrix} e^{-1/2}\cos(\frac{\pi\sqrt{2}}{2}) & -\sqrt{2}/2e^{-1/2}\sin(\frac{\pi\sqrt{2}}{2}) \\ \sqrt{2}e^{-1/2}\sin(\frac{\pi\sqrt{2}}{2}) & e^{-1/2}\cos(\frac{\pi\sqrt{2}}{2}) \end{bmatrix}.$$

From equation (11), the eigenvalues are $\mu = \rho e^{\iota\theta}$ and $\bar{\mu} = \rho e^{-\iota\theta}$, where $\rho = e^{-1/2}$, $\theta = \pi(-1+\sqrt{2}/2)$. A set of eigenvectors is

$$\mathbf{V}_\Phi = \begin{bmatrix} \frac{1}{2}\iota\sqrt{2} & -\frac{1}{2}\iota\sqrt{2} \\ 1 & 1 \end{bmatrix}.$$

The Floquet exponents are

$$\lambda = \frac{1}{T}(\ln(r) + \iota\theta) = -1 + \iota\pi(-2+\sqrt{2}),$$

and we obtain the Floquet matrix

$$\begin{aligned} \mathbf{F}_{T_0} &= \mathbf{V}_\Phi \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} \mathbf{V}_\Phi^{-1} \\ &= \begin{bmatrix} -1 & \pi(\sqrt{2}-1) \\ -2\pi(\sqrt{2}-1) & -1 \end{bmatrix}. \end{aligned}$$

As expected, $\mathbf{F}_{T_0}$ satisfies $\mathbf{\Phi}(T,0) = e^{T\mathbf{F}_{T_0}}$ and $\mathbf{L}_{T_0}(.,0)$ is T-periodic.

4.2.2 Derivation of $\mathbf{F}_{T_1^*}$

We use the theorem to build another real matrix matrix, $\mathbf{F}_{T_1^*}$, corresponding to a T-antiperiodic matrix $\mathbf{L}_{T^*}(.,0)$. To this end, we create a new matrix with eigenvalues[1] $\{\lambda + \iota\pi/T, \bar{\lambda} - i\pi/T\}$. We obtain

$$\begin{aligned} \mathbf{F}_{T_1^*} &= \mathbf{V}_\Phi \begin{bmatrix} \lambda + i\pi/T & 0 \\ 0 & \bar{\lambda} - i\pi/T \end{bmatrix} \mathbf{V}_\Phi^{-1} \\ &= \begin{bmatrix} -1 & -\pi \\ 2\pi & -1 \end{bmatrix}. \end{aligned}$$

Note that this is the matrix given in equation (9), and the corresponding $\mathbf{L}_{T_1^*}(.)$ in equation (8) is

$$\mathbf{L}_{T_1^*} = \begin{bmatrix} \cos(2\pi t) & \sin(2\pi t) \\ -\sin(2\pi t) & \cos(2\pi t) \end{bmatrix},$$

which is T-antiperiodic as expected.

4.2.3 Derivation of $\mathbf{L}_{T_0}(.,0)$

We can use the above expression of $\mathbf{L}_{T^*}(.,0)$ to derive a closed-form expression for $\mathbf{L}_{T_0}(.,0)$. First, we write the two computed real factorizations of the state transition matrix $\mathbf{\Phi}(.,0)$,

$$\mathbf{\Phi}(t,0) = \mathbf{L}_{T_0}(t,0)e^{t\mathbf{F}_{T_0}} = \mathbf{L}_{T_1^*}(t,0)e^{t\mathbf{F}_{T_1^*}},$$

[1] Of course, adding any other odd multiple of π/T would have lead to a T-antiperiodic factorization.

and use it to express $\mathbf{L}_{T_0}$ in terms of already known quantities, i.e.,

$$\mathbf{L}_{T_0}(t) = \mathbf{L}_{T_1^*}(t)e^{t\mathbf{F}_{T_1^*}}e^{-t\mathbf{F}_{T_0}}.$$

Recalling how $\mathbf{F}_{T_1^*}$ was constructing, we can write

$$e^{t\mathbf{F}_{T_1^*}} = \mathbf{V}_\Phi e^{t\mathbf{\Lambda}_{F_{T_0}}} e^{2\iota\pi t \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \mathbf{V}_\Phi^{-1}, \qquad (12)$$

and the new matrix $\mathbf{L}_{T_0}(.)$ has components

$$\begin{aligned} l_{11}(t) &= \cos^2(2\pi t) + \sqrt{2}\sin^2(2\pi t), \\ l_{12}(t) &= \frac{1}{2}\sin(4\pi t)(1 - \frac{\sqrt{2}}{2}), \\ l_{21}(t) &= \frac{1}{2}\sin(4\pi t)(\sqrt{2}-1), \\ l_{22}(t) &= \cos^2(4\pi t) + \frac{\sqrt{2}}{2}\sin^2(2\pi t), \end{aligned}$$

with period $T = \frac{1}{2}$, as expected. A plot of the variations of the matrices is given in Figure 2 for $\eta = \pi, \alpha = 2\pi$, and $\epsilon = 1$.

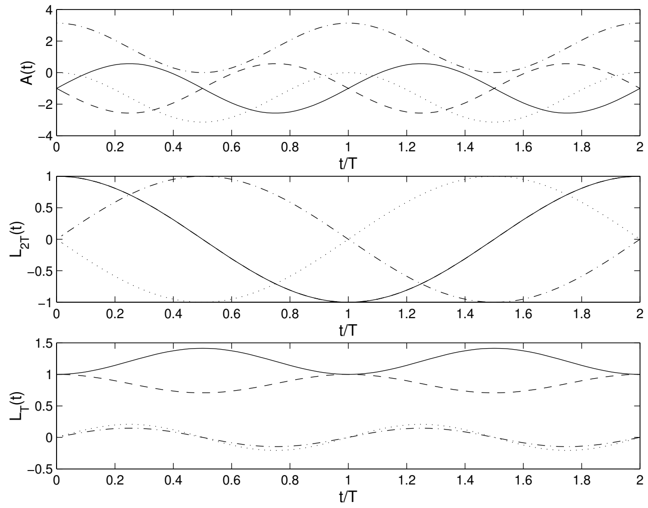

Figure 2: Periodic quantities when $\eta < \alpha\epsilon$. Top: T-periodic $\mathbf{A}(t)$; Middle: $2T$-periodic $\mathbf{L}_{2T}(t)$ (Note: $L_{2T_{11}}(t) = L_{2T_{22}}(t)$); Bottom: T-periodic $\mathbf{L}_T(t)$.

4.2.4 Paired Jordan Blocks

Since there was a single Jordan block in the example of Section 4.1, the real decomposition was unique and $\mathbf{L}(.)$ was T-antiperiodic because the multiplier $\mu = -e^{-\pi/\eta}$ was negative. We modify this example by building a new 4×4 system matrix $\mathbf{A}_4(.)$ based on the previous $\mathbf{A}(.)$, namely $\mathbf{A}_4(t) = \begin{bmatrix} \mathbf{A}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{A}(t) \end{bmatrix}$. The monodromy matrix is

$$\mathbf{\Phi}_4\left(\frac{\pi}{\alpha}\right) = -\begin{bmatrix} e^{-\pi/\eta} & 0 & 0 & 0 \\ \pi e^{-\pi/\eta} & e^{-\pi/\eta} & 0 & 0 \\ 0 & 0 & e^{-\pi/\eta} & 0 \\ 0 & 0 & \pi e^{-\pi/\eta} & e^{-\pi/\eta} \end{bmatrix},$$

with Jordan canonical form $\mathbf{\Phi}_4(\frac{\pi}{\alpha}) = \mathbf{W}_4 \mathbf{J}_{\Phi_4}^{-1} \mathbf{W}_4^{-1}$, where $\mathbf{J}_{\Phi_4} = \begin{bmatrix} \mathbf{J}_\Phi(\mu) & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \mathbf{J}_\Phi(\mu) \end{bmatrix}$, and

$$\mathbf{W}_{\Phi_4} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\pi e^{-\pi/\eta} & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ -\pi e^{-\pi/\eta} & 0 & -\pi e^{-\pi/\eta} & 0 \end{bmatrix}.$$

Based on the example in Section 4.1, we derive

$$\begin{aligned} \mathbf{F}_{T_0} &= \frac{1}{T}\mathbf{W}_{\Phi_4} \ln\left(-\mathbf{J}_{\Phi_4}\right) \mathbf{W}_{\Phi_4}^{-1} \\ &= \begin{bmatrix} -\alpha/\eta & 0 & 0 & 0 \\ \alpha & -\alpha/\eta & 0 & 0 \\ 0 & 0 & -\alpha/\eta & 0 \\ 0 & 0 & \alpha & -\alpha/\eta \end{bmatrix}. \end{aligned}$$

The Yakubovich matrix associated to $\mathbf{F}_{T_0^*}$ is $\mathbf{Y}_0 = -\mathbf{I}_n$, indicating that the corresponding $\mathbf{L}_{T_0^*}(.)$ will be T-antiperiodic, as was the case in Section 4.1. However, since the Jordan blocks are paired, it is possible to derive another real pair $\{\mathbf{F}_{T_1}, \mathbf{L}_{T_1}(.)\}$, where $\mathbf{L}_{T_1}(.)$ is T-periodic. We build $\mathbf{J}_{T_1} = \begin{bmatrix} \lambda & \frac{1}{T\mu} & \pi/T & 0 \\ 0 & \lambda & 0 & \pi/T \\ -\pi/T & 0 & \lambda & \frac{1}{T\mu} \\ 0 & -\pi/T & 0 & \lambda \end{bmatrix}$, and we derive $\mathbf{F}_{T_1} = \mathbf{W}_{\Phi_4}\mathbf{J}_{T_1}\mathbf{W}_{\Phi_4}^{-1}$. When $\eta = \alpha = 2\pi$,

$$\mathbf{F}_{T_1} = \begin{bmatrix} -(2\pi+1) & 0 & 2\pi & 0 \\ 2\pi & -(2\pi+1) & 0 & 2\pi \\ -4\pi & 0 & 2\pi - 1 & 0 \\ 0 & -4\pi & 2\pi & 2\pi - 1 \end{bmatrix}.$$

Using $\mathbf{L}_k(t,0) = \mathbf{\Phi}(t,0)e^{-t\mathbf{F}_k}$, we get $\mathbf{L}_{T_1}(t)$, where

$$\begin{aligned} l_{11}(t) &= l_{22}(t) = \cos(\alpha t)[\cos(\alpha t) + \sin(\alpha t)], \\ l_{12}(t) &= -l_{21}(t) = \sin(\alpha t)[\cos(\alpha t) + \sin(\alpha t)], \\ l_{13}(t) &= l_{24}(t) = -\cos(\alpha t)\sin(\alpha t), \\ l_{14}(t) &= -l_{23}(t) = -\sin^2(\alpha t) \\ l_{31}(t) &= l_{42}(t) = \sin(2\alpha t) \\ l_{32}(t) &= -l_{41}(t) = -\sin(2\alpha t) \\ l_{33}(t) &= l_{44}(t) = \cos(\alpha t)[\cos(\alpha t) - \sin(\alpha t)], \\ l_{34}(t) &= -l_{43}(t) = \sin(\alpha t)[\cos(\alpha t) - \sin(\alpha t)]. \end{aligned}$$

We note that $\mathbf{L}_{T_1}(.)$ is T-periodic, as expected.

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