University of Saskatchewan
Department of Mathematics and Statistics

Numerical Analysis I
(MATH 314)
Instructor: Dr. Raymond J. Spiteri

ASSIGNMENT 01
Due: 10:00 a.m. Tuesday, October 01, 2013

1. [**15 marks**]

   (a) For each of the following constant-coefficient systems $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$, determine whether the system is stable or unstable; if it is stable determine whether it is asymptotically stable.

   $$\text{(i)} \quad \mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -100 \end{bmatrix} \qquad \text{(ii)} \quad \mathbf{A} = \begin{bmatrix} -1 & 10 \\ 0 & -2 \end{bmatrix}$$
   $$\text{(iii)} \quad \mathbf{A} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \qquad \text{(iv)} \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

   (b) Convert the following high-order non-autonomous differential equation for the variable $u = u(t)$ to autonomous first-order form:

   $$\frac{d^4u}{dt^4} + u\left(\frac{du}{dt}\right)^2 + \frac{g}{l}\sin u = \frac{1}{2\pi}\cos(2\pi\omega t).$$

   (c) Compute the eigenvalues of the time-dependent matrix

   $$\mathbf{A}(t) = \begin{bmatrix} -\frac{1}{4} + \frac{3}{4}\cos 2t & 1 - \frac{3}{4}\sin 2t \\ -1 - \frac{3}{4}\sin 2t & -\frac{1}{4} - \frac{3}{4}\cos 2t \end{bmatrix}.$$

   Determine whether the variable-coefficient system $\dot{\mathbf{y}} = \mathbf{A}(t)\mathbf{y}$ is stable or unstable. Does this contradict your result about the eigenvalues of $\mathbf{A}(t)$?

2. [**15 marks**] Consider the scalar IVP

   $$\dot{y} = -5ty^2 + \frac{5}{t} - \frac{1}{t^2}, \qquad y(1) = 1,$$

   for $1 \leq t \leq 25$.

   Show that the exact solution is $y(t) = 1/t$.

   Compute the error at $t = 25$ when using the forward Euler method for step sizes $h = 0.1, 0.05$, and 0.025. By comparing the relative sizes of the errors, determine the convergence rate of forward Euler. Repeat your calculations for the following explicit

Runge-Kutta method that advances the solution to the IVP $\dot{y} = f(t, y)$, $y(0) = y_0$ from time $t = t_{n-1}$ to time $t = t_n = t_{n-1} + h$:

$$
\begin{aligned}
K_1 &= f(t_{n-1}, y_{n-1}) \\
K_2 &= f(t_{n-1} + h/2, y_{n-1} + hK_1/2) \\
K_3 &= f(t_{n-1} + h/2, y_{n-1} + hK_2/2) \\
K_4 &= f(t_{n-1} + h, y_{n-1} + hK_3) \\
y_n &= y_{n-1} + \frac{h}{6}(K_1 + 2(K_2 + K_3) + K_4)
\end{aligned}
$$

3. **[25 marks]** Show that the linear, constant-coefficient, second-order differential equation for the harmonic oscillator

$$\ddot{\theta} + \theta = 0, \qquad \theta(0) = r, \quad \dot{\theta}(0) = 0,$$

can be written as

$$
\begin{aligned}
\dot{x} &= y, \qquad & x(0) &= r \qquad (1a) \\
\dot{y} &= -x, \qquad & y(0) &= 0. \qquad (1b)
\end{aligned}
$$

Show that the exact solution of this IVP is $\theta(t) = r \cos t$. Does the (maximum) amplitude of the exact solution grow, decay, or remain constant?

Write a Matlab program to integrate (1) from $t = 0$ to $t = 120$ with stepsize $h = 0.02$ using forward Euler, backward Euler, and trapezoidal method. What happens to the (maximum) amplitude of the numerical solution in each case? Explain your results. [**Hint:** One way to do this is to show that for the exact solution we have $x_n^2 + y_n^2 = x_{n-1}^2 + y_{n-1}^2$. Then for each of the three schemes, derive an expression for $x_n^2 + y_n^2$ in terms of $x_{n-1}^2 + y_{n-1}^2$ and compare to the exact relationship.]

4. **[45 marks]**

(a) The *fundamental solution* $\mathbf{Y}(t)$ of the linear system

$$\dot{\mathbf{y}} = \mathbf{A}(t)\mathbf{y} + \mathbf{q}(t) \qquad (2)$$

of size m is the $m \times m$ matrix that satisfies

$$\dot{\mathbf{Y}} = \mathbf{A}(t)\mathbf{Y}, \qquad \mathbf{Y}(0) = \mathbf{I},$$

i.e., the ith column of $\mathbf{Y}$ is the solution to (2) with initial condition equal to the ith column of the identity matrix.

Floquet Theory tells us that the stability of a linear time-periodic homogeneous system (i.e., (2) with $\mathbf{A}(t + T) = \mathbf{A}(t)$ and $\mathbf{q}(t) \equiv \mathbf{0}$) can be determined by the eigenvalues of the matrix logarithm of the *monodromy matrix* $\mathbf{Y}(T)$: if all these eigenvalues have nonpositive real parts, the system is stable; otherwise it is unstable. Using the Matlab functions `ode45`, `logm`, and `eig`, determine the stability of the linear time-periodic system in Question 1(c).

(b) For the forward Euler method applied to the test equation $\dot{y} = \lambda y, \; \mathrm{Re}(\lambda) < 0$, we obtained the following condition for A-stability:

$$|1 + h\lambda| \leq 1. \tag{3}$$

However, this condition does not necessarily suppress spurious oscillation in the numerical solution (see Figure 3.3 of the text for solution with $h = 0.21$ and λ real and negative). What is meant by *spurious oscillation*? Derive a condition similar to (3) that guarantees there will be no spurious oscillation in the numerical solution.

(c) Apply *forward Euler* to (1a) and *backward Euler* to (1b). Is the resulting *implicit-explicit* (IMEX) method implicit or explicit? Modify your Matlab program from Question 3 to integrate (1) from $t = 0$ to $t = 120$ with stepsize $h = 0.02$ using this method. What happens to the (maximum) amplitude of the numerical solution in this case? Explain your results.

5. **[0 marks]** Declare whether you intend to work on the project for this course on your own or as part of a group. If you intend to work as part of a group, please list all the group members. Please hand your answer in on a separate piece of paper, which will be kept as a record of your declaration.