University of Saskatchewan
Department of Mathematics and Statistics

Numerical Analysis III
(MATH 314)
Instructor: Dr. Raymond J. Spiteri

ASSIGNMENT 03
Due: 10:00 a.m. Tuesday, November 05, 2013

1. **[25 marks]**

   (a) Derive the 3-step Adams–Bashforth method

   $$y_n = y_{n-1} + \frac{\Delta t}{12}\left(23 f_{n-1} - 16 f_{n-2} + 5 f_{n-3}\right). \tag{1}$$

   (b) Derive the 2-step Adams–Moulton method

   $$y_n = y_{n-1} + \frac{\Delta t}{12}\left(5 f_n + 8 f_{n-1} - f_{n-2}\right). \tag{2}$$

   (c) Derive the 3-step BDF method

   $$y_n - \frac{18}{11} y_{n-1} + \frac{9}{11} y_{n-2} - \frac{2}{11} y_{n-3} = \Delta t \frac{6}{11} f_n. \tag{3}$$

   What is its order? What is its error constant?

2. **[25 marks]** Consider the following implicit Runge–Kutta method

$$\begin{array}{c|cc} \frac{3-\sqrt{3}}{6} & \frac{1}{4} & \frac{3-2\sqrt{3}}{12} \\ \frac{3+\sqrt{3}}{6} & \frac{3+2\sqrt{3}}{12} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}.$$

   Determine the convergence rate of the method by solving the following problem on $0 \le t \le 1$ for step-sizes $\Delta t = 0.2, 0.1, 0.05$, and 0.025:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

3. **[25 marks]**

   (a) Derive the 7-step BDF method (of order 7) and show it is *not* strongly stable.

   (b) Derive the two-stage Radau Runge–Kutta method from the fact that it is collocation method at the canonical points $c_1 = \frac{1}{3}$ and $c_2 = 1$.

   (c) Show *directly* (i.e., from the definition) that the two-stage Radau Runge–Kutta method has stiff decay.

   (d) Show that for a linear multistep method to be consistent, we must have $\rho(1) = 0$ **and** $\rho'(1) = \sigma(1)$.

4. **[25 marks]** The displacement $x(t)$ at time t of a cork inside the neck of a bottle of fermenting liquid satisfies

$$\ddot{x}(t) = g(1+q)\left[\left(1+\frac{x}{d}\right)^{-\gamma} + \frac{Rt}{100} - 1 + \frac{qx}{L(1+q)}\right],$$

   where $g = 9.81$, $q = 20$, $d = 5$, $\gamma = 1.4$, and $R = 4$. If the cork starts at rest at $x = 0$ and the neck has length $L = 3.75$, find the time t^* when the cork leaves the neck of the bottle. What is the cork's speed then?

5. **[0 marks]** Provide a brief interim report for your project.

   The report should include a description and motivation for the topic and an update on progress toward the goals for the project and timelines for achieving the goals.

   Please use LaTeXwith the standard `article` document class, double spacing, and 12-point font.

   The interim report should be between 5 and 10 pages in length (10 and 20 pages for teams of two). Much of this report should be usable for the final report. Recall that the final report is expected to be about 20 pages (40 pages for teams of two). Completion of the interim report can thus represent completion of 10–50% of your final report.