CHAPTER 2: *On problem stability*

Stability: small changes to input

$$\Downarrow$$

small changes to output

But other definitions possible, depending on context.

2.1 Test equation (Dahlquist)

$$\dot{y} = \lambda y$$
$\lambda =$ constant (possibly complex !)

$\updownarrow$

eigenvalues of a matrix

Solution for initial condition $y(0)$:

$$y(t) = e^{\lambda t} y(0), \quad t \geq 0. \qquad \text{(verify!)}$$

For initial condition $\tilde{y}(0)$:

$$\tilde{y}(t) = e^{\lambda t} \tilde{y}(0), \quad t \geq 0.$$

Difference:

$$\begin{aligned} \|y(t) - \tilde{y}(t)\| &= \|\big(y(0) - \tilde{y}(0)\big) e^{\lambda t}\| \\ &= \|y(0) - \tilde{y}(0)\| e^{(\mathcal{R}e\lambda)t} \end{aligned}$$

Verify:

$$\|e^{\lambda t}\| = e^{(\mathcal{R}e\lambda)t}$$

Pretend:

$y(t)$ "exact solution"

$\tilde{y}(t)$ solution to problem with perturbed ICs

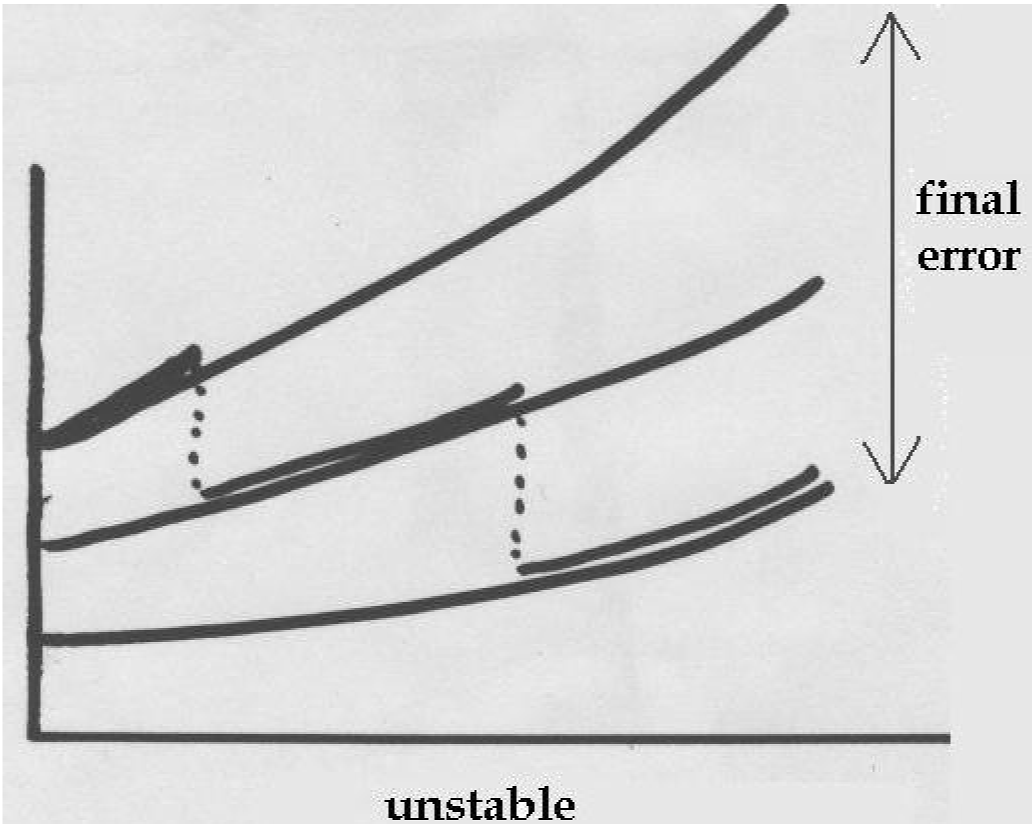
$\mathcal{R}e\lambda \leq 0$: difference bounded (stable)
$\mathcal{R}e\lambda < 0$: difference decays (asymptotically stable)
$\mathcal{R}e\lambda > 0$: difference unbounded (unstable)

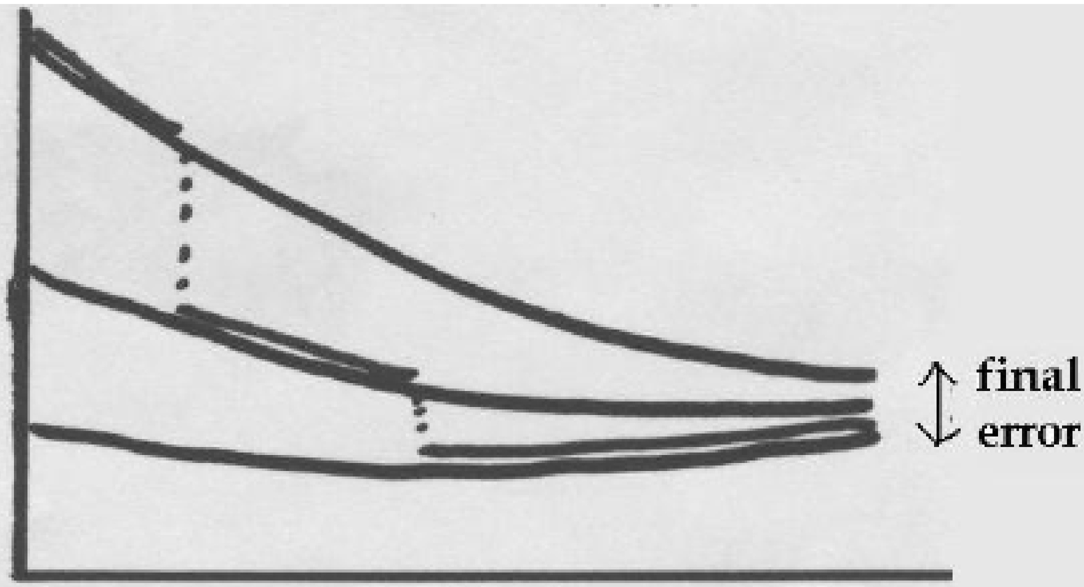
- Technical point: The solution $\mathbf{y}(t)$ to $\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$ is stable if for all $\epsilon > 0$, there is a $\delta > 0$ such that if $\hat{\mathbf{y}}(t)$ satisfies the ODE and $\|\mathbf{y}(0) - \hat{\mathbf{y}}(0)\| \leq \delta$ then

$$\|\mathbf{y}(t) - \hat{\mathbf{y}}(t)\| \leq \epsilon \quad \text{for all } t \geq 0.$$

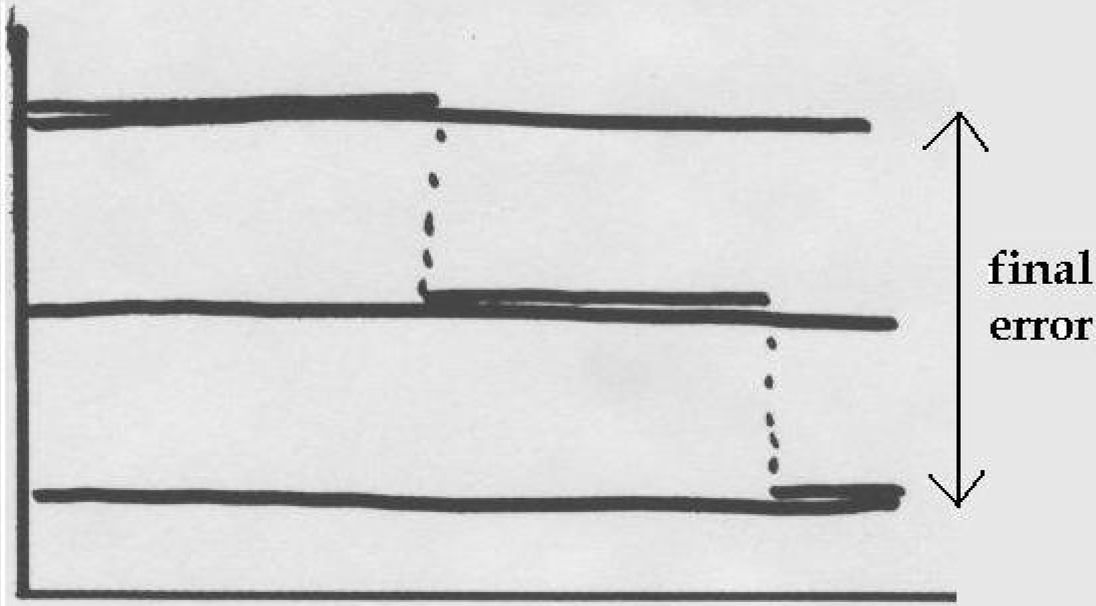
Asymptotically stable = stable + $\|\mathbf{y}(t) - \hat{\mathbf{y}}(t)\| \to 0$ as $t \to \infty$.

Example 1.





asymptotically stable



stable

Review: eigenvalues (of a real $m \times m$ matrix $\mathbf{A}$)

- Definition: λ is an eigenvalue (scalar, complex) and $\mathbf{x} \neq \mathbf{0}$ is an eigenvector (up to scaling factor) if $\quad \mathbf{A}\mathbf{x} = \lambda \mathbf{x}.$

  Note 1. *λ is real if $\mathbf{A}$ is symmetric $(\mathbf{A} = \mathbf{A}^T)$.*

- Similarity transformation: If $\exists$ $\mathbf{T}$ nonsingular such that
$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T},$$
  then $\mathbf{B}$ is similar to $\mathbf{A}$ (has same eigenvalues).
  If $\mathbf{B}$ is diagonal

$$\mathbf{B} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_m \end{pmatrix},$$

then
$$\mathbf{T} = [\mathbf{x}_1 \,|\, \mathbf{x}_2 \,|\, \cdots \,|\, \mathbf{x}_m],$$
and $\mathbf{A}$ is called diagonalizable.

Note 2. *Any symmetric matrix is diagonalizable by an orthogonal matrix* $\mathbf{T}$ *(i.e.,* $\mathbf{T}^{-1} = \mathbf{T}^T$*).*

TECHNICAL POINTS:

- An orthogonal similarity matrix can only generally bring a matrix to an upper triangular form:

$$\begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}.$$

Eigenvalues are still on the diagonal!

- For general $\mathbf{A}$, there is always a similarity transformation that takes $\mathbf{A}$ to a Jordan canonical form $\mathbf{B}$, where

$$\mathbf{B} = \begin{bmatrix} \mathbf{\Lambda}_1 & & & \\ & \mathbf{\Lambda}_2 & & \\ & & \ddots & \\ & & & \mathbf{\Lambda}_s \end{bmatrix}, \quad s \leq m,$$

and

$$\mathbf{\Lambda}_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}, \quad \sum_{i=1}^{s} \dim(\mathbf{\Lambda}_i) = m.$$

2.2 Linear constant-coefficient systems

$$\begin{array}{cc} \dot{y} = \lambda y & \dot{\mathbf{y}} = \mathbf{A}\mathbf{y} \\ \downarrow & \downarrow \\ y(t) = e^{\lambda t} y(0) & \mathbf{y}(t) = e^{t\mathbf{A}} \mathbf{y}(0) \end{array}$$

with $\mathbf{A} = m \times m$ constant matrix.

Note 3. $e^{t\mathbf{A}} = \mathbf{I} + (t\mathbf{A}) + \frac{(t\mathbf{A})^2}{2!} + \frac{(t\mathbf{A})^3}{3!} + \cdots$

Suppose $\mathbf{A}$ is diagonalizable.

$$\Rightarrow \exists\ \mathbf{T} \text{ s.t. } \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix}.$$

Change variables:

$$\mathbf{w} = \mathbf{T}^{-1}\mathbf{y}.$$

Then

$$\dot{\mathbf{w}} = \mathbf{\Lambda}\mathbf{w}.$$

$\rightarrow$ decoupled system ! $\dot{w}_i = \lambda_i w_i$

$\rightarrow$ Stability of $\mathbf{w}$ (and hence $\mathbf{y}$) determined by eigenvalues of $\mathbf{A}$.

- All $\mathcal{R}e(\lambda_i) < 0$ asymptotically stable
- All $\mathcal{R}e(\lambda_i) \leq 0$ stable
- One (or more) $\mathcal{R}e(\lambda_i) > 0$ unstable

TECHNICAL NOTE: What if $\mathbf{A}$ is <u>not</u> diagonalizable? Then we have a Jordan form

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} \mathbf{\Lambda}_1 & & & \\ & \mathbf{\Lambda}_2 & & \\ & & \ddots & \\ & & & \mathbf{\Lambda}_s \end{bmatrix},$$

where

$$\mathbf{\Lambda}_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}.$$

In this case,
stable $\Leftrightarrow$ each λ_i satisfies either $\mathcal{R}e(\lambda_i) < 0$,
or $\mathcal{R}e(\lambda_i) = 0$ and λ_i is simple.

asymptotically stable $\Leftrightarrow$ all λ_i satisfy $\mathcal{R}e(\lambda_i) < 0$.

Example 1. *(Vibrating spring, again)*

$$-\ddot{u} + u = 0.$$

As a first-order system

$$\dot{\mathbf{y}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{y}, \quad \mathbf{y} = \begin{pmatrix} u \\ \dot{u} \end{pmatrix}. \quad \textit{(verify!)}$$

Eigenvalues of $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *:* $\lambda_1 = -1, \; \lambda_2 = 1.$

(verify!)

Problem is unstable !

Example 2. *(General homogeneous constant-coefficient scalar ODE)*

$$a_k u + a_{k-1}\dot{u} + \cdots + a_0 u^{(k)} = 0, \quad a_0 > 0,$$

or

$$\sum_{j=0}^{k} a_j \frac{d^{k-j}}{dt^{k-j}} u = 0.$$

$\rightarrow$ *convert to first order, or guess* $y = e^{\lambda t}$.

Either way, λ *satisfies the characteristic polynomial*

$$\phi(\lambda) = a_0\lambda^k + a_1\lambda^{k-1} + \cdots + a_k = 0.$$

$\Rightarrow$ *Solution is* stable *iff all* λ *satisfy either* $\mathcal{R}e(\lambda) < 0$*, or* $\mathcal{R}e(\lambda) = 0$ *and* λ *is simple.*

Asymptotically stable iff all λ *satisfy* $\mathcal{R}e\lambda < 0$*.*

2.3 Linear, variable-coefficient systems

$$\dot{\mathbf{y}} = \mathbf{A}(t)\mathbf{y} + \mathbf{q}(t)$$

Now eigenvalues are irrelevant !!

Example 3.

$$\dot{y} = (\cos t)y$$

Eigenvalue: $\lambda(t) = \cos t \rightarrow$ *sometimes positive.*

Solution $\qquad y(t) = e^{\sin t} y(0)$

$\rightarrow \quad \|y(t)\| < e\|y(0)\|$ *problem is stable (verify)*

More can be said for periodic $\mathbf{A}(t)$*;*
i.e., $\exists\, T > 0$ *such that* $\mathbf{A}(t+T) = \mathbf{A}(t)$*.*
$\rightarrow$ *Floquet–Lyapunov theory*

2.4 Nonlinear problems

For nonlinear problems, stability depends on which specific solution trajectory is considered!

Given an isolated solution $\mathbf{y}(t)$,
ODE can be linearized about $\mathbf{y}(t)$,
then analyse stability of perturbations.

Let $\mathbf{y}(t)$ solve $\quad \dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}), \quad \mathbf{y}(0) = \mathbf{y}_0,$
Let $\hat{\mathbf{y}}(t)$ solve $\quad \dot{\hat{\mathbf{y}}} = \mathbf{f}(t, \hat{\mathbf{y}}), \quad \hat{\mathbf{y}}(0) = \hat{\mathbf{y}}_0,$
$\hat{\mathbf{y}}_0$ close to $\mathbf{y}_0$.

Then because $\quad \mathbf{f}(t, \hat{\mathbf{y}}) \approx \mathbf{f}(t, \mathbf{y}) + \frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\hat{\mathbf{y}} - \mathbf{y}),$
we look at the linear variational equation

$$
\begin{aligned}
\dot{\mathbf{z}} &= \mathbf{A}(t, \mathbf{y})\mathbf{z}, \\
\mathbf{z} &= \hat{\mathbf{y}} - \mathbf{y}, \quad \text{(perturbation)}
\end{aligned}
$$

where $\mathbf{A}(t, \mathbf{y}) = \frac{\partial \mathbf{f}}{\partial \mathbf{y}}(t, \mathbf{y})$ is called the Jacobian.

Example 4. *Often, we are interested in the stability of equilibria or steady-state solutions;*
i.e., when $\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) = \mathbf{0}$*.*

Consider $\dot{y} = y(1 - y)$ *(model of logistic growth)*
Equilibria: $y = 0$ *and* $y = 1$
Jacobian: $\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{y}} = 1 - 2y$

When $y = 0, \mathbf{A} = 1 > 0$*, solution is unstable.*
When $y = 1, \mathbf{A} = 1 - 2(1) = -1 < 0$*, solution is stable.*

$\rightarrow$ *Solutions with* $y(0) = c$ *with* $0 < c \ll 1$ *are repelled from* $y = 0$ *and are attracted to* $y = 1$*.*

Exercise: use Matlab's ode45 to verify this.