CHAPTER 6: *BVP Theory and Applications*

$$\underbrace{\mathbf{y}' = \mathbf{f}(x, \mathbf{y})}_{m \text{ components}}, \qquad a < x < b,$$

$$\underbrace{\mathbf{g}(\mathbf{y}(a), \mathbf{y}(b))}_{m \text{ two-point BCs.}} = \mathbf{0}. \qquad \mathbf{g} \text{ can be nonlinear.}$$

$\downarrow$

only involve $\mathbf{y}(a), \mathbf{y}(b)$.

(Otherwise we have *interface conditions*.)

Often $\mathbf{g}$ is linear; i.e., BCs can be written as

$$\mathbf{B}_a \mathbf{y}(a) + \mathbf{B}_b \mathbf{y}(b) = \boldsymbol{\beta}.$$

$\uparrow \qquad\qquad \uparrow$

constant $m \times m$ matrices.

Also, BCs are often separated:
$\rightarrow$ each BC ($\leftrightarrow$ component of $\mathbf{g}$) involves $\mathbf{y}(a)$ or $\mathbf{y}(b)$ but not both.

$\leftrightarrow$ For each $i, \quad i = 1, 2, \cdots, m$,
either i^{th} row of $\mathbf{B}_a$ or $\mathbf{B}_b$ is zero.

Example 1. $\quad -(p(x)u')' + q(x)u = r(x)$,
$u(a) = 0, \quad u'(b) = 0$,
$p(x) > 0, \quad q(x) \geq 0$, *for all* $a \leq x \leq b$.

$\rightarrow$ *Convert to first-order system:*

- *The usual way* $\quad y_1 = u, \quad y_2 = u'$.

- *If p has discontinuities, a better choice is*

$$y_1 = u, \quad y_2 = py_1'.$$

(y_2 is called the flux *and is better (well) behaved.)*

Then

$$\mathbf{f}(x, \mathbf{y}) = \begin{pmatrix} \frac{1}{p} y_2 \\ q y_1 - r \end{pmatrix}. \quad \textit{(verify)}$$

Either way, BCs are linear and separated:

$$\mathbf{B}_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{B}_b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \text{ \textit{(verify)}}$$

No nice existence/uniqueness theory for BVPs (unlike for IVPs).

Suppose $\mathbf{y}(x) = \mathbf{y}(x; \mathbf{y}_a)$ solves IVP
with IC: $\mathbf{y}(a; \mathbf{y}_a) = \mathbf{y}_a$.

$\mathbf{y}(x; \mathbf{y}_a)$: solution at x starting from $\mathbf{y}_a$.

Then the BCs imply

$$\mathbf{g}(\mathbf{y}_a, \mathbf{y}(b; \mathbf{y}_a)) = \mathbf{0}.$$

$\to$ m nonlinear algebraic equations for m unknowns (components of $\mathbf{y}_a$).

$\to$ No solution, one solution, many solutions!

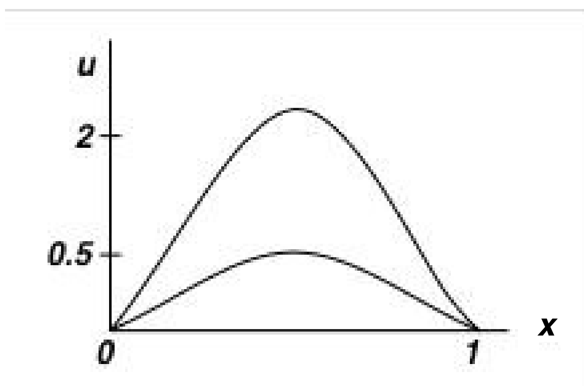
Example 2. $u' + e^{u+1} = 0$,
$u(0) = u(1) = 0$.

Two solutions:

$$u(x) = -2\ln\left\{\frac{\cosh\frac{1}{2}(x-\frac{1}{2})\theta}{\cosh\frac{\theta}{4}}\right\},$$

where θ *satisfies*

$$\theta = \sqrt{2e}\cosh\left(\frac{\theta}{4}\right).$$



With BVPs, you must start with an initial guess for the whole solution.

$\rightarrow$ When the solution is not unique, which solution you converge to depends on the initial guess!

6.2 Stability of BVPs

Consider $\quad y_1' = \lambda y_1, \quad a < x < b, \quad b \gg 1.$

IVP (say with $y_1(a) = 1$) is stable

$$\text{if} \quad \mathcal{R}e(\lambda) \leq 0.$$

Now change the direction of x:

$$\xi = (a + b) - x.$$

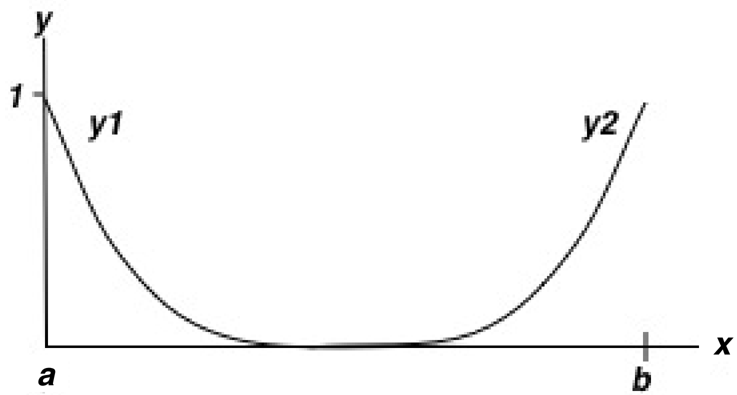
The same problem now reads

$$\begin{aligned} \frac{dy_2}{d\xi} &= -\lambda y_2, & \text{(verify)} \\ y_2(b) &= 1, & y_2 = y_2(\xi). \end{aligned}$$

$\rightarrow$ Terminal-value problem (integrate from b to a).

$\rightarrow$ Changing direction (i.e., re-writing problem with a new variable) does not affect original stability.

i.e., TVP is stable for $\mathcal{R}e(-\lambda) \geq 0$.
(stability $\leftrightarrow$ effect of small perturbations on solution.)



Putting them together, the BVP

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

$y_1(a) = 1, \quad y_2(b) = 1$ is stable.

Stability depends on how BCs are specified!

Note 1. *The pure IVP is not stable for $\mathcal{R}e(\lambda) \neq 0$. There is not well-defined sense of direction for BVPs as there is with IVPs.*

6.3 BVP Stiffness

For IVPs, stiffness corresponded to stepsize restrictions due to stability considerations from fast-decaying modes ($\mathcal{R}e(\lambda) < 0$).

This makes sense for IVPs because of the well-defined sense of integration.

This is not the case for BVPs !
Because there is no directionality, we must also account for rapidly increasing modes.
One can define a stable BVP to be stiff

$$\text{if} \qquad (b-a)|\mathcal{R}e(\lambda)| \gg 1.$$

$\rightarrow$ Whether you integrate forward or backward, you see at least one rapidly decaying component.

- What about numerical methods?

- There is no magical method like BE as there was for IVPs.

- Because of the bi-directionality of BVPs, symmetric methods (e.g., collocation at Gauss points) are preferred (but theory is shaky!)

- Riccati methods attempt to explicitly decouple growing and decaying modes (and integrate them in the right direction !).
  They seem to work well on specific problems, but not in general.
  (It is hard to decouple the modes!)

6.4 Conversion to Standard Form

Very often, BVPs are not posed in standard form. (We are used to converting high-order DEs to first order.)

But there are a couple of well-known tricks for BVPs that commonly arise in practice.

Example 3. *The flow in a channel can be modelled by the BVP*

$$f''' - R[(f')^2 - ff''] + cR = 0,$$
$$f(0) = f'(0) = 0, \quad f(1) = f'(1) = 0.$$

f	-	*Potential function*	*(unknown)*
R	-	*Reynolds number*	*(known)*
c	-	*Constant*	*(unknown)*

Note 2. *There are 3 derivatives on f, but 4 BCs ! Problem is overdetermined if not for c.*

Augment the system with

$c' = 0 \quad \rightarrow$ *now well determined! (4 DEs, 4BCs)*

- The end-point of integration (T) may be unknown; e.g., find the period of a system.
  Change variable: $\tau = \frac{t}{T}$

$$\left.\begin{array}{r}\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}), \\ 0 < t < T,\end{array}\right\} \rightarrow \begin{array}{rcl}\frac{d\mathbf{y}}{d\tau} & = & T\mathbf{f}(T\tau, \mathbf{y}), \\ \frac{dT}{d\tau} & = & 0, \qquad 0 < \tau < 1.\end{array}$$

- Some codes only handle separated BCs

$$\left.\begin{array}{rcl}\mathbf{y}' & = & \mathbf{f}(x, \mathbf{y}), \\ \mathbf{g}(\mathbf{y}(a), \mathbf{y}(b)) & = & \mathbf{0},\end{array}\right\} \rightarrow$$

$$\begin{array}{rcl}\mathbf{y}' & = & \mathbf{f}(x, \mathbf{y}), \\ \boldsymbol{\alpha}' & = & \mathbf{0},\end{array}$$

$$\left.\begin{array}{rcl}\mathbf{y}(a) & = & \boldsymbol{\alpha}(a), \\ \mathbf{g}(\boldsymbol{\alpha}(b), \mathbf{y}(b)) & = & \mathbf{0}.\end{array}\right\} \text{separated}$$

  Of course, we are substituting a constant vector $\mathbf{y}(a)$ by an unknown (but constant) vector function $\boldsymbol{\alpha}(x)$, so this is not economical!