Two Barriers on Strong-Stability-Preserving Time Discretization Methods

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Abstract

Strong-stability-preserving (SSP) time discretization methods are popular and effective algorithms for the simulation of hyperbolic conservation laws having discontinuous or shock-like solutions. They are (nonlinearly) stable with respect to general convex functionals including norms such as the total-variation norm and hence are often referred to as total-variation-diminishing (TVD) methods. For SSP Runge-Kutta (SSPRK) methods with positive coefficients, we present results that fundamentally restrict the achievable CFL coefficient for linear, constant-coefficient problems and the overall order of accuracy for general nonlinear problems. Specifically we show that the maximum CFL coefficient of an s-stage, order-p SSPRK method with positive coefficients is $s - p + 1$ for linear, constant-coefficient problems. We also show that it is not possible to have an s-stage SSPRK method with positive coefficients and order $p > 4$ for general nonlinear problems.

1 Introduction

Popular time-stepping schemes are almost exclusively based on linear stability analysis; i.e., the behaviour of the scheme on the test equation $dy/dt = \lambda y$ for $\Re(\lambda) << 0$. Indeed such analysis is very effective on problems having smooth solutions. Unfortunately however, these schemes often perform poorly on problems having discontinuous or shock-like solutions.

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This poor performance can manifest itself in the form of spurious oscillations, overshoots, or progressive smearing. On the other hand, strong-stability-preserving (SSP) time discretization methods [6, 7, 4, 8] are based on a *nonlinear* stability property that makes them particularly suitable for the simulation of partial differential equations having nonsmooth solutions. This property can be viewed as a generalization of the more well-known total-variation-diminishing (TVD) property to convex functionals or norms other than the total variation [4].

Consider the system of ODEs

$$\dot{U} = L(U), \tag{1}$$

subject to appropriate initial conditions, obtained from the method of lines applied to the hyperbolic conservation law

$$u_t + f(u)_x = 0 \tag{2}$$

subject to appropriate initial and boundary conditions. We assume that (2) has been suitably discretized in its spatial variables. Then when properly combined with a SSP time-stepping scheme and appropriate time-step restriction, the numerical solution obtained does not exhibit nonlinear instabilities. However, nonlinear instabilities can occur in a numerical solution obtained with, e.g., a SSP spatial discretization scheme, but a standard (i.e., linearly stable) time-stepping scheme [3].

The favourable properties of SSPRK methods are derived only from convexity arguments. In particular, if the forward Euler method is strongly stable in a given norm with a certain CFL coefficient, higher-order SSPRK methods with a modified CFL coefficient can be constructed as convex combinations of forward Euler steps with various step sizes [7].

In this paper we prove two fundamental barriers on SSPRK methods with positive coefficients: one on the maximum CFL coefficient attainable for linear, constant-coefficient problems, and one on the maximum order attainable for any method applied to a general nonlinear problem. We note that the first theorem is particularly useful because it implies that any bound obtained for general problems can be no better than this bound on the CFL coefficient for linear, constant-coefficient problems. The second theorem is significant in that it naturally implies that negative coefficients must be used in order to obtain SSPRK methods for a sufficiently high order when applied to a general nonlinear problem. This has a direct impact on researchers designing high-order methods as well as on practitioners because of the way schemes with negative coefficients are implemented (see below).

Some previous results on barriers for CFL coefficients for SSPRK methods with positive coefficients are as follows. In [6], the maximum CFL coefficient of an s-stage, first-order SSPRK scheme was shown to be s; see also [8] for an alternative proof. In [8], the maximum CFL coefficient of an s-stage, second-order SSPRK scheme was shown to be $s-1$; see also [3] for an alternative proof of the $s=2$ case. However linear, constant-coefficient differential equations play an important role in application and results are often given specifically for this special case (see e.g., [4]). In this paper we show that the maximum CFL coefficient of an s-stage, order-p SSPRK scheme applied to a linear, constant-coefficient problem is $s-p+1$. Gottlieb and Gottlieb [2] give a similar result with a different method of proof. This theorem generalizes the result of [4] that the maximum CFL coefficient attainable for an s-stage, order-s SSPRK method applied to a linear, constant-coefficient system is 1.

The following order barriers for SSPRK methods have also been proven. Gottlieb and Shu [3] proved that no four-stage, fourth-order SSPRK method exists having positive coefficients; i.e., involving just evaluations of $L(\cdot)$ (see Section 2). A fourth-order accurate method was also given in [3] at the additional expense of introducing two additional evaluations of a related operator $\tilde{L}(\cdot)$. Unfortunately the introduction of $\tilde{L}(\cdot)$ typically leads to sub-optimal efficiency both in terms of the effective CFL coefficient and memory usage of the method. In [8], the present authors give a five-stage, fourth-order SSPRK method with positive coefficients; results for up to 8 stages have also been obtained [9]. However, this is where the possibility of high-order SSPRK methods with positive coefficients ends: In this paper we show no such method exists with order greater than 4.

The remainder of this paper is organized as follows. In Section 2, we briefly describe SSP schemes and their use. In Section 3, we prove that the maximum CFL coefficient of an s-stage, order-p SSPRK method with positive coefficients applied to a linear, constant-coefficient problem is $s-p+1$. Representations of schemes that attain this maximal bound are given in [2]. In Section 4, we prove that an SSPRK method with positive coefficients and order $p>4$ cannot exist for general nonlinear problems.

2 SSP Schemes

We begin by recalling the definition of strong stability:

Definition 2.1 A sequence $\{U^n\}$ is said to be *strongly stable* in a given norm $||\cdot||$ provided that $||U^{n+1}|| \leq ||U^n||$ for all $n \geq 0$.

Here we assume that U^n represents a vector of solution values on a mesh obtained from a method-of-lines approach to solving a PDE. The choice of norm is arbitrary, with the TV-norm and the infinity norm being two natural possibilities. The choice of a relevant norm depends on the problem to be solved. For example, strong stability is relevant to the solution of

(2) because exact solutions to the scalar problem have a range-diminishing property. Thus the strong-stability property is a useful property to require of a numerical solution to (2), even in the case of systems of conservation laws.

In this paper we take our canonical SSPRK method to be an s-stage, explicit Runge-Kutta method written in the form

$$\begin{aligned} U^{(0)} &= U^n && \text{(3a)} \\ U^{(i)} &= \sum_{k=0}^{i-1}(\alpha_{ik}U^{(k)} + \Delta t \beta_{ik} L(U^{(k)})), \qquad i = 1, 2, \ldots, s, && \text{(3b)} \\ U^{n+1} &= U^{(s)}, && \text{(3c)} \end{aligned}$$

where all the $\alpha_{ik},\ \beta_{ik} \geq 0$ and $\alpha_{ik} = 0$ only if $\beta_{ik} = 0$ [6]. This representation of a Runge-Kutta method corresponds to a unique Butcher array form (see e.g., [5]). However, a given Butcher array form generally corresponds to many algebraically equivalent forms (3) [7, 8].

Throughout this article, we give representations (3) that naturally allow stability restrictions to be read from the coefficients of the scheme.

For consistency, we must have that $\sum_{k=0}^{i-1} \alpha_{ik} = 1$, for all $i = 1, 2, \ldots, s$. Hence, if both sets of coefficients $\alpha_{ik},\ \beta_{ik}$ are positive, then (3) is a convex combination of forward Euler steps with various step sizes $\frac{\beta_{ik}}{\alpha_{ik}}\Delta t$. The Runge-Kutta scheme written in this form is particularly convenient to make use of the following result [7, 4]:

Theorem 2.2 If the forward Euler method is strongly stable under the CFL restriction $\Delta t \leq \Delta t_{FE}$, then the Runge-Kutta method (3) with $\beta_{ik} \geq 0$ is SSP provided

$$\Delta t \leq C \Delta t_{FE},$$

where C is the CFL coefficient

$$C \equiv \min_{i,k} \frac{\alpha_{ik}}{\beta_{ik}}.$$

A similar result holds with β_{ik} replaced by $|\beta_{ik}|$ in the definition of c provided $L(\cdot)$ is replaced with the related operator $\tilde{L}(\cdot)$ for each $\beta_{ik} < 0$, where $\tilde{L}(\cdot)$ is assumed to be strongly stable for Euler's method solved *backwards* in time for a suitable time-step restriction.

3 CFL Barrier for Linear, Constant-Coefficient Problems

Theorem 3.1 Consider the family of s-stage, order-p SSPRK methods (3) with $\alpha_{ik}, \beta_{ik} \geq 0$ applied to (1) with a *linear, constant-coefficient* $L(\cdot)$. The

CFL restriction then satisfies $\Delta t \leq (s-p+1)\Delta t_{FE}$.

Proof. Define

$$
\begin{aligned}
P_0(\alpha,\beta) &= 1,\\
P_r(\alpha,\beta) &= \sum_{k=0}^{r-1} \alpha_{rk} P_k(\alpha,\beta),\\
P_1^1(\alpha,\beta) &= \beta_{10},
\end{aligned}
$$

and for $r>1$ define

$$
P_r^{k_1,k_2,\ldots,k_\ell}(\alpha,\beta) = \begin{cases} \sum_{k=0}^{r-1} \beta_{rk} P_k(\alpha,\beta) & r=k_\ell, \ell=1,\\ \sum_{k=k_{\ell-1}}^{r-1} \beta_{rk} P_k^{k_1,k_2,\ldots,k_{\ell-1}}(\alpha,\beta) & r=k_\ell, \ell>1,\\ \sum_{k=k_\ell}^{r-1} \alpha_{rk} P_k^{k_1,k_2,\ldots,k_\ell}(\alpha,\beta) & \text{otherwise.} \end{cases}
$$

Then the SSPRK method (3) applied to (1) with linear, constant-coefficient $L(U)$ may be written

$$
\begin{aligned}
U^{n+1} &= P_s(\alpha,\beta)U^n + \sum_{k_1=1}^{s} P_s^{k_1}(\alpha,\beta)\Delta t L(U^n)\\
&+ \sum_{k_1=1}^{s} \sum_{k_2>k_1}^{s} P_s^{k_1,k_2}(\alpha,\beta)(\Delta t)^2 L^2(U^n)\\
&+ \cdots + \sum_{k_1=1}^{s} \sum_{k_2>k_1}^{s} \cdots \sum_{k_s>k_{s-1}}^{s} P_s^{k_1,k_2,\ldots,k_s}(\alpha,\beta)(\Delta t)^s L^s(U^n).
\end{aligned}
$$

Denote the coefficient of $(\Delta t)^q L^q(U^n)$ by $A_q(\alpha,\beta)$, $0 \leq q \leq s$. Because the scheme is linear and order p we also know that

$$
A_q(\alpha,\beta) = \frac{1}{q!}
$$

for all $0 \leq q \leq p$. To obtain a contradiction, suppose that the CFL coefficient is greater than $(s-p+1)$. Then

$$
\beta_{ij} \leq \frac{1}{s-p+1}\alpha_{ij} \tag{4}
$$

for all α_{ij} and β_{ij} with equality holding if and only if $\alpha_{ij}=0$. Thus,

$$
P_r^{k_1,k_2,\ldots,k_\ell}(\alpha,\beta) \leq \frac{1}{s-p+1} E_{k_q}(P_r^{k_1,k_2,\ldots,k_\ell}(\alpha,\beta)), \tag{5}
$$

where $E_{k_q}(P_r^{k_1,k_2,\ldots,k_\ell}(\alpha,\beta))$ replaces all instances of $\beta_{k_q\ell}$ by $\alpha_{k_q\ell}$, $0 \leq k_q \leq s-1$, and $0 \leq \ell \leq k_q - 1$ where applicable in the polynomial $P_r^{k_1,k_2,\ldots,k_\ell}(\alpha,\beta)$. Thus,

$$
\begin{aligned}
A_p(\alpha,\beta) &= \sum_{k_1=1}^{s} \sum_{k_2>k_1}^{s} \cdots \sum_{k_p>k_{p-1}}^{s} P_s^{k_1,k_2,\ldots,k_p}(\alpha,\beta) \\
&= \sum_{k_1=1}^{s} \sum_{k_2>k_1}^{s} \cdots \sum_{k_p>k_{p-1}}^{s} \frac{1}{p} \sum_{q=1}^{p} P_s^{k_1,k_2,\ldots,k_p}(\alpha,\beta) \\
&\leq \frac{1}{s-p+1} \sum_{k_1=1}^{s} \sum_{k_2>k_1}^{s} \cdots \sum_{k_p>k_{p-1}}^{s} \frac{1}{p} \sum_{q=1}^{p} E_{k_q}(P_s^{k_1,k_2,\ldots,k_p}(\alpha,\beta))
\end{aligned}
$$

with equality holding if and only if $A_p(\alpha,\beta) = 0$. But $A_p(\alpha,\beta) \neq 0$ for a method of order p so

$$
\begin{aligned}
A_p(\alpha,\beta) &< \frac{1}{s-p+1} \sum_{k_1=1}^{s} \sum_{k_2>k_1}^{s} \cdots \sum_{k_p>k_{p-1}}^{s} \frac{1}{p} \sum_{q=1}^{p} E_{k_q}(P_s^{k_1,k_2,\ldots,k_p}(\alpha,\beta)) \\
&= \frac{1}{s-p+1} \left(\frac{1}{p}\right) \sum_{k_1=1}^{s} \sum_{k_2>k_1}^{s} \cdots \sum_{k_{p-1}>k_{p-2}}^{s} \sum_{q\notin\{k_1,k_2,\ldots,k_{p-1}\}} E_q(P_s^{k_1,k_2,\ldots,k_{p-1},q}(\alpha,\beta)) \\
&\leq \frac{1}{s-p+1} \left(\frac{1}{p}\right) \sum_{k_1=1}^{s} \sum_{k_2>k_1}^{s} \cdots \sum_{k_{p-1}>k_{p-2}}^{s} \sum_{q\notin\{k_1,k_2,\ldots,k_{p-1}\}} P_s^{k_1,k_2,\ldots,k_{p-1}}(\alpha,\beta) \\
&= \frac{1}{s-p+1} \left(\frac{1}{p}\right) \sum_{k_1=1}^{s} \sum_{k_2>k_1}^{s} \cdots \sum_{k_{p-1}>k_{p-2}}^{s} (s-p+1) P_s^{k_1,k_2,\ldots,k_{p-1}}(\alpha,\beta) \\
&= \left(\frac{1}{p}\right) A_{p-1}(\alpha,\beta) \\
&= \frac{1}{p!},
\end{aligned}
$$

which is a contradiction. We therefore conclude that the CFL coefficient must be less than or equal to $(s-p+1)$.

We note that the order conditions that are relevant for linear, constant-coefficient problems are the so-called *sub-quadrature* or *tall-tree* [5] conditions given in Butcher notation by:

$$
b^T c^k = \frac{1}{k+1}, \qquad k = 0, 1, \tag{6a}
$$

$$
b^T A^k c = \frac{1}{(k+2)!} \qquad k = 1, 2, \ldots, p-2, \tag{6b}
$$

where $c^0 \equiv (1, 1, \ldots, 1)^T$.

These high-stage schemes are useful in practice because the additional computational cost per step is more than offset by the gain in stable step size. Moreover, this efficiency gain increases with increasing order.

For example, the optimal four-stage, order-three SSPRK scheme denoted SSPRK$(4,3)$ in [8] costs 33% more than the more well-known optimal three-stage, order-3 scheme [7] denoted SSPRK(3,3) but offers a 100% larger CFL coefficient. This leads to an *effective CFL coefficient* of $C_{\text{eff}} = 3/2$ for SSPRK$(4,3)$ versus $C_{\text{eff}} = 1$ for SSPRK$(3,3)$. We also note that these third-order schemes as well as all optimal schemes of orders 1 and 2 are presented in [8], where they are shown to be optimal for nonlinear problems as well; i.e., the optimality bounds for linear, constant-coefficient problems coincide with those for general nonlinear problems in these special cases.

4 Order Barrier for SSPRK Schemes applied to Nonlinear Problems

In this section we prove a fundamental restriction on SSPRK methods with positive coefficients: their overall order for general nonlinear problems cannot exceed 4. This result now forces research into high-order SSPRK methods to focus on including negative coefficients; we report elsewhere on investigations along these lines.

In order to relate the representation (3) to the standard Butcher coefficients, we introduce the following notation (cf. [3]):

$$U^{(0)} = U^n, \tag{7a}$$

$$U^{(i)} = U^{(0)} + \Delta t \sum_{k=0}^{i-1} \kappa_{ik} L(U^{(k)}), \quad i = 1, 2, \ldots, s, \tag{7b}$$

$$U^{n+1} = U^{(s)}. \tag{7c}$$

The coefficients κ_{ik} are related to the $\alpha_{ik},\ \beta_{ik}$ recursively by

$$\kappa_{ik} = \beta_{ik} + \sum_{j=k+1}^{i-1} \alpha_{ij}\kappa_{jk}. \tag{8}$$

It is also easy to see that the coefficients κ_{ik} are related to the Butcher array quantities $a_{ik},\ b_k$ by

$$\begin{aligned} a_{ik} &= \kappa_{i-1,k-1}, & k &= 1, 2, \ldots, i-1, \quad i = 1, 2, \ldots, s-1, \\ b_k &= \kappa_{s,k-1}, & k &= 1, 2, \ldots, s. \end{aligned}$$

Theorem 4.1 There is no s-stage SSPRK method (3) with $\beta_{ik} \geq 0$, $k = 0, 1, \ldots, i-1$, $i = 1, 2, \ldots, s$ with order $p > 4$.

Proof. The non-existence proof relies on two results, one which is known and one which will be proven here. Using standard Butcher notation, the known result is that *every* Runge-Kutta method of order $p > 4$ must satisfy $b_i \leq 0$ for at least one i. This result is a consequence of a fifth-order condition. Following the notation used in [1], define the vector $\gamma_2 = \frac{c^2}{2} - Ac$, where $(\cdot)^2$ is understood to be componentwise, $s \geq 4$, and c and A are the standard Butcher quantities. Using this notation, one of the fifth-order conditions can be written as

$$b^T \gamma_2^2 = 0. \tag{9}$$

The vanishing of vector γ_2 is the defining condition for a Runge-Kutta method to have *stage order* 2; i.e., all internal stages $u^{(i)}$, $i = 1, 2, \ldots, s-1$, will be (at least) second-order approximations to the solution at times $t = t_n + c_i \Delta t$. Clearly, the $i = 1$ stage of any explicit Runge-Kutta method cannot have stage order greater than 1; thus at least one component of γ_2 must be non-zero. Thus for (9) to hold, we must have at least one $b_i \leq 0$.

The result proven here is that a SSPRK method (3) with $\beta_{ik} \geq 0$ forces the corresponding Butcher array to satisfy $b_i > 0$ for all $i = 1, 2, \ldots, s$. The contradiction with the fifth-order condition (9) is then immediate.

Lemma 4.2 An s-stage SSPRK method of the form (3) with $\alpha_{ik},\ \beta_{ik} \geq 0$ has Butcher array coefficients satisfying $b_i > 0$ for all $i = 1, 2, \ldots, s$.

Proof. In the proof we make repeated use of the following results:

- Because all $\alpha_{ik}, \beta_{ik} \geq 0$, all $\kappa_{ik} \geq 0$; i.e., $\kappa_{ik} \neq 0 \iff \kappa_{ik} > 0$.
- If for a given $l \in \{0, 1, \ldots, s-1\}$, the quantities $\beta_{il} = 0$, $i = l+1, l+2, \ldots, s$, then the method (3) does not truly have s stages.
- We require $\alpha_{ik} = 0 \implies \beta_{ik} = 0$; thus also $\beta_{ik} \neq 0 \implies \alpha_{ik} \neq 0$.

We begin by noting $b_s = \kappa_{s,s-1} = \beta_{s,s-1} \neq 0$; otherwise the method would not truly have s stages.

Now given $l \in \{0, 1, \ldots, s-2\}$, assume $\kappa_{sk} \neq 0$ for $k = s-1, s-2, \ldots, l+1$. Our strategy is to show that the assertion $\kappa_{sl} = 0$ leads to $\beta_{il} = 0$ for $i = l+1, l+2, \ldots, s$, which contradicts the assumption that the method has s stages. That is, $b_k = \kappa_{sk} \neq 0$ for $k = s-1, s-2, \ldots, l+1 \implies b_l = \kappa_{sl} \neq 0$. The proof of the lemma then follows by induction.

Suppose $\kappa_{sl} = 0$. Then from (8) $\beta_{sl} = 0$ and

$$\text{at least one of } \{\alpha_{sk}, \kappa_{kl}\} \text{ must vanish for each } k = l+1, l+2, \ldots, s-1. \tag{10}$$

In particular since $b_s = \kappa_{s,s-1} = \beta_{s,s-1} \neq 0 \implies \alpha_{s,s-1} \neq 0$ we have from (10) that $\kappa_{s-1,l} = 0$. Thus $\beta_{s-1,l} = 0$ and

$$\text{at least one of } \{\alpha_{s-1,k}, \kappa_{kl}\} \text{ must vanish for each } k = l+1, l+2, \ldots, s-2. \tag{11}$$

Of course, if $l = s-2$, the remainder of the proof is not relevant.

We note that from (11) it is easy to see that if $\kappa_{kl} = 0$, $k = l+1, l+2, \ldots, s-2$, we immediately obtain $\beta_{kl} = 0$, $k = l+1, l+2, \ldots, s-2$, which together with $\beta_{sl} = 0$ from (10) leads to the desired contradiction. We now show that $\kappa_{kl} = 0$ generically in (10) regardless of whether or not α_{sk} vanishes. This will complete the proof.

The only possibility for $\kappa_{s-2,l} \neq 0$ in (11) is for $\alpha_{s-1,s-2} = 0$. Then $\beta_{s-1,s-2} = 0$, which by definition means that $\kappa_{s-1,s-2} = 0$. But by hypothesis $\kappa_{s,s-2} = \alpha_{s,s-1}\kappa_{s-1,s-2} + \beta_{s,s-2} \neq 0 \implies \beta_{s,s-2} \neq 0 \implies \alpha_{s,s-2} \neq 0$. Now from (10) $\kappa_{s-2,l} = 0$, and so we must have $\beta_{s-2,l} = 0$ and

$$\text{at least one of } \{\alpha_{s-2,k}, \kappa_{kl}\} \text{ must vanish for each } k = l+1, l+2, \ldots, s-3. \tag{12}$$

Of course, if $l = s-3$, the proof is complete.

Now the only possibility for $\kappa_{s-3,l}$ not to vanish in (12) is for $\alpha_{s-2,s-3}$ to vanish. Then $\beta_{s-2,s-3} = 0$, which by definition means that $\kappa_{s-2,s-3} = 0$. But by hypothesis $\kappa_{s,s-3} = \alpha_{s,s-2}\kappa_{s-2,s-3} + \alpha_{s,s-1}\kappa_{s-1,s-3} + \beta_{s,s-3} \neq 0$. Because $\kappa_{s-2,s-3} = 0$ and $\alpha_{s,s-1} \neq 0$, this implies $\kappa_{s-1,s-3} \neq 0$, $\beta_{s,s-3} \neq 0$, or both. We now show that either condition leads to $\kappa_{s-3,l} = 0$.

If $\kappa_{s-1,s-3} = \alpha_{s-1,s-2}\kappa_{s-2,s-3} + \beta_{s-1,s-3} \neq 0$ then $\beta_{s-1,s-3} \neq 0 \implies \alpha_{s-1,s-3} \neq 0$, which from (11) means that $\kappa_{s-3,l} = 0$.

If $\beta_{s,s-3} \neq 0$ then $\alpha_{s,s-3} \neq 0$ and from (10) we have $\kappa_{s-3,l} = 0$.

Thus in any event we must have $\beta_{s-3,l} = 0$ and

$$\text{at least one of } \{\alpha_{s-3,k}, \kappa_{kl}\} \text{ must vanish for each } k = l+1, l+2, \ldots, s-4. \tag{13}$$

Of course, if $l = s-4$, the proof is complete.

Similarly we can obtain conditions of the form $\beta_{k^*,l} = 0$ and

$$\text{at least one of } \{\alpha_{k^*,k}, \kappa_{kl}\} \text{ must vanish for each } k = l+1, l+2, \ldots, k^*-1, \tag{14}$$

$k^* \in \{s-4, s-5, \ldots, l+2\}$. By considering the only possibility for $\kappa_{k^*-1,l}$ not to vanish in (14) is to have α_{k^*,k^*-1} vanish, then β_{k^*,k^*-1} vanishes, which by definition leads to $\kappa_{k^*,k^*-1} = 0$. Then from the hypotheses $\kappa_{s,k^*-1} \neq 0$, we obtain conditions of the form

$$\kappa_{j,k^*-1} \neq 0, j = k^*+1, k^*+2, \ldots, s-1, \text{ or } \beta_{s,k^*-1} \neq 0, \tag{15}$$

or any subset thereof. We conclude by showing that $\beta_{s,k^*-1} \neq 0 \implies \kappa_{k^*-1,l} = 0$ and that every $\kappa_{j,k^*-1} \neq 0$ eventually leads to $\kappa_{l+1,l} = 0$. Combining the first condition over all cases of k^* in succession with the second condition yields the desired result.

The condition $\beta_{s,k^*-1} \neq 0 \implies \alpha_{s,k^*-1} \neq 0$, which using $\kappa_{sl} = 0 \implies \kappa_{k^*-1,l} = 0$ from (10). Now if $\kappa_{j,k^*-1} \neq 0$, then

$$\kappa_{j^*,k^*-1} \neq 0,\ j^* = k^*+1, k^*+2, \ldots, j-2, \text{ or } \beta_{j,k^*-1} \neq 0, \tag{16}$$

or any subset thereof, using $\alpha_{j,j-1} = 0$ from a previous step and $\kappa_{k^*,k^*-1} = 0$. This set is clearly contained in (15), so we can proceed in this manner until we reach $\kappa_{l+3,l+1} \neq 0$ if l is odd or $\kappa_{l+4,l+1} \neq 0$ if l is even. It has already been shown how the corresponding non-vanishing conditions on $\beta_{l+3,l+1}$ and $\beta_{l+4,l+1}$ can be converted to $\kappa_{l+1,l} \equiv \beta_{l+1,l} = 0$. It is also easy to see from here that $\kappa_{l+3,l+1} = \alpha_{l+3,l+2}\kappa_{l+2,l+1} + \beta_{l+3,l+1} \neq 0 \implies \beta_{l+3,l+1} \neq 0$ since $\kappa_{l+2,l+1} = 0$ from a previous step. Thus we have $\alpha_{l+3,l+1} \neq 0$ and using $\kappa_{l+3,l} = 0$ from a previous step we get $\kappa_{l+1,l} \equiv \beta_{l+1,l} = 0$. Similarly, $\kappa_{l+4,l+1} = \alpha_{l+4,l+3}\kappa_{l+3,l+1} + \alpha_{l+4,l+2}\kappa_{l+2,l+1} + \beta_{l+4,l+1} \neq 0 \implies \beta_{l+4,l+1} \neq 0$ since $\kappa_{l+2,l+1} = 0$ and $\alpha_{l+4,l+3} = 0$ from previous steps. Thus we have $\alpha_{l+4,l+1} \neq 0$ and using $\kappa_{l+4,l} = 0$ from a previous step we get $\kappa_{l+1,l} \equiv \beta_{l+1,l} = 0$. This completes the proof.

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