Good Vibrations: Vibrating Circular Membrane and Bessel Functions

Heather Cameron and Dana Mullins

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Abstract

In nature, vibrations occur in many forms. If it were not for them, we would not be able to hear anything, not even the beat of a drum. The vibrations of a bound circular membrane, like those of a drumhead, can be modelled with the two-dimensional wave equation, which is a partial differential equation. This partial differential equation is the subject of this report. The solution method for solving this equation required a coordinate transformation from Cartesian to polar coordinates and the use of the method of separation of variables to obtain three ordinary differential equations. The three resultant solutions were then combined to form the product solution of the original partial differential equation. The initial conditions governing the problem were then applied and various plots were produced. A drum was used as a model and the results of the model were compared with the corresponding plots of the vibrational modes.

Contents

1 Introduction

Vibrations can be modelled using a partial differential equation called the wave equation. The one-dimensional wave equation is derived as a result of Newton's equation of motion applied to a vibrating string. The wave equation is a hyperbolic equation, one of the three majors classes of partial differential equations. It can be applied to a variety of physical problems with many different boundary conditions. A few examples of physical phenomena that can be modelled with the wave equation are sound waves, electromagnetic waves, and water waves; these are all cases involving one-dimensional waves, called plane waves. However, not only one-dimensional, but two- and three-dimensional waves can be modelled with this equation. In fact, almost any oscillatory disturbance can be modelled using this partial differential equation. The type of waves that will be investigated in this report are two-dimensional waves, called cylindrical waves. A vibrating circular membrane, like a drumhead, displays this cylindrical wave motion.

There are many real-life applications which involve vibrating membranes, that is why understanding these vibrations is important. For example, drum makers need to understand how a drum works in order to design drums with various pitches; they must understand what happens to the surface of a drum when it vibrates. Another very important real-life application is resonance effects resulting from vibrations. Resonance effects must be accounted for during design and construction processes for various structures such as bridges and buildings.

The purpose of this project is to investigate the vibrations of a circular membrane through the use of mathematics to model what is physically happening when a drumhead vibrates. The mathematical model discussed herein can be applied to other geometries, such as rectangular and triangular membranes. However, we will only consider the circular case.

The difficult aspects of this particular problem include: the circular geometry which will necessitate a transformation to polar coordinates, solving the two-dimensional wave equation, and interpreting the Fourier-Bessel series solutions which result. We also created a physical model with a drum to demonstrate the modes of vibration. Pictures of the results obtained using our model are contained herein.

2 Problem Formulation

The mathematical problem takes the form of the two-dimensional wave equation, $u_{tt} = c^2\nabla^2 u$. Due to the circular geometry of this problem, a transformation to polar coordinates, from Cartesian coordinates, is convenient. The method of separation of variables according to polars is then used because we seek product solutions of the form $u(r,\theta,t) = \phi(r,\theta)T(t)$. The initial result of this separation is a two-dimensional eigenvalue problem and an ordinary differential equation. The method of separation of variables is applied again to the two-dimensional eigenvalue problem yielding two more ordinary differential equations, one of which has solutions in the form of Fourier-Bessel series solutions. Finally, after solving for certain coefficients in the solution expression, plots of the different modes of vibration of the circular membrane will be produced.

3 Solving the Problem

3.1 The Wave Equation

The wave equation in one dimension is the partial differential equation

$$u_{tt} = c^2 u_{xx}. \tag{1}$$

This equation can be used to represent the vertical displacement of a vibrating string. Extending this equation to two dimensions, it becomes

$$u_{tt} = c^2 \nabla^2 u, \tag{2}$$

the two-dimensional wave equation. Note that $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and is called the Laplacian operator. We can use (2) to model the vertical displacement of a vibrating bound circular membrane.

Due to the circular geometry involved, it is convenient to transform the spatial part in (2), $\nabla^2 u$, from Cartesian to polar coordinates leaving the temporal part u_{tt} unchanged. The outcome of this transformation is

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}. \tag{3}$$

This coordinate transformation can be found in A-I. From (3), we realize that r represents the radius of the circular membrane and that θ represents the angular distance around the circumference. When (3) is substituted into (2), the result is the linear partial differential equation

$$u_{tt} = c^2\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right). \tag{4}$$

The conversion to polar coordinates simplifies the boundary condition governing this problem. Letting the radius of the circular membrane equal a, and assuming that the boundary is fixed in place at zero, the boundary condition is

$$\text{BC:} \quad u(a, \theta, t) = 0.$$

At time $t = 0$, initial conditions can be defined based on the initial position and velocity of the circular membrane.

$$\text{IC:} \quad \begin{aligned} u(r, \theta, 0) &= \alpha(r, \theta) \\ u_t(r, \theta, 0) &= \beta(r, \theta) \end{aligned}$$

Because the boundary condition above and the partial differential equation (4) are both linear and homogeneous, the method of separation of variables can be applied.

3.2 Separation of Variables

The time variable is first separated out by seeking product solutions of the form

$$u(r, \theta, t) = \phi(r, \theta)T(t). \tag{5}$$

This form of $u(r, \theta, t)$ is then substituted into (2) to give

$$\phi(r, \theta)T''(t) = c^2\nabla^2\phi(r, \theta)T(t) \tag{6}$$

Equation (6) is then divided by $\phi(r, \theta)T(t)$ from equation (5) and by c^2 for convenience when later solving the boundary value problem for ϕ. This separation of the variables yields

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{\nabla^2\phi(r,\theta)}{\phi(r,\theta)}. \tag{7}$$

A separation constant $-\lambda$ is now introduced and is defined as

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{\nabla^2\phi(r,\theta)}{\phi(r,\theta)} = -\lambda, \quad \lambda > 0 \tag{8}$$

This constant is chosen to be negative because for vibrations to occur, oscillatory solutions are expected. If the separation constant was instead chosen to be zero or positive, only trivial, not oscillatory, solutions would result. An ordinary differential equation and a partial differential equation are obtained using the separation of variables method. $T(t)$ satisfies the ordinary differential equation

$$T''(t) + \lambda c^2 T(t) = 0 \tag{9}$$

and $\phi(r,\theta)$ satisfies the linear, homogeneous partial differential equation

$$\nabla^2\phi(r,\theta) + \lambda\phi(r,\theta) = 0, \tag{10}$$

which is a two-dimensional eigenvalue problem.

The boundary condition $u(a,\theta,t) = 0$ can be rewritten by separating the variables in the following manner

$$\text{BC:} \quad u(a,\theta,t) = \phi(a,\theta)T(t) = 0. \tag{11}$$

In (11) we notice that $T(t)$ cannot be equal to zero for all values of t if vibrations are to occur; therefore,

$$\phi(a,\theta) = 0. \tag{12}$$

The next task is to solve (10) by separating the variables with the Laplacian in polar form. First, we separate the variables in $\phi(r, \theta)$, seeking product solutions of the form

$$\phi(r, \theta) = R(r)\Theta(\theta), \tag{13}$$

where $0 < r < a$ and $-\pi < \theta < \pi$ for the circular membrane. Next, we take the Laplacian of (13) in polar form yielding

$$\nabla^2\phi(r, \theta) = \nabla^2(R(r)\Theta(\theta)) = \frac{1}{r}\frac{d}{dr}(rR'(r))\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta). \tag{14}$$

Substituting (14) into (10) we obtain

$$\frac{1}{r}\frac{d}{dr}(rR'(r))\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) + \lambda R(r)\Theta(\theta) = 0 \tag{15}$$

Dividing (15) by $\frac{R(r)\Theta(\theta)}{r^2}$ permits the separation of r and θ yielding

$$\frac{r}{R(r)}\frac{d}{dr}(rR'(r)) + \frac{\Theta''(\theta)}{\Theta(\theta)} + r^2\lambda = 0.$$

Separating the variables we obtain

$$\frac{r}{R(r)}\frac{d}{dr}(rR'(r)) + \lambda r^2 = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \mu. \tag{16}$$

Note that a second positive separation constant μ must be introduced because we expect oscillations in θ; that is, the ordinary differential equation

$$\Theta''(\theta) + \mu\Theta(\theta) = 0 \tag{17}$$

yields oscillatory solutions for $\Theta(\theta)$. Also, from (16) we obtain

$$r\frac{d}{dr}(rR'(r)) + R(r)(\lambda r^2 - \mu) = 0. \tag{18}$$

Therefore, we now have three ordinary differential equations ((9), (17), (18)) with two separation constants:

$$T''(t) + \lambda c^2 T(t) = 0$$

$$\Theta''(\theta) + \mu\Theta(\theta) = 0$$

$$r\frac{d}{dr}(rR'(r)) + R(r)(\lambda r^2 - \mu) = 0.$$

Equation (9) is a second-order, linear ordinary differential equation. We expect solutions of the form $T(t) = e^{rt}$ resulting in the auxiliary equation $r^2 + \lambda c^2 = 0$. Using the quadratic formula to solve for r, we obtain $r = \pm c\sqrt{\lambda}i$. Therefore, the solution for $T(t)$ obtained as a result of the imaginary r values is:

$$T(t) = c_1 \cos c\sqrt{\lambda}t + c_2 \sin c\sqrt{\lambda}t$$

or in a more compact from this can be written as:

$$T(t) = \left\{ \begin{matrix} \cos c\sqrt{\lambda}t \\ \sin c\sqrt{\lambda}t \end{matrix} \right\}. \tag{19}$$

Equations (17) and (18) are eigenvalue problems. The only boundary condition that is specified is $R(a) = 0$ (the zero displacement condition on the circumference of the circular membrane) which results from $\phi(a, \theta) = R(a)\Theta(\theta) = 0$. Recalling that θ and r are defined over the intervals $0 < r < a$ and $-\pi < \theta < \pi$, boundary conditions should exist at both ends of these intervals. Therefore, one more boundary condition is required for $R(r)$ and two boundary conditions are required for $\Theta(\theta)$.

To deduce what type of boundary conditions should exist at the ends, it is important to first notice that at $r = 0$ the polar coordinates are singular. This means that the displacement of the membrane could be any value at this point. However, this is not what we want. The displacement must be finite so we impose a singularity condition at $r = 0$,

$$|R(0)| < \infty. \tag{20}$$

The solution in θ is periodic in nature, so this suggests using periodic boundary conditions:

$$\begin{aligned}\Theta(-\pi) &= \Theta(\pi)\\ \Theta'(-\pi) &= \Theta'(\pi).\end{aligned}$$

3.3 The Eigenvalue Problems

In the previous section, two eigenvalue problems ((17) and (18)) were obtained after using the separation of variables method.

(i) The eigenvalues and eigenfunctions can be easily obtained for the familiar θ-eigenvalue problem

$$\Theta(\theta) + \mu\Theta(\theta) = 0,$$

subject to

$$\Theta(-\pi) = \Theta(\pi),$$

$$\Theta'(-\pi) = \Theta'(\pi).$$

This problem is not a regular Sturm-Liouville problem because it has periodic boundary conditions. However, we can still solve for the eigenvalues and eigenfunctions by analyzing the three cases for the separation constant μ to obtain non-trivial solutions (Pinsky, 1984). The three cases are:

$$\textit{Case } 1 : \mu > 0, \quad \Theta(\theta) = c_1 \cos(\sqrt{\mu}\theta) + c_2 \sin(\sqrt{\mu}\theta)$$

$$\textit{Case } 2 : \mu = 0, \quad \Theta(\theta) = c_1 + c_2\theta$$

$$\textit{Case } 3 : \mu < 0, \quad \Theta(\theta) = c_1 \cosh(\sqrt{\mu}\theta) + c_2 \sinh(\sqrt{\mu}\theta)$$

where c_1 and c_2 are constants.

Case 2 and 3 yield trivial solutions when the periodic boundary conditions are imposed. Therefore, we will only deal with Case 1 for which the boundary conditions require that

$$c_1 \cos(\sqrt{\mu}(-\pi)) + c_2 \sin(\sqrt{\mu}(-\pi)) = c_1 \cos(\sqrt{\mu}(\pi)) + c_2 \sin(\sqrt{\mu}(\pi)) \tag{21}$$

$$-c_1\sqrt{\mu} \sin(\sqrt{\mu}(-\pi)) + c_2\sqrt{\mu} \cos(\sqrt{\mu}(-\pi)) = -c_1\sqrt{\mu} \sin(\sqrt{\mu}(\pi)) + c_2\sqrt{\mu} \cos(\sqrt{\mu}(\pi)) \tag{22}$$

Equations (21) and (22) yield $c_2 \sin(\sqrt{\mu}(\pi)) = 0$ and $c_1\sqrt{\mu} \sin(\sqrt{\mu}(\pi)) = 0$, respectively. These equations are satisfied if $\sqrt{\mu} = 0, 1, 2, ...$ If $\sqrt{\mu}$ is not an integer, $c_1 = c_2 = 0$, the trivial solution. The eigenvalues are $\mu = m^2$, where $m = 0, 1, 2, ...$ and we obtain from this the eigenfunctions

$$\Theta(\theta) = c_1 \cos(m\theta) + c_2 \sin(m\theta) \tag{23}$$

or in a more compact from this can be written as:

$$\Theta(\theta) = \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix}. \tag{24}$$

When $m = 0$, only one eigenfunction is obtained, i.e. a constant. A full Fourier series in θ is created by this eigenvalue problem. In the θ-direction, m denotes the number of crests of the vibrational motion.

(ii) Equation (18) allows us to define an eigenvalue problem for λ for each integral value of m. Substituting m^2 in for μ gives

$$r\frac{d}{dr}(rR'(r)) + (\lambda r^2 - m^2)R(r) = 0 \tag{25}$$

subject to

$$R(a) = 0, \quad |R(0)| < \infty$$

Equation (25) is not easy to evaluate because it has non-constant coefficients; however, dividing it by r, puts it into Sturm-Liouville form:

$$\frac{d}{dr}(rR'(r)) + \left(\lambda r - \frac{m^2}{r}\right) R(r) = 0. \tag{26}$$

If we compare (26) to the general Sturm-Liouville form differential equation,

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\phi + \lambda\sigma(x)\phi = 0,$$

with r as the independent variable, we notice that $x = r$, $p(x) = r$, $q(x) = \frac{-m^2}{r}$, and $\sigma(x) = r$.

This problem is not a regular Surm-Liouville problem. The main reason for this has to do with the behavior at $r = 0$. Consequently, we note the following:

1. $|R(0)| < \infty$ is not the correct form for the boundary condition at $r = 0$.

2. $p(r)$ and $\sigma(r)$ are not positive everywhere because they both equal zero at $r = 0$.

3. As r approaches zero, $q(r)$ approaches ∞, so $q(r)$ is discontinuous for $m \neq 0$. Thus, $R(r)$ satisfies a singular Sturm-Liouville problem. Despite the singular nature of this problem, statements pertaining to regular Sturm-Liouville problems are still applicable for this singular Sturm-Liouville problem. Taking this into account, for each m there exists an infinite number of eigenvalues. Each eigenvalue is specified by λ_{nm}, where $m = 0, 1, 2, ...$ and $n = 1, 2, ...$ and each corresponding eigenfunction is given by $R_{nm}(r)$.

These eigenfunctions are orthogonal with weight r for each fixed value of m. Therefore,

$$\int_0^a R_{mn_1} R_{mn_2} r dr = 0 \quad \text{for } n_1 \neq n_2. \tag{27}$$

3.4 Bessel's Differential Equation of Order m

The singular Sturm-Liouville differential equation can be altered using the product rule of differentiation and by multiplying through by r. This manipulation leads to

$$r^2 R''(r) + rR'(r) + (\lambda r^2 - m^2)R(r) = 0. \tag{28}$$

We can see that the parameters m and λ are present in (28). We have shown previously that m is an integer, but we still do not know the acceptable values for λ. It is a tedious process to substitute different integral values of m into (28) to solve for the numerous values of λ. Therefore, we employ an independent variable transformation, $s = \sqrt{\lambda} r$ for $m \neq 0$, which will eliminate the λ-dependence in the differential equation. So now we have:

$$s^2R''(r) + sR'(r) + (s^2 - m^2)R(s) = 0. \tag{29}$$

This equation is now conveniently recognized as Bessel's differential equation of order m. This equation is very well known and has been investigated for over 150 years (Haberman, 1998). With this in mind, we will assume we know the special solutions to this equation, called Bessel functions.

In the neighbourhood of $s = 0$, two types of solutions result, well-behaved and singular. For each distinct value of m a specific differential equation is obtained. A well-behaved solution is denoted, $J_m(s)$, the Bessel function of the first kind of order m. Whereas, a singular solution is denoted, $Y_m(s)$, the Bessel function of the second kind of order m. It is a well known fact that as $s \to 0$, $Y_m(s)$ approaches $\pm\infty$ (Haberman, 1998). Thus, the general solution to Bessel's differential equation is a linear combination of Bessel functions,

$$R(r) = c_1J_m(\sqrt{\lambda}r) + c_2Y_m(\sqrt{\lambda}r), \quad \text{where } s = \sqrt{\lambda}r. \tag{30}$$

The eigenvalues can be determined using the homogeneous boundary conditions, $R(a) = 0$ and $|R(0)| < \infty$. The second condition states that $R(0)$ must be finite; however, $Y_m(0)$ is infinite, resulting in $c_2 = 0$. The solution now becomes

$$R(r) = c_1J_m(\sqrt{\lambda}r). \tag{31}$$

Using the other condtion $R(a) = 0$, the eigenvalues can be determined from:

$$J_m(\sqrt{\lambda}a) = 0. \tag{32}$$

This means that $\sqrt{\lambda}a$ is a zero of the Bessel function $J_m(s)$, of which there are an infinite number. If we let s_{mn} be the nth zero of $J_m(s)$, then we obtain

$$\lambda_{mn} = \left(\frac{s_{mn}}{a}\right)^2. \tag{33}$$

There is an infinite number of eigenvalues for each value of m leading to the eigenfunctions:

$$J_m\left(\sqrt{\lambda_{mn}}r\right) = J_m\left(s_{mn}\frac{r}{a}\right), \quad \text{for} \quad m = 0, 1, 2, ..., \; n = 1, 2, ... \tag{34}$$

Similarly, there is an infinite set of eigenfunctions for each value of m. The eigenfunctions are orthogonal with weight r for each specific value of m as previously discussed:

$$\int_0^a J_m\left(\sqrt{\lambda_{mp}}r\right) J_m\left(\sqrt{\lambda_{mq}}r\right) r dr = 0, \quad p \neq q.$$

This set of eigenfunctions (m fixed) is complete. Therefore, we can represent any piecewise smooth function of r using a Fourier series (Haberman, 1998). In this case, the eigenfunctions are:

$$\alpha(r) = \sum_{n=1}^{\infty} a_n J_m\left(\sqrt{\lambda_{mn}}r\right), \quad \text{for fixed } m. \tag{35}$$

This is a Fourier-Bessel series and using orthogonality of the Bessel functions we can determine the coefficients. To do this, both sides of (35) are multiplied by $J_m(\sqrt{\lambda_{mn}}r)r$ and integrated from 0 to a with respect to r:

$$\int_0^a \alpha(r) J_m\left(\sqrt{\lambda_{mn}}r\right) r dr = \int_0^a a_n J_m^2\left(\sqrt{\lambda_{mn}}r\right) r dr.$$

Rearranging the previous equation for a_n we obtain:

$$a_n = \frac{\int_0^a \alpha(r) J_m\left(\sqrt{\lambda_{mn}}r\right) r\,dr}{\int_0^a J_m^2\left(\sqrt{\lambda_{mn}}r\right) r\,dr}. \tag{36}$$

3.5 The Final Solution

The three product solutions $R(r)$, $\Theta(\theta)$, and $T(t)$ can now be combined to form the final product solution:

$$u(r,\theta,t) = J_m\left(\sqrt{\lambda_{mn}}r\right) \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} \begin{Bmatrix} \cos c\sqrt{\lambda_{mn}}t \\ \sin c\sqrt{\lambda_{mn}}t \end{Bmatrix}. \tag{37}$$

We now return to our intial condition for velocity $u_t(r,\theta,0) = \beta(r,\theta)$. For practical purposes, we will assume that the membrane is initially at rest; therefore,

$$u_t(r,\theta,0) = \beta(r,\theta) = 0.$$

After differentiating (37) and applying the initial velocity condition, we notice that the $\sin c\sqrt{\lambda_{mn}}t$ terms are no longer needed. We now use the principle of superposition to form an infinite linear combination of the remaining product solutions in order to satisfy the initial value problem:

$$\begin{aligned} u(r,\theta,t) = & \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_m\left(\sqrt{\lambda_{mn}}r\right) \cos m\theta \cos c\sqrt{\lambda_{mn}}t \\ & + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} J_m\left(\sqrt{\lambda_{mn}}r\right) \sin m\theta \cos c\sqrt{\lambda_{mn}}t. \end{aligned} \tag{38}$$

Applying the initial position condition $u(r,\theta,0) = \alpha(r,\theta)$, we obtain:

$$\alpha(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_m\left(\sqrt{\lambda_{mn}}r\right) \cos m\theta \;+\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} J_m\left(\sqrt{\lambda_{mn}}r\right) \sin m\theta. \tag{39}$$

4 Normal Modes and Nodal Curves

In this section, we explore various functions for the initial position $\alpha(r, \theta)$ and make Maple plots of these results. First, we need to state a couple definitions related to the vibrational modes of a membrane. A *node* on a membrane is a point or a line that remains stationary while the rest of the membrane is vibrating (Russell,1999). Specifically, a line that remains stationary is referred to as a *nodal line.* Similarly, a *nodal circle* is a circle which remains stationary as the standing waves are set up on the membrane (Karoly, 1995).

The next two subsections contain various Maple plots of the vibrational modes of a circular membrane for two different initial position functions. These initial position conditions were chosen as examples; however, any initial position condition can be applied to our problem solution. In our examples, we only consider cases with an initial velocity of zero. However, a non-zero velocity initial condition could also have been employed.

4.1 Initial Condition: $\alpha(r,\theta) = J_m\left(\sqrt{\lambda_{mn}}r\right)$

4.2 Initial Condition: $\alpha(r,\theta) = \cos\frac{\pi r}{2a}$

5 The Drumhead Model

We created a model to investigate the vibrational modes of a circular membrane. We used a tom drum, sand (approximately a millimeter in diameter), a function generator, a speaker, and a digital camera. The setup of the experiment involved removing the bottom skin of the drum and placing the speaker underneath the drum as close to the drumhead as possible. Next, the sand was spread evenly over the surface of the drumhead. We then prepared for the painful noise we were about to hear by putting in our earplugs! The function generator was turned on and the frequency was varied until resonant frequencies were found. The resonant frequencies made the drumhead vibrate and thus caused the sand to jump about forming "rings" of sand. This phenomenon is due to standing waves being set up on the drumhead. Standing waves are produced when the waves reflect off the circular boundary. There are points or lines in the vibrating mode patterns where there are no vibrations. This is a result of the cancellation of waves through superposition. The *rim* of the drum is considered to be a *nodal circle* because it is unable to move. Recall that a nodal circle is simply a circle which remains stationary as the standing waves are set up on the drumhead. Three different resonant frequencies were found to produce well-defined vibrating mode patterns in the sand on the drumhead. Figures 1, 2, and 3 show the resonant patterns obtained. Refer to section 4 to compare the Maple plots with the drumhead model results which follow.

Using our model we were able to observe nodal circles (rings in the sand) but we were not able to observe nodal lines that would result from more complicated modes set up on the drumhead. We used the full range of frequencies on the function generator, including those that were beyond our hearing range, but only found a narrow range of frequencies that would resonate the drumhead.

Figure 1: This picture shows the mode pattern obtained at a frequency of 0.375 kHz. Three "rings" of sand can be seen clearly in this picture indicating that this is a normal mode of vibration ($m = 0$) with $n = 3$.

Figure 2: This picture shows the mode pattern obtained at a frequency of 1.75 kHz. Seven "rings" of sand can be seen in this picture indicating that this is a normal mode of vibration ($m = 0$) with $n = 7$.

Figure 3: This picture shows the mode pattern obtained at a frequency of 2 kHz. Ten "rings" of sand can be seen clearly in this picture indicating that this is a normal mode of vibration ($m = 0$) with $n = 10$. The strongest vibrations of the drumhead occurred in the neighbourhood of 2 kHz.

6 Additional Comments

We feel that we fared well with respect to the original goals that we set out to achieve in this project. Our original goals were to solve the two-dimensional wave equation modelling the vibrations of a bound circular membrane, obtain plots of our results, create a physical model, and possibly explore spherical problems and Legendre polynomials. We solved the partial differential equation governing our problem using the method of separation of variables, and were able to produce plots of our results using Maple. We were pleased that we were able to create a drumhead model and compare the results of our model to the plots obtained with Maple. When we began this project, we stated that we would have liked to investigate spherical problems and Legendre polynomials if time permitted. We were unable to meet this particular goal, however we feel that our project is complete with respect to the vibrating circular membrane. While compiling our report, we consulted a large number of resources, mostly textbooks and internet sites. We did not find any inconsistencies in the resources that we used only slightly different methods to solve the same problem. Some books made simplifying assumptions concerning the coefficients and others used different notation when solving the problem. As an example, $\sqrt{\lambda_{mn}}$ was written in a variety of different ways, it was referred to as k_{mn} in one resource and α_{mn} in another. We had to be careful not become confused when consulting a resource that used different notation.

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Appendix

A-I The Laplacian in Polar Coordinates

The Laplacian of a function u is defined as the divergence of the gradient of u, that is,

$$\nabla^2 u = \nabla \cdot (\nabla u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

To express the Laplacian in polar coordinates we must first recall the equations for transforming from Cartesian to polar coordinates for a point in the xy plane:

$$x = r\cos\theta \tag{40}$$

$$y = r\sin\theta \tag{41}$$

Now we let $u(x, y)$ be a function in Cartesian coordinates and $U(r, \theta)$ be the corresponding function in polar coordinates (Pinsky,1984). Therefore,

$$U(r, \theta) = u(r\cos\theta, r\sin\theta).$$

Next, we need to express $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ in terms of $\frac{\partial^2 U}{\partial r^2}$, $\frac{\partial^2 U}{\partial \theta^2}$, $\frac{\partial^2 U}{\partial r \partial \theta}$, $\frac{\partial U}{\partial r}$, and $\frac{\partial U}{\partial \theta}$. Using the chain rule we obtain:

$$\frac{\partial u}{\partial x} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial x} \tag{42}$$

$$\frac{\partial u}{\partial y} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial y} \tag{43}$$

We now determine $\frac{\partial r}{\partial x}$, $\frac{\partial \theta}{\partial x}$, $\frac{\partial r}{\partial y}$, and $\frac{\partial \theta}{\partial y}$ by taking the partial derivatives with respect to x and y of $r^2 = x^2 + y^2$:

$$2r\frac{\partial r}{\partial x} = 2x \quad \text{and} \quad 2r\frac{\partial r}{\partial y} = 2y$$

Rearranging the previous equations for $\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial y}$ and substituing in equations (40) and (41) for x and y, respectively, yields:

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \quad \text{and} \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta.$$

By taking the derivative of both sides of (40) and (41) with respect to y and x, respectively, we obtain:

$$0 = \frac{\partial r}{\partial y}\cos\theta - r\sin\theta\frac{\partial\theta}{\partial y} \tag{44}$$

and

$$0 = \frac{\partial r}{\partial x}\sin\theta + r\cos\theta\frac{\partial\theta}{\partial x} \tag{45}$$

Substituting for $\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial y}$ in (44) and (45) yields:

$$0 = \sin\theta\cos\theta - r\sin\theta\frac{\partial\theta}{\partial y} \tag{46}$$

and

$$0 = \cos\theta\sin\theta + r\cos\theta\frac{\partial\theta}{\partial x}. \tag{47}$$

Rearranging (46) and (47) for $\frac{\partial\theta}{\partial y}$ and $\frac{\partial\theta}{\partial x}$, respectively, we obtain:

$$\frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r} \tag{48}$$

and

$$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}. \qquad (49)$$

Equations (42) and (43) now become:

$$\frac{\partial u}{\partial x} = \cos\theta \frac{\partial U}{\partial r} - \frac{\sin\theta}{r}\frac{\partial U}{\partial \theta}$$

and

$$\frac{\partial u}{\partial y} = \sin\theta \frac{\partial U}{\partial r} + \frac{\cos\theta}{r}\frac{\partial U}{\partial \theta}.$$

We now apply the $\partial/\partial x$ and $\partial/\partial y$ operators in polar coordinates again to transform $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ into polar coordinates:

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\cos\theta \frac{\partial U}{\partial r} - \frac{\sin\theta}{r}\frac{\partial U}{\partial \theta}\right) \\
&= \frac{\partial}{\partial x}\left(\cos\theta \frac{\partial U}{\partial r}\right) - \frac{\partial}{\partial x}\left(\frac{\sin\theta}{r}\frac{\partial U}{\partial \theta}\right) \\
&= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial U}{\partial r}\right) - \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\frac{\sin\theta}{r}\frac{\partial U}{\partial \theta}\right)
\end{aligned}$$

After much manipulation involving multiplying through each term and using the product rule for differentiation, we finally obtain:

$$\frac{\partial^2 u}{\partial x^2} = \cos^2\theta\frac{\partial^2 U}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial U}{\partial \theta} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 U}{\partial r \partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial U}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 U}{\partial \theta^2}. \quad (50)$$

Using a similar method for $\frac{\partial^2 u}{\partial y^2}$ we obtain:

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta\frac{\partial^2 U}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial U}{\partial \theta} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 U}{\partial r \partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial U}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 U}{\partial \theta^2}. \quad (51)$$

(50) and (51) are now added together and the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$

is then applied to obtain:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}&=\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\\&=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}.\end{aligned}\tag{52}$$

Letting $U = u$ for convenience, we obtain the expression for the Laplacian of u in polar coordinates:

$$\nabla^2 u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

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