DYNAMICS AND CONTROL OF TIME-PERIODIC MECHANICAL SYSTEMS VIA FLOQUET-LYAPUNOV THEORY

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Abstract

Many practical problems in engineering can be modelled as linear dynamical systems with periodically varying coefficients. This thesis proposes a new design method for the control of these linear time-periodic systems.

First, Floquet-Lyapunov theory is used to derive the Floquet factors of the state-transition matrix of a given system. We introduce a novel approach to obtain every real representation. It is demonstrated that the periodicity of the periodic factor can be determined a priori using a constant matrix, which we call the *Yakubovich matrix,* based upon the signs of the eigenvalues of the monodromy matrix. We then introduce a novel method for the numerical computation of the Floquet factors, relying upon a boundary-value problem formulation and the Yakubovich matrix.

In the second part, we use the invertibility of the controllability Gramian and a specific form for the feedback gain matrix to build a novel control law for the closed-loop system. The new controller can be full-state or observer-based and allows the control engineer to assign all the invariants of the system, i.e. the full monodromy matrix. Deriving the feedback matrix requires first solving a matrix integral equation for the periodic Floquet factor of the new state-transition matrix of the closed-loop system. This is achieved via a spectral method, which can then be further refined by a boundary-value problem formulation. Computational efficiency of the scheme may be further improved by performing the controller synthesis on the transformed system obtained from the reducibility theorem.

Finally, the effectiveness of the method is illustrated with an application to a quick-return mechanism using a software toolbox developed for `MATLAB`™.

Résumé

Beaucoup de problèmes en génie peuvent être modélisés par un système linéaire dynamique à coefficients périodiques. Cette thèse propose une nouvelle méthode de conception pour le contrôle de ces systèmes linéaires périodiques.

Tout d'abord, la théorie de Floquet-Lyapunov est utilisée pour obtenir les facteurs de Floquet de la matrice de transition d'un système donné. Nous introduisons une nouvelle approche pour obtenir toutes les représentations réelles. Il est démontré que la périodicité du facteur périodique peut être déterminé a priori en utilisant une matrice constante, que nous appelons la *matrice de Yakubovich,* basée sur le signe des valeurs propres de la matrice de monodromie. Nous introduisons ensuite une nouvelle méthode pour le calcul numérique des facteurs de Floquet, en utilisant un problème aux valeurs limites et la matrice de Yakubovich.

Dans la seconde partie, nous nous servons de l'invertibilité du grammien de controllabilité et d'une forme particulière de la matrice de gain de retour pour construire une nouvelle loi de contrôle pour le système en boucle fermée. Le nouveau contrôleur peut être adapté pour utiliser l'état observé et permet à l'automaticien de placer tous les invariants du système, c'est à dire la matrice de monodromie au complet. Le calcul de la matrice de gain nécessite d'abord de résoudre une équation intégrale matricielle pour obtenir le facteur de Floquet périodique du système en boucle fermée. Pour cela, nous utilisons une méthode spectrale dont la solution peut ensuite être améliorée grâce à un problème aux valeurs limites. On gagne en efficacité lorsque la synthèse du contrôleur se fait à partir de la transformée du système.

Finalement, cette méthode est illustrée à travers une application à un mécanisme de retour rapide en utilisant le logiciel pour `MATLAB`$^{\mathrm{TM}}$ que nous avons developé.

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Notation

Abbreviations

App.	appendix
BC	boundary condition
BVP	boundary-value problem
CPF	continuous state periodic feedback
Def.	definition
Eq.	equation
Fig.	figure
IVP	initial-value problem
LTI	linear time-invariant
LTP	linear time-periodic
PMF	primary matrix function
STM	state-transition matrix
a.k.a.	also known as

Mathematical Symbols

$\otimes$	Kronecker product, p.8
$\triangleq$	by definition
$\approx$	spectral approximation, p.9
$\det(\mathbf{M})$	determinant of $\mathbf{M} \in \mathbb{R}^{n \times n}$

$\mathrm{diag}\{d_1, \cdots, d_n\}$	$n \times n$ diagonal matrix whose diagonal entries are $d_1, \cdots, d_n$
$\exp(\cdot)$	matrix exponential
$\mathrm{Im}(\cdot)$	imaginary part of a complex number
$\log(\cdot)$	matrix logarithm, p.7
$\ln(\cdot)$	scalar logarithm
$\mathbf{M}^T$	transpose, $\mathbf{M}^T = [M_{ji}]$ is $n \times m$ if $\mathbf{M} = [M_{ij}]$ is $m \times n$
$\mathbf{M}^{-1}$	inverse of nonsingular square matrix $\mathbf{M}$
$\mathbf{M}^{\dagger}$	generalized inverse of $m \times n$ matrix $\mathbf{M}$
$\mathrm{Re}(\cdot)$	real part of a complex number
$\mathrm{tr}\,\mathbf{M}$	trace of $\mathbf{M}$
$\mathrm{vec}(\mathbf{M})$	vector of stacked columns of $\mathbf{M}$, p.8

Sets

$\mathcal{A}_{T_A}$	set of matrix functions, p.15
$\mathbb{C}$	the complex numbers
$\mathbb{C}^{n\times m}$	the space of all complex $n \times m$ matrices
$\mathbb{K}^*$	$\mathbb{K} - \{0\}$, $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}, \mathbb{Z}\}$
$\mathcal{L}_{T_L}^{\mathbb{K}}$	set of matrix functions, p.15
$\mathcal{L}og\{\mathbf{M}\}$	the set of all logarithms of $\mathbf{M}$, p.7
$\mathcal{RL}og\{\mathbf{M}\}$	the set of all real logarithms of $\mathbf{M}$, p.8
$\mathbb{R}$	the real numbers
$\mathbb{R}^+$	the real positive numbers
$\mathbb{R}^n$	Cartesian n-space
$\mathbb{R}^{n\times m}$	the space of all real $n \times m$ matrices
$\mathbb{Z}$	the set of signed integers

Greek Letters

$\mathbf{\Phi}(\cdot, \cdot)$	state-transition matrix of $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$, p.6

δ_{ij}	Kronecker symbol: $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$
θ_i	argument of μ_i, p.30
ι	imaginary number, $\iota^2 = -1$
λ_i	eigenvalues of $\mathbf{F}$, p.30
μ_i	eigenvalues of $\mathbf{\Phi}(T_A, 0)$, p.30
ξ_i	poles of the auxiliary system, p.101
ρ_i	modulus of μ_i, p.30
$\boldsymbol{\sigma}(\cdot)$	spectral basis, p.9

Bold Roman Letters

$\mathbf{A}(\cdot)$	T_A-periodic system matrix, p.6
$\mathbf{A}_O(\cdot)$	system matrix of the observer, p.63
$\mathbf{B}(\cdot)$	T_A-periodic input matrix, p.6
$\tilde{\mathbf{B}}$	constant input matrix of the auxiliary system, p.96
$\mathbf{C}(\cdot)$	T_A-periodic output matrix, p.6
$\mathbf{F}$	constant Floquet factor, p.17
$\mathbf{F}_K$	constant Floquet factor of the closed loop system, p.45
$\mathbf{G}$	operational matrix of integration, p.10
$\mathbf{I}$	identity matrix when the dimension is implicit in the context
$\mathbf{I}_n$	$n \times n$ identity matrix
$\mathbf{K}(\cdot)$	feedback gain matrix, p.45
$\bar{\mathbf{K}}$	constant feedback matrix gain of the auxiliary system, p.96
$\mathbf{K}_c$	real constant $n \times n$ matrix, p.48
$\mathbf{K}_O(\cdot)$	feedback gain matrix for the observer, p.63
$\mathbf{K}_z(\cdot)$	feedback gain matrix for the transformed system, p.59
$\mathbf{L}(\cdot)$	periodic Floquet factor, p.17
$\mathbf{L}_K(\cdot)$	periodic Floquet factor of the closed loop system, p.45
$\mathbf{M}, \mathbf{N}, \cdots$	matrices, $\mathbf{M} \stackrel{\Delta}{=} [M_{ij}]$
$\mathbf{0}$	zero matrix when the dimension is implicit in the context

$\mathbf{0}_{n\times m}$	$n \times m$ zero matrix
$\mathbf{Q}(\cdot)$	operational matrix of product, p.10
$\mathbf{R}(\cdot)$	rotation matrix, p.28
$\mathbf{W}_C(\cdot,\cdot)$	controllability Gramian, reachability Grammian, p.7
$\mathbf{W}_O(\cdot,\cdot)$	observability Gramian, p.24
$\mathbf{W}_R(\cdot,\cdot)$	reachability Grammian, p.7
$\mathbf{W}_J$	generalized eigenvectors
$\mathbf{Y}$	Yakubovich matrix, p.34
$\mathbf{u}(\cdot)$	input variable, p.6
$\mathbf{x}(\cdot)$	state vector, p.6
$\hat{\mathbf{x}}(\cdot)$	observer state vector, p.63
$\mathbf{y}(\cdot)$	ouput variable, p.6
$\mathbf{z}(\cdot)$	state vector of the transformed system, p.19
$\tilde{\mathbf{z}}(\cdot)$	state vector of the auxiliary system, p.96

Other Roman Letters

C_m	viscous damping of the motor, p.66
E_k	kinetic energy of the system, p.68
E_p	potential energy of the system, p.68
I_c	moment of inertia of the cam, p.68
I_f	moment of inertia of the follower, p.68
I_q	moment of inertia of the belt-pulley mechanism, p.68
I_m	moment of inertia of the motor, p.66
I_s	moment of inertia of the shaft, p.68
K_m	electro-mechanical constant of the motor, p.66
N_A	amplifier ratio, p.67
N_R	reducer ratio, p.67
Q_α, Q_θ	generalized forces, p.69
R_m	resistance of the motor, p.66

T_A	period of the open-loop system, p.6
T_{A_K}	period of the closed-loop system, p.45
T_c	cycle time, p.68
T_s	non-dimensional period, p.73
V_k	armature voltage, p.68
i	armature current, p.68
i_s	current source, p.68
k_s	stiffness of the shaft, p.66

CHAPTER 1

Introduction

1.1. Historical Perspective

The study of periodically varying systems dates back to the nineteenth century and an excellent detailed account is given by Richards (1983). Faraday (1831) is credited with the first recorded demonstration of parametric behaviour using a vibrating membrane. With Mathieu (1868), research started to focus more specifically on periodic systems, and notably produced his eponymous equation. Floquet (1883) and Lyapunov (1992) independently established the fundamental form of solutions of linear time-periodic (LTP) systems. Their work is at the core of what is now known as Floquet-Lyapunov theory. Finally, Hill (1886) and Poincaré (1892) opened the field to celestial mechanics and were the first to discuss the motion of (natural) satellites.

1.2. Motivation

Linear time-invariant (LTI) systems have been extensively studied, with many different strategies developed throughout the years for their control. Yet, modelling real-world processes often leads to a LTP system. In engineering alone, many mechanical systems work under a periodic regime in steady-state conditions and can be reduced to a LTP formulation under "small perturbations." This is the case, for instance, of cam mechanisms (Spiteri, Montagnier, and Angeles 1998), manipulators performing repetitive tasks (Streit, Krousgrill, and Bajaj 1985), helicopters (Calico and Wiesel 1986; Friedmann and Silverthorn 1974; Peters and Hohenemser 1971; Webb, Calico, and Wiesel 1991), and satellites (Calico Jr. and Yeakel 1983; Cole and Calico 1992; Kane and Barba 1966; Mingori 1969).

Results established for LTI systems do not usually hold for time-varying systems. General time-varying systems have to be treated individually. Techniques devised for one type of systems cannot be generalized for use with another. LTP systems are an exception in that they all exhibit similar behaviour, and thus, form a unified class. Moreover, several aspects of Floquet-Lyapunov theory hint at similarities with LTI systems, raising the prospect of being able to take advantage of that abundant source of knowledge.

1.3. Thesis Overview

This thesis is organized as follows: Chapters 2 and 3 review the most important results for the study of LTP systems within the framework of Lyapunov-Floquet theory. The canonical representation and reducibility theorems are recalled, followed by their consequences in terms of system stability and controllability. The remainder of the thesis is divided into three parts. The first part, Chapters 4 and 5, focuses on providing novel results on the existence and computation of real Floquet representations, which are utilised in the second part, Chapter 6, for the synthesis of a novel controller for LTP systems. The third and final part, Chapter 7, illustrates the findings of the thesis by studying the dynamics of a quick-return mechanism.

Chapter 4 is devoted to the Floquet theorems restricted to the real field. The chapter builds on existing results to identify the set of all real Floquet factors, and pays special attention to their periodicity. This leads to the introduction to the concept of *Yakubovich matrices* for the characterization of those real factorizations.

Chapter 5 studies the computational aspect of the Floquet-Lyapunov theory, i.e. the derivation of the Floquet factors for a given system. The two main computational approaches, direct numerical integration and spectral methods, are briefly introduced. Both have in common that they rely upon the initial computation of the STM, which is then decomposed in a pair of Floquet factors. A novel boundary-value problem formulation is introduced to take advantage of the remarkable properties of a pair of factors, one element of the pair being periodic, the second constant. This formulation is enhanced by the results of Chapter 4.

After having reviewed previous relevant attempts at the assignment of invariants for LTP systems, Chapter 6 describes the synthesis of a Gramian-based controller, at the core of which is a boundary-value formulation similar to the one adopted in Chapter 5. This approach leads to a matrix integral equation, whose spectral solution is detailed. Section 6.7 further explores the design-related issues for this controller. It discusses the use of the reducibility theorem as an intermediary step in the synthesis. Finally, full-state feedback is extended to observer-based feedback by integrating this controller to existing results.

Chapter 7 provides a real-life application for this controller in the form of a quick-return mechanism. This system is first modelled and put in state-space form, its stability analysis is performed, and a controller based upon the previous chapters is designed.

1.4. Contributions

This thesis claims the following major contributions to the field of LTP systems:

- The characterization of the real Floquet factorizations of a LTP system (Chapter 4).
- The BVP formulation for the direct derivation of the Floquet factors (Chapter 5).
- The proof of Lemma 6.4.1 on the form of $\mathbf{L}_K(\cdot)$.
- The form of the feedback matrix defined in Eq. (6.14) providing a controller which can assign all the invariants of a LTP system.
- Theorem 6.5.1 and its proof, regarding the integral equation satisfied by $\mathbf{L}_K(\cdot)$.
- The solution methods for Eq. (6.17) (§6.6, §6.7.2).
- The advantages obtained by working on the transformed system (§6.7.3).

- The derivation of the mathematical model of a quick-return mechanism (Chapter 7).
- The proof of the shortcomings of an existing feedback strategy for the control of LTP systems (App. A).
- The development of a software toolbox implementing the procedures established in this thesis (App. B).

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CHAPTER 2

Fundamental Concepts

2.1. Periodicity

DEFINITION 2.1.1. It is necessary to be precise about what we mean by a "periodic function."

- A function is *T-periodic* if $T \in \mathbb{R}^{+*}$ and $\forall t,\ f(t+T) = f(t)$.
- A function is *T-antiperiodic* if $T \in \mathbb{R}^{+*}$ and $\forall t,\ f(t+T) = -f(t)$.
- A function is *basically T-periodic* if it is T-periodic or T-antiperiodic.
- A function is *primarily T-periodic* or periodic with primary period T if $T \in \mathbb{R}^{+*}$ is the smallest number such that $f(\cdot)$ is T-periodic.
- A function is *generally $2T$-periodic* if it is primarily $2T$-periodic and it is not T-antiperiodic.

2.2. System Theory

DEFINITION 2.2.1 (Linear Time-Periodic System). A *linear time-periodic* (LTP) system is a dynamical system represented by the following realization

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{2.1a}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t), \tag{2.1b}$$

where $\mathbf{x}(\cdot) \in \mathbb{R}^n$, $\mathbf{u}(\cdot) \in \mathbb{R}^r$, and $\mathbf{y}(\cdot) \in \mathbb{R}^m$; $\mathbf{A}(\cdot)$, $\mathbf{B}(\cdot)$, and $\mathbf{C}(\cdot)$ are respectively, $n \times n$, $n \times r$, and $m \times n$ matrices whose elements are known, piecewise continuous, real-valued, primarily T_A-periodic functions defined on $\mathbb{R}^+$.

Definition 2.2.2 (Fundamental Matrix). Any nonsingular solution of the homogeneous differential system

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t) \tag{2.2}$$

is called a *fundamental matrix* of (2.2).

Theorem 2.2.1. *(D'Angelo 1970, Theorem 2.6) The product of a constant nonsingular $n \times n$ matrix $\mathbf{K}$ with a fundamental matrix $\mathbf{X}(t)$ is also a fundamental matrix; any fundamental matrix can be formed from a given fundamental matrix $\mathbf{X}(t)$ and some nonsingular $\mathbf{K}$.*

Definition 2.2.3 (State-Transition Matrix). There exists a unique fundamental matrix $\mathbf{\Phi}(t, t_0)$ of system (2.2) such that $\mathbf{\Phi}(t_0, t_0) = \mathbf{I}$; this matrix is called the *state-transition matrix* (STM) of the system.

Without loss of generality we shall assume henceforth $t_0 = 0$.

Definition 2.2.4 (Controllability).

- system (2.1) is *controllable* (respectively *reachable*) *on* $[t_1, t_2]$ if each initial condition $\mathbf{x}(t_1) = \mathbf{x}_1$ can be driven to $\mathbf{x}(t_2) = 0$ (respectively $\mathbf{x}(t_2) = \mathbf{x}_2$)) at t_2.
- system (2.1) is *controllable* (respectively *reachable*) *at* t_1 if there exists $t_2 > t_1$ such that it is controllable (respectively reachable) on $[t_1, t_2]$.
- system (2.1) is *controllable* (respectively *reachable*) if it is controllable (respectively reachable) at each time.

DEFINITION 2.2.5. We can now define the *controllability Gramian* $\mathbf{W}_C(t_2, t_1)$ and the *reachability Gramian* $\mathbf{W}_R(t_2, t_1)$ on $[t_1, t_2]$ as

$$\mathbf{W}_C(t_2, t_1) \triangleq \int_{t_1}^{t_2} \mathbf{\Phi}(t_1, \tau)\mathbf{B}(\tau)\mathbf{B}^T(\tau)\mathbf{\Phi}^T(t_1, \tau)d\tau, \text{ and} \tag{2.3a}$$

$$\mathbf{W}_R(t_2, t_1) \triangleq \int_{t_1}^{t_2} \mathbf{\Phi}(t_2, \tau)\mathbf{B}(\tau)\mathbf{B}^T(\tau)\mathbf{\Phi}^T(t_2, \tau)d\tau. \tag{2.3b}$$

The following results can expressed equivalently for $\mathbf{W}(t_2, t_1) = \mathbf{W}_R(t_2, t_1)$ or $\mathbf{W}(t_2, t_1) = \mathbf{W}_C(t_2, t_1)$.

THEOREM 2.2.2. *(D'Angelo 1970, Theorem 4.1) system (2.1) is controllable on* $[t_1, t_2]$ *if and only if* $\mathbf{W}(t_2, t_1)$ *is nonsingular.*

THEOREM 2.2.3. *(Kalman, Falb, and Arbib 1969, §2.8, p.33),(Weiss 1968, Theorem 5) The rank of* $\mathbf{W}(t_2, t_1)$ *is nondecreasing as the range of the integration is increased.*

In the special case where the matrices $\mathbf{A}(\cdot)$ and $\mathbf{B}(\cdot)$ of the system are *constant,* controllability can also be evaluated from the following result.

DEFINITION 2.2.6 (Controllability Index). Let $\mathbf{M}$ and $\mathbf{N}$ be two $n \times n$ and $n \times r$ constant matrices and define

$$\mathbf{U}_k = \begin{bmatrix} \mathbf{N} \mid & \mathbf{MN} \mid & \cdots \mid & \mathbf{M}^k\mathbf{N} \end{bmatrix} \tag{2.4}$$

The controllability index of the pair $\{\mathbf{M}, \mathbf{N}\}$ is the smallest integer k_0 such that $\mathbf{U}_{k_0-1}$ has rank n.

THEOREM 2.2.4. *The pair* $\{\mathbf{M}, \mathbf{N}\}$ *is controllable if and only if* $k_0 \leq n$.

2.3. Matrix Computations

2.3.1. Matrix Logarithm

2.3.1.1. *Existence*

THEOREM 2.3.1. *(Henrici 1974, Theorem 2.6h) The matrix equation* $\exp(\mathbf{X}) = \mathbf{M}$, $\mathbf{M} \in \mathbb{C}^{n\times n}$ *has infinitely many solutions* $\mathbf{X} \in \mathbb{C}^{n\times n}$ *if and only if* $\mathbf{M}$ *is nonsingular.*

We call any such solution $\mathbf{X}$ a *logarithm* of $\mathbf{M}$ and write $\mathbf{X} = \ln(\mathbf{M})$. We will denote the set of all solutions by $\mathcal{L}og\{\mathbf{M}\} = \{\mathbf{X} : \exp(\mathbf{X}) = \mathbf{M}\}$, and the subset of all real solutions by $\mathcal{RL}og\{\mathbf{M}\}$.

2.3.1.2. *Real Matrix Logarithm*

THEOREM 2.3.2. *(Culver 1966, Theorem 1) Let* $\mathbf{M}$ *be a real square matrix. There exists a real solution* $\mathbf{X}$ *to the equation* $\exp(\mathbf{X}) = \mathbf{M}$ *if and only if* $\mathbf{M}$ *is nonsingular and each Jordan block of* $\mathbf{M}$ *belonging to a negative eigenvalue occurs an even number of times.*

As a consequence, we have:

COROLLARY 2.3.3. *(Lukes 1982, Lemma 8.1.1) For each nonsingular* $\mathbf{M} \in \mathbb{R}^{n\times n}$ *there exists a real solution* $\mathbf{R} \in \mathbb{R}^{n\times n}$ *to the equation* $\mathbf{M}^2 = \exp(\mathbf{R})$.

PROOF: Since the eigenvalues of $\mathbf{M}^2$ are the squares of the corresponding eigenvalues of $\mathbf{M}$, any negative eigenvalue of $\mathbf{M}^2$ must come from a conjugate pair of purely imaginary eigenvalues of $\mathbf{M}$. Moreover, these purely imaginary eigenvalues must come in complex conjugate pairs of Jordan blocks since $\mathbf{M}$ is real. Thus, there are an even number of Jordan blocks of $\mathbf{M}^2$ associated with each negative eigenvalue and Theorem 2.3.2 leads to the result. □

2.3.2. Kronecker Product

DEFINITION 2.3.1. Let $\mathbf{M}$ and $\mathbf{N}$ be two matrices of order $m_1 \times n_1$ and $m_2 \times n_2$ respectively. The *Kronecker product* (a.k.a. *tensor product*) $\mathbf{M}\otimes\mathbf{N}$ is the $m_1 m_2 \times n_1 n_2$ matrix defined as

$$\mathbf{M} \otimes \mathbf{N} \triangleq \begin{bmatrix} M_{1,1}\mathbf{N} & M_{1,2}\mathbf{N} & \cdots & M_{1,n1}\mathbf{N} \\ \vdots & \vdots & & \vdots \\ M_{m_1,1}\mathbf{N} & M_{m_1,2}\mathbf{N} & \cdots & M_{m_1,n1}\mathbf{N} \end{bmatrix}. \tag{2.5}$$

DEFINITION 2.3.2. With each matrix $\mathbf{M} = [M_{i,j}] \in \mathbb{R}^{m\times n}$, we associate the vector $\mathrm{vec}(\mathbf{M}) \in \mathbb{R}^{mn}$ defined by

$$\mathrm{vec}(\mathbf{M}) \triangleq \begin{bmatrix} M_{1,1} & \cdots & M_{m_1,1} & M_{1,2} & \cdots & M_{m_1,2} & \cdots & M_{1,n} & \cdots & M_{m_1,n_1} \end{bmatrix}^T. \quad (2.6)$$

LEMMA 2.3.1. *(Horn and Johnson 1994, Lemma 4.3.1) Let* $\mathbf{M} \in \mathbb{R}^{m\times n}$, $\mathbf{N} \in \mathbb{R}^{p\times q}$, *and* $\mathbf{P} \in \mathbb{R}^{m\times q}$ *be given and let* $\mathbf{X} \in \mathbb{R}^{n\times p}$ *be unknown. The matrix equation*

$$\mathbf{MXN} = \mathbf{P}$$

is equivalent to the system of qm *equations in* np *unknowns given by*

$$\left[\mathbf{N}^T \otimes \mathbf{M}\right] \mathrm{vec}(\mathbf{X}) = \mathrm{vec}(\mathbf{P})$$

that is, $\mathrm{vec}(\mathbf{MXN}) = \left[\mathbf{N}^T \otimes \mathbf{M}\right] \mathrm{vec}(\mathbf{X})$.

2.4. Operational Matrices in Spectral Methods

Operational matrices are a useful tool in spectral methods (cf., e.g. Trefethen 2001) and are reviewed in this section.

2.4.1. Expansion of a function The s-order approximation of a scalar function $f(\cdot)$ can be written

$$f(t) \approx \boldsymbol{\sigma}^T(t)\mathbf{f}^s = \mathbf{f}^{sT}\boldsymbol{\sigma}(t), \quad (2.7)$$

where $\mathbf{f}^s \in \mathbb{R}^s$ is the vector of the *expansion coefficients* of $f(\cdot)$, and

$$\boldsymbol{\sigma}(t) = \begin{bmatrix} \sigma_0(t) & \sigma_1(t) & \cdots & \sigma_{s-1}(t) \end{bmatrix}^T \quad (2.8)$$

is the chosen spectral basis.

The operations of scalar multiplication and scalar addition are preserved by the expansion operation:

$$g(\cdot) = \alpha f(\cdot),\ \alpha \in \mathbb{R} \qquad \Rightarrow \qquad \mathbf{g}^s = \alpha \mathbf{f}^s, \quad (2.9a)$$

$$h(\cdot) = f(\cdot) + g(\cdot) \qquad \Rightarrow \qquad \mathbf{h}^s = \mathbf{f}^s + \mathbf{g}^s. \quad (2.9b)$$

2.4.2. Integration The integration operation can be approximated by a constant matrix $\mathbf{G}$ known as the *operational matrix of integration.* This matrix $\mathbf{G}$ depends upon the spectral basis only and is defined as the real constant matrix which satisfies

$$\int_0^t f(\tau)d\tau \approx \boldsymbol{\sigma}^T(t)\mathbf{G}^T\mathbf{f}^s. \tag{2.10}$$

Note that $\mathbf{G}$ has to be computed for each choice of basis.

2.4.3. Product The product of two scalar functions $f(\cdot)$ and $g(\cdot)$ can be approximated using an *operational matrix of product* $\mathbf{Q}(\cdot)$. This matrix $\mathbf{Q}(\cdot)$ depends upon the spectral basis and its entries are functions of the approximation of either function. In fact, $\mathbf{Q}(\cdot)$ can be defined as the $s \times s$ real matrix which satisfies

$$f(t)g(t) \approx \mathbf{f}^{sT}\boldsymbol{\sigma}(t)\boldsymbol{\sigma}^T(t)\mathbf{g}^s = \boldsymbol{\sigma}^T(t)\mathbf{Q}(\mathbf{f}^s)\mathbf{g}^s. \tag{2.11}$$

The entries of $\mathbf{Q}(\cdot)$ have to be computed symbolically for each choice of basis. Since $f(\cdot)g(\cdot) = g(\cdot)f(\cdot)$, note that another valid approximation is

$$f(t)g(t) \approx \boldsymbol{\sigma}^T(t)\mathbf{Q}(\mathbf{g}^s)\mathbf{f}^s. \tag{2.12}$$

The operator $\mathbf{Q}(\cdot)$ is linear:

$$\forall\alpha \in \mathbb{R}, \quad \mathbf{Q}(\alpha\mathbf{f}^s) = \alpha\mathbf{Q}(\mathbf{f}^s), \tag{2.13a}$$

$$\mathbf{Q}(\mathbf{f}^s + \mathbf{g}^s) = \mathbf{Q}(\mathbf{f}^s) + \mathbf{Q}(\mathbf{g}^s). \tag{2.13b}$$

2.4.4. Illustration

2.4.4.1. *Fourier Basis* Denote the complex Fourier basis by

$$\boldsymbol{\sigma}(t) = \begin{bmatrix}\sigma_{-s}(t) & \sigma_{-s+1}(t) & \cdots & \sigma_{-1}(t) & \sigma_0 & \sigma_1(t) & \cdots & \sigma_s(t)\end{bmatrix}^T, \tag{2.14a}$$

$$\sigma_k(t) = \mathrm{e}^{\imath k2\pi t/T}. \tag{2.14b}$$

When integrating one of the basis functions $\sigma_k(t)$, the following relation holds:

$$\int_0^t \sigma_k(\tau)d\tau = \begin{cases} \frac{1}{2k\pi\iota}[\sigma_k(t) - \sigma_0] & \text{if } k \neq 0, \\ \frac{1}{2}\sigma_0 + \sum_{j=-\infty, j\neq 0}^{\infty} \frac{-1}{2j\pi\iota}\sigma_j(t) & \text{if } k = 0. \end{cases} \tag{2.14c}$$

This can also be written in vector form:

$$\int_0^t \boldsymbol{\sigma}(\tau)d\tau = \mathbf{G}\boldsymbol{\sigma}(\tau)d\tau,$$

where $\mathbf{G}$ is the $(2s+1) \times (2s+1)$ operational matrix of integration, viz.

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{g}_1 & \mathbf{0}_{s\times s} \\ \mathbf{g}_1^T & \frac{1}{2} & \mathbf{g}_2^T \\ \mathbf{0}_{s\times s} & \mathbf{g}_2 & \mathbf{G}_{22} \end{bmatrix}, \tag{2.14d}$$

$$\mathbf{g}_1 = \begin{bmatrix} \frac{1}{2s\pi i} & \frac{1}{2(s-1)\pi i} & \cdots & \frac{1}{2\pi i} \end{bmatrix}^T, \qquad \mathbf{G}_{11} = \operatorname{diag}(\mathbf{g}_1), \tag{2.14e}$$

$$\mathbf{g}_2 = \begin{bmatrix} \frac{-1}{2\pi i} & \cdots & \frac{1}{2(s-1)\pi i} & \frac{-1}{2s\pi i} \end{bmatrix}^T, \qquad \mathbf{G}_{22} = \operatorname{diag}(\mathbf{g}_2). \tag{2.14f}$$

The product of two Fourier basis functions is

$$\sigma_j(t)\sigma_k(t) = \sigma_{j+k}(t). \tag{2.14g}$$

In vector form, taking into account the truncation of elements of order higher than s obtained from Eq. (2.14g) when $j+k > s$, we obtain the (symmetric) Hankel matrix

$$\boldsymbol{\sigma}(t)\boldsymbol{\sigma}^T(t) = \begin{bmatrix} 0 & \sigma_{-s}(t) & \cdots & \cdots & \sigma_{-1}(t) & \sigma_0(t) \\ \sigma_{-s}(t) & \sigma_{-s+1}(t) & \cdots & \cdots & \sigma_0(t) & \sigma_1(t) \\ \sigma_{-s+1}(t) & & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \vdots & \vdots \\ \sigma_{-1}(t) & & & & \sigma_{s-1}(t) & \sigma_s(t) \\ \sigma_0(t) & \sigma_1(t) & \cdots & \cdots & \sigma_s(t) & 0 \end{bmatrix}. \tag{2.14h}$$

Consider now the product of the s-order expansions of two functions $f(\cdot)$ and $g(\cdot)$, viz.

$$f(t) \approx \boldsymbol{\sigma}^T(t)\mathbf{f}^s, \tag{2.14i}$$

$$g(t) \approx \boldsymbol{\sigma}^T(t)\mathbf{g}^s, \tag{2.14j}$$

$$f(t)g(t) \approx \mathbf{f}^{sT}\boldsymbol{\sigma}(t)\boldsymbol{\sigma}^T(t)\mathbf{g}^s, \tag{2.14k}$$

$$= \boldsymbol{\sigma}^T(t)\mathbf{Q}(\mathbf{f}^s)\mathbf{g}^s. \tag{2.14l}$$

$\mathbf{Q}(\mathbf{f}^s)$ is a $(2s+1)\times(2s+1)$ Toeplitz matrix and is a function of the vector $\mathbf{f}^s$:

$$\mathbf{Q}(\mathbf{f}^s) = \begin{bmatrix} f_0 & f_{-1} & \cdots & \cdots & f_{-s} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & & \ddots & \ddots & \vdots \\ f_{s-1} & & f_0 & f_{-1} & & & f_{-s} & 0 \\ f_s & \ddots & & f_0 & f_{-1} & & & f_{-s} \\ 0 & f_s & \ddots & f_1 & f_0 & & & f_{-s+1} \\ \vdots & \ddots & \ddots & & & \ddots & & \vdots \\ & & & & & & & f_{-1} \\ 0 & \cdots & \cdots & 0 & f_s & f_{s-1} & \cdots & f_0 \end{bmatrix}. \tag{2.14m}$$

2.4.4.2. *Chebyshev Basis* Denote the s-order basis of shifted Chebyshev polynomials of the first kind by

$$\boldsymbol{\sigma}(t) = \begin{bmatrix}\sigma_0(t) & \sigma_1(t) & \cdots & \sigma_{s-1}(t)\end{bmatrix}^T, \tag{2.15a}$$

$$\sigma_0(t) = 1, \quad \sigma_1(t) = 2t-1, \quad \sigma_{k+1}(t) = 2(2t-1)\sigma_k(t) - \sigma_{k-1}(t), \quad \forall k \geq 1. \tag{2.15b}$$

The general recursive formula for the integration of the shifted Chebyshev polynomials of the first kind can be written as

$$\int_0^t \sigma_k(\tau)d\tau = \frac{1}{4}\left[\frac{\sigma_{k+1}(t)}{k+1} - \frac{\sigma_{k-1}(t)}{k-1}\right] - \frac{(-1)^k}{2(k^2-1)}, \quad k = 0, 2, 3, 4, \ldots \tag{2.16a}$$

$$\int_0^t \sigma_1(\tau)d\tau = \frac{\sigma_2(t) - \sigma_0(t)}{8}. \tag{2.16b}$$

This can be written in vector form as

$$\int_0^t \boldsymbol{\sigma}(\tau)d\tau = \mathbf{G}\boldsymbol{\sigma}(t), \tag{2.16c}$$

and $\mathbf{G}$ is the $s \times s$ matrix with elements given by

$$g_{1j} = \begin{cases} 1/2 & , \quad j = 1, 2 \\ 0 & , \quad j \geq 3 \end{cases} \tag{2.16d}$$

$$g_{2j} = \begin{cases} -1/8 & , \quad j = 1 \\ 0 & , \quad j = 2 \\ 1/8 & , \quad j = 3 \\ 0 & , \quad j \geq 4 \end{cases} \tag{2.16e}$$

and, for $i \geq 3$,

$$g_{ij} = \begin{cases} -\dfrac{(-1)^{(i-1)}}{2((i-1)^2-1)} & , \quad j = 1 \\ -\dfrac{1}{4((i-1)-1)} & , \quad j = i-1 \\ \dfrac{1}{4((i-1)+1)} & , \quad j = i+1 \\ 0 & , \quad \text{elsewhere.} \end{cases} \tag{2.16f}$$

The scalar product of two shifted Chebyshev polynomials of the first kind is given as

$$\sigma_i(t)\sigma_j(t) = \frac{1}{2}\left\{\sigma_{i+j}(t) + \sigma_{|i-j|}(t)\right\}, \quad 0 \leq i, j \leq s-1. \tag{2.16g}$$

In vector form, taking into account the truncations of elements of order higher than $(s-1)$, we obtain the symmetric matrix

$$\boldsymbol{\sigma}(t)\boldsymbol{\sigma}^T(t) = \begin{bmatrix} \sigma_0 & \sigma_1 & \sigma_2 & \cdots & \sigma_{s-1} \\ \sigma_1 & \frac{1}{2}(\sigma_0+\sigma_2) & \frac{1}{2}(\sigma_1+\sigma_3) & \cdots & \frac{1}{2}\sigma_{s-2} \\ \sigma_2 & \frac{1}{2}(\sigma_1+\sigma_3) & \frac{1}{2}(\sigma_0+\sigma_4) & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \sigma_{s-1} & \frac{1}{2}\sigma_{s-2} & \frac{1}{2}\sigma_{s-3} & \cdots & \frac{1}{2}\sigma_0 \end{bmatrix}. \tag{2.16h}$$

Finally, if we consider two functions $f(t)$ and $g(t)$,

$$f(t) \approx \boldsymbol{\sigma}^T(t)\mathbf{f}^s, \quad g(t) \approx \boldsymbol{\sigma}^T(t)\mathbf{g}^s, \tag{2.16i}$$

then,

$$f(t)g(t) \approx \begin{bmatrix} f_0 & f_1 & f_2 & \cdots & f_{s-1} \end{bmatrix} \boldsymbol{\sigma}\boldsymbol{\sigma}^T \begin{bmatrix} g_0 & g_1 & g_2 & \cdots & g_{s-1} \end{bmatrix}^T, \tag{2.16j}$$

which can be rewritten as

$$f(t)g(t) \approx \boldsymbol{\sigma}^T \mathbf{Q}(\mathbf{f}^s)\mathbf{g}^s, \tag{2.16k}$$

where $\mathbf{Q}(\mathbf{f}^s)$ has the following form:

$$\mathbf{Q}(\mathbf{f}^s) = \begin{bmatrix} f_0 & \mathbf{v}^T/2 \\ \mathbf{v} & \mathbf{U} \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} f_1 & f_2 & \cdots & f_{s-1} \end{bmatrix}^T, \tag{2.16l}$$

and the $(s-1) \times (s-1)$ symmetric block $\mathbf{U}$ is

$$\mathbf{U} = \begin{bmatrix} f_0 + f_2/2 & (f_1 + f_3)/2 & \cdots & f_{s-2}/2 \\ (f_1 + f_3)/2 & f_0 + f_4/2 & \cdots & f_{s-3}/2 \\ \vdots & \vdots & \ddots & \vdots \\ f_{s-2}/2 & f_{s-3}/2 & \cdots & f_0 \end{bmatrix}. \tag{2.16m}$$

CHAPTER 3

Review of LTP Systems

3.1. Preliminaries

Although the following theory (e.g., Lukes 1982; Yakubovich and Starzhinskii 1975) applies to any fundamental matrix of solutions, for simplicity we will limit our discussion in terms of the STM.

Definition 3.1.1. We define the following sets of matrices:

- $\mathcal{A}_{T_A}$ is the set of $n \times n$ matrices $\mathbf{A}(\cdot)$ real, piecewise continuous, primarily T_A-periodic;
- $\mathcal{L}_{T_L}^{\mathbb{K}}$ is the set of $n \times n$ matrices $\mathbf{L}(\cdot)$ of $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, where $\mathbf{L}(t)$ are T_L-periodic, nonsingular for all t, continuous with a piecewise continuous derivative, and such that $\mathbf{L}(0) = \mathbf{I}$.

Theorem 3.1.1. *If $\mathbf{A}(\cdot) \in \mathcal{A}_{T_A}$, then the STM $\mathbf{\Phi}(\cdot, 0)$ of the system*

$$\forall t \in \mathbb{R}^+, \quad \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \tag{3.1}$$

will satisfy

$$\forall t \in \mathbb{R}^+, \quad \mathbf{\Phi}(t + T_A, 0) = \mathbf{\Phi}(t, 0)\mathbf{\Phi}(T_A, 0). \tag{3.2}$$

Conversely, let $\mathbf{X}(\cdot)$, such that $\mathbf{X}(0) = \mathbf{I}$, be a nonsingular continuous $n \times n$ matrix function of time, having a piecewise continuous derivative. If T_A is the smallest *number for which* $\mathbf{X}(\cdot)$ *satisfies Eq. (3.2), then* $\mathbf{X}(\cdot)$ *is the STM of some system of the form (3.1) where* $\mathbf{A}(\cdot)$ *is a periodic function of time whose primary period is* T_A.

PROOF: See Yakubovich and Starzhinskii (1975), Chapter II § 1.7 and § 2.1. □

NOTE 3.1.1. The STM $\mathbf{\Phi}(\cdot, 0)$ of a LTP system is usually not periodic, but it is apparent that

$$\forall (t,u) \in \mathbb{R}^{+2}, \quad \mathbf{\Phi}(t+T_A, u+T_A) = \mathbf{\Phi}(t,u). \tag{3.3}$$

PROOF: Consider

$$\mathbf{\Psi}(t,u) \stackrel{\Delta}{=} \mathbf{\Phi}(t+T_A, u+T_A).$$

$\mathbf{\Psi}(t,u)$ satisfies Eq. (3.1) and $\mathbf{\Psi}(0,0) = \mathbf{I}$. By the uniqueness of the STM, $\mathbf{\Psi}(t,u)$ and $\mathbf{\Phi}(t,u)$ are equal. □

NOTE 3.1.2. By transitivity, Eq. (3.2) of Theorem 3.1.1 can be generalized to

$$\forall t \in \mathbb{R}^{+}, \quad \mathbf{\Phi}(t+kT_A, 0) = \mathbf{\Phi}(t,0)\mathbf{\Phi}^k(T_A, 0). \tag{3.4}$$

PROOF: Eq. (3.4) comes from the semigroup property of the STM, viz.

$$\forall (t_1,t_2,t_3) \in \mathbb{R}^{+3}, \quad \mathbf{\Phi}(t_1,t_2) = \mathbf{\Phi}(t_1,t_3)\mathbf{\Phi}(t_3,t_2), \tag{3.5a}$$

which leads to

$$\forall t \in \mathbb{R}^{+}, \quad \mathbf{\Phi}(t+kT_A, 0) = \mathbf{\Phi}(t+kT_A, kT_A)\mathbf{\Phi}(kT_A, 0), \tag{3.5b}$$

and recall that

$$\mathbf{\Phi}(kT_A,0) = \mathbf{\Phi}(kT_A,(k-1)T_A)\mathbf{\Phi}((k-1)T_A,(k-2)T_A)\cdots\mathbf{\Phi}(2T_A,T_A)\mathbf{\Phi}(T_A,0) \quad = \mathbf{\Phi}^k(T_A,0) \tag{3.5c}$$

from Note 3.1.1. □

3.2. Floquet Theorems

THEOREM 3.2.1 (Representation Theorem). *The STM $\mathbf{\Phi}(\cdot, 0)$ of Eq. (3.1) can be factored as*

$$\mathbf{\Phi}(t, 0) \triangleq \mathbf{L}(t) e^{t\mathbf{F}}, \tag{3.6}$$

where $\mathbf{L}(\cdot) \in \mathcal{L}_{T_A}^{\mathbb{C}}$, and $\mathbf{F} \in \mathbb{C}^{n \times n}$.

Conversely, given $\mathbf{L}(\cdot) \in \mathcal{L}_{T_A}^{\mathbb{C}}$ and an arbitrary $\mathbf{F} \in \mathbb{C}^{n \times n}$, matrix $\mathbf{\Phi}(\cdot, 0)$ defined by Eq. (3.6) is the STM of some system (3.1) with $\mathbf{A}(\cdot) \in \mathcal{A}_{T_A}$.

PROOF: Consider matrix $\mathbf{X}(t) \triangleq \mathbf{\Phi}(t + T_A, 0)$. The relation

$$\forall t \in \mathbb{R}^+, \quad \dot{\mathbf{X}}(t) = \dot{\mathbf{\Phi}}(t + T_A, 0) = \mathbf{A}(t + T_A)\mathbf{\Phi}(t + T_A, 0) = \mathbf{A}(t)\mathbf{X}(t) \tag{3.7a}$$

shows that $\mathbf{X}(t)$ is a fundamental matrix of system (3.1). Thus, there exists a nonsingular matrix $\mathbf{K}$ (cf. Theorem 2.2.1) such that

$$\forall t \in \mathbb{R}^+, \quad \mathbf{X}(t) = \mathbf{\Phi}(t + T_A, 0) = \mathbf{\Phi}(t, 0)\mathbf{K}. \tag{3.7b}$$

The existence of a complex $n \times n$ matrix $\mathbf{F}$ such that $\mathbf{K} = \exp(T_A \mathbf{F})$ follows from Theorem 2.3.1. Define $\mathbf{L}(\cdot)$ by

$$\forall t \in \mathbb{R}^+, \quad \mathbf{L}(t) \triangleq \mathbf{\Phi}(t, 0) \exp(-t\mathbf{F}), \tag{3.7c}$$

and note that $\mathbf{L}(\cdot)$ is continuous and nonsingular for all t. Substituting from Eq. (3.7b) in Eq. (3.7c) results in

$$\begin{aligned} \forall t \in \mathbb{R}^+, \quad \mathbf{L}(t + T_A) &= \mathbf{\Phi}(t, 0) \exp(T_A \mathbf{F}) \exp(-T_A \mathbf{F}) \exp(-t\mathbf{F}) \\ &= \mathbf{\Phi}(t, 0) \exp(-t\mathbf{F}) \\ &= \mathbf{L}(t), \end{aligned}$$

and thus, $\mathbf{L}(t)$ is T_A periodic. Right-multiplying Eq. (3.7c) by $\exp(t\mathbf{F})$ yields Eq. (3.6). Finally, differentiating Eq. (3.7c) with respect to t yields

$$\begin{aligned} \forall t \in \mathbb{R}^+, \quad \dot{\mathbf{L}}(t) &= \dot{\mathbf{\Phi}}(t,0)\exp(-t\mathbf{F}) + \mathbf{\Phi}(t,0)\exp(-t\mathbf{F})(-\mathbf{F}) \\ &= \mathbf{A}(t)\mathbf{L}(t) - \mathbf{L}(t)\mathbf{F}. \end{aligned} \tag{3.7d}$$

The derivative of $\mathbf{L}(.)$ is thus piecewise continuous, completing the proof that $\mathbf{L}(.) \in \mathcal{L}_{T_A}^{\mathbb{C}}$.

Conversely, given $\mathbf{L}(.) \in \mathcal{L}_{T_A}^{\mathbb{C}}$ and $\mathbf{F} \in \mathbb{C}^{n\times n}$, $\mathbf{\Phi}(t,0) = \mathbf{L}(t)\exp(t\mathbf{F})$ satisfies

$$\begin{aligned} \mathbf{\Phi}(0,0) &= \mathbf{I}, \quad \text{and} \\ \forall t \in \mathbb{R}^+, \quad \mathbf{\Phi}(t+T_A,0) &= \mathbf{L}(t+T_A)\exp[(t+T_A)\mathbf{F}] \\ &= \mathbf{L}(t)\exp(t\mathbf{F})\exp(T_A\mathbf{F}) \\ &= \mathbf{\Phi}(t,0)\mathbf{\Phi}(T_A,0). \end{aligned}$$

Referring to Theorem 3.1.1, $\mathbf{\Phi}(\cdot,0)$ is thus the STM of a system of the form (3.1) where $\mathbf{A}(\cdot) \in \mathcal{A}_{T_A}$. □

At this point it is important to stress that the STM of a *real* matrix $\mathbf{A}(\cdot)$ will generally have *complex* Floquet factors $\mathbf{L}(\cdot)$ and $\mathbf{F}$. A well-known result follows:

Corollary 3.2.2 (Real Floquet Factors). *If $\mathbf{A}(\cdot) \in \mathcal{A}_{T_A}$, then it is always possible to obtain a pair of real Floquet factors by regarding $\mathbf{A}(\cdot) \in \mathcal{A}_{T_R}$, where $T_R = 2T_A$.*

Note 3.2.1. As we now apply Theorem 3.2.1 with $\mathbf{A}(\cdot) \in \mathcal{A}_{T_R}$, the time-varying matrix $\mathbf{L}(t,0)$ can then only be assumed to be T_R-periodic as well:

$$\forall t \in \mathbb{R}^+, \quad \mathbf{L}_{2T_A}(t+2T_A) = \mathbf{L}_{2T_A}(t), \quad \mathbf{L}(0) = \mathbf{I}. \tag{3.8}$$

In other words, the real T_A-periodic system of Eq. (3.1) always admits at least one pair of Floquet factors $\{\mathbf{L}_{2T_A}(\cdot), \mathbf{F}\} \in \mathcal{L}_{2T_A}^{\mathbb{R}} \times \mathbb{R}^{n\times n}$.

PROOF: Let $\mathbf{A}(\cdot) \in \mathcal{A}_{T_A}$. Then,

$$\begin{aligned}
\mathbf{A}(\cdot) \in \mathcal{A}_{T_A} &\Rightarrow \mathbf{A}(\cdot) \in \mathcal{A}_{2T_A}. \\
\mathbf{A}(\cdot) \in \mathcal{A}_{2T_A} &\Rightarrow \exists\{\mathbf{L}(\cdot), \mathbf{F}\} \in \mathcal{L}^{\mathbb{C}}_{2T_A} \times \mathbb{C}^{n\times n}; \mathbf{\Phi}(t,0) = \mathbf{L}(t)\exp(t\mathbf{F}) \quad \text{(Theorem 3.2.1)}. \\
\mathbf{L}(t) \in \mathcal{L}^{\mathbb{C}}_{2T_A} &\Rightarrow \mathbf{L}(2T_A) = \mathbf{I}. \\
\mathbf{L}(2T_A) = \mathbf{I} &\Rightarrow \mathbf{\Phi}(2T_A, 0) = \exp(2T_A\mathbf{F}).
\end{aligned}$$

On the other hand, recalling Eq. (3.4),

$$\mathbf{\Phi}(2T_A, 0) = \mathbf{\Phi}(2T_A, T_A)\mathbf{\Phi}(T_A, 0) = \mathbf{\Phi}^2(T_A, 0), \tag{3.9a}$$

and we derive

$$\mathbf{F}_{2T_A} \triangleq \frac{1}{2T_A}\ln \mathbf{\Phi}(2T_A, 0) = \frac{1}{2T_A}\ln \mathbf{\Phi}^2(T_A, 0). \tag{3.9b}$$

From Lemma 2.3.3 we know that

$$\mathbf{F}_{2T_A} \in \mathbb{R}^{n\times n}, \text{ thus}$$

$$\mathbf{L}(t) \triangleq \mathbf{\Phi}(t,0)\exp(-tF_{2T_A}) \in \mathcal{L}^{\mathbb{R}}_{2T_A}.$$

□

THEOREM 3.2.3 (Reducibility Theorem). *Let* $\mathbf{L}(\cdot)$ *and* $\mathbf{F}$ *be the same Floquet factors as in Theorem 3.2.1. The substitution*

$$\mathbf{x}(t) = \mathbf{L}(t)\mathbf{z}(t) \tag{3.10a}$$

transforms the LTP homogeneous system (3.1) into the LTI system

$$\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t). \tag{3.10b}$$

PROOF: Substituting Eq. (3.10a) into Eq. (3.1) results in

$$\dot{\mathbf{L}}(t)\mathbf{z}(t) + \mathbf{L}(t)\dot{\mathbf{z}}(t) = \mathbf{A}(t)\mathbf{L}(t)\mathbf{z}(t), \tag{3.11a}$$

which can be rewritten as

$$\dot{\mathbf{z}}(t) = \mathbf{L}^{-1}(t)\left(\mathbf{A}(t)\mathbf{L}(t) - \dot{\mathbf{L}}(t)\right)\mathbf{z}(t). \tag{3.11b}$$

Equations (3.11b) and (3.7d) yield Eq. (3.10b). □

NOTE 3.2.2. In the case of a general nonhomogenous LTP system (2.1), the change of variable defined in Eq. (3.10a) will transform the system equations into

$$\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t) + \mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{u}(t), \tag{3.12a}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{L}(t)\mathbf{z}(t), \tag{3.12b}$$

where only $\mathbf{A}(\cdot)$ is reduced to a constant matrix, $\mathbf{F}$, in the new system.

3.3. Stability Analysis

DEFINITION 3.3.1 (Monodromy Matrix). The value of the STM of homogeneous system (3.1) after one period, $\mathbf{\Phi}(T_A, 0)$, is called the *monodromy matrix.* The eigenvalues of the monodromy matrix are called the *Floquet multipliers* (a.k.a. Poincaré multipliers, characteristic multipliers, or characteristic roots) of the system.

DEFINITION 3.3.2. The eigenvalues of Floquet factor $\mathbf{F}$ in Theorem 3.2.1 are called the *Floquet exponents* (a.k.a. Poincaré exponents or characteristic exponents) of the system.

THEOREM 3.3.1. *(e.g., Brockett 1970) The Floquet multipliers of a LTP system are unique, whereas its Floquet exponents are not. Moreover, a LTP system is asymptotically stable if and only if all its Floquet multipliers lie within the unit circle, which is equivalent to its Floquet exponents having negative real parts.*

EXAMPLE 3.3.1. Let $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$, with

$$\mathbf{A}(t) = \frac{1}{\epsilon}\begin{bmatrix} -1 + \frac{1}{2}\eta\sin(2\alpha t) & \frac{1}{2}\eta(\cos(2\alpha t) + 1) \\ \frac{1}{2}\eta(\cos(2\alpha t) - 1) & -1 - \frac{1}{2}\eta\sin(2\alpha t) \end{bmatrix},$$

$$(\alpha, \epsilon, \eta) \in \mathbb{R}^{+*} \times \mathbb{R}^{+*} \times \mathbb{R}^{+}, \quad T_A = \frac{\pi}{\alpha}. \tag{3.13a}$$

One real pair of Floquet factors is given by Kreiss (1978) as

$$\mathbf{\Phi}(t,0) = \mathbf{L}(t)\exp(t\mathbf{F}), \text{ where} \tag{3.13b}$$

$$\mathbf{L}(t) = \begin{bmatrix} \cos\alpha t & \sin\alpha t \\ -\sin\alpha t & \cos\alpha t \end{bmatrix}, \text{ and } \quad \mathbf{F} = \begin{bmatrix} -1/\epsilon & \eta/\epsilon - \alpha \\ \alpha & -1/\epsilon \end{bmatrix}. \tag{3.13c}$$

We can therefore derive the value of the monodromy matrix:

$$\mathbf{\Phi}\left(\frac{\pi}{\alpha},0\right) = \mathbf{L}\left(\frac{\pi}{\alpha}\right)\exp\left(\frac{\pi}{\alpha}\mathbf{F}\right) = -\exp\left(\begin{bmatrix} -\frac{\pi}{\alpha}\frac{1}{\epsilon} & \frac{\pi}{\alpha}\frac{\eta}{\epsilon} - \pi \\ \pi & -\frac{\pi}{\alpha}\frac{1}{\epsilon} \end{bmatrix}\right). \tag{3.13d}$$

The Floquet multipliers of the system are

$$\mu_{\pm} = -\mathrm{e}^{-\frac{\pi}{\alpha\epsilon}\beta_{\pm}}, \text{ where } \beta_{\pm} = 1 \pm \sqrt{\alpha\epsilon(\eta - \alpha\epsilon)}. \tag{3.13e}$$

Thus, $\mathbf{A}(t)$ has constant (negative) eigenvalues equal to $-\frac{1}{\epsilon}$ for any t; however, choosing $\eta > \alpha\epsilon$ can lead to Floquet multipliers lying outside the unit circle. This is a necessary and sufficient condition for the existence of unstable solutions, thereby illustrating the irrelevance of the eigenvalues of a time-varying matrix for stability analysis.

3.4. Dual Nature of LTP Systems

The results of the previous sections illustrate the dual nature of LTP systems which can be interpreted either from the discrete or the continuous-time point of view:

- At time $t = kT_A$, Eq. (3.4) becomes

$$\mathbf{\Phi}(kT_A, 0) = \mathbf{\Phi}^k(T_A, 0), \tag{3.14}$$

and any solution to the differential equation Eq. (3.1) satisfies

$$\begin{aligned}\mathbf{x}\left((k+1)T_A\right) &= \mathbf{\Phi}^{k+1}(T_A,0)\mathbf{x}(0)\\ &= \mathbf{\Phi}(T_A,0)\mathbf{\Phi}^{k}(T_A,0)\mathbf{x}(0)\\ &= \mathbf{\Phi}(T_A,0)\mathbf{x}(kT_A).\end{aligned} \tag{3.15}$$

Because of this correspondence with a discrete time-invariant system, $\mathbf{\Phi}(T_A, 0)$ is sometimes called (D'Angelo 1970) the *discrete transition matrix.* The system is thus described in the z-plane and the stability criterion of the Floquet multipliers being within the unit circle corresponds indeed to discrete LTI systems;

- On the other hand, the transformed system of Eq. (3.10b) obtained with the Reducibility Theorem gives a description of the system in the s-plane and the stability criterion (exponents with negative real parts) is similar to continuous LTI system theory.

Going from one interpretation to the other using Eq. (3.10a) can be put in parallel with sampling system (3.10b) with a T_A sampling time, resulting in system (3.15). In this paradigm $\mathbf{L}(\cdot)$ becomes a mapping from the z-plane to the s-plane. More specifically, $\mathbf{L}(\cdot)$ maps the z-plane to an horizontal strip of width $2\pi/T_A$ (cf. Fig. 3.1), dependent upon which branch of the logarithm is selected for $\mathbf{F} = 1/T_A \ln(\mathbf{\Phi}(T_A, 0))$.

3.5. Controllability

The classical notion of controllability due to Kalman (e.g. Kalman 1963) has been refined over the years for the specific framework of LTP systems (Bittanti, Colaneri, and Guardabassi 1984; Bittanti, Guardabassi, Maffezzoni, and Silverman 1978; Brunovský 1969; Hewer 1975; Kabamba 1986; Kano and Nishimura 1985; Silverman and Meadows 1967).

When matrices $\mathbf{A}(\cdot)$, $\mathbf{B}(\cdot)$, together with their $(n-2)$ and $(n-1)$ derivatives respectively, are continuous, controllability can be derived from the controllability

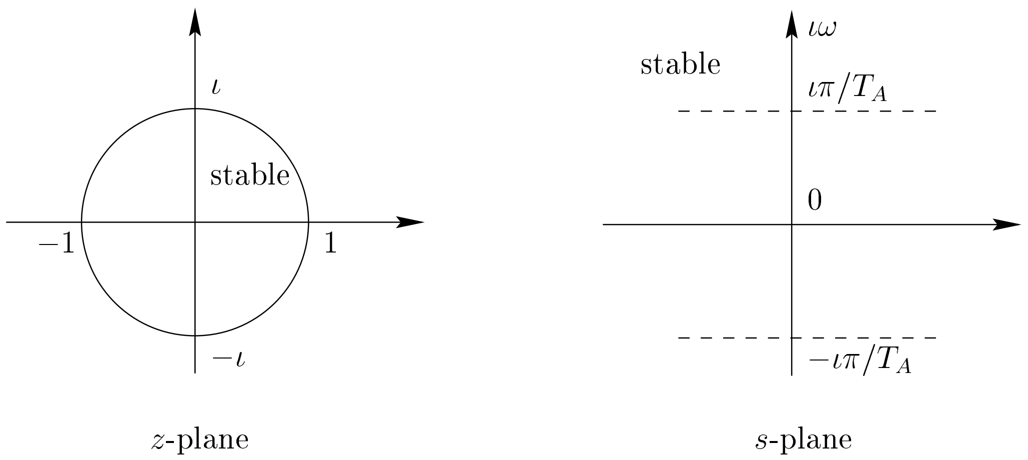


Figure 3.1: z-plane and s-plane interpretations

matrix (Silverman and Meadows 1967):

$$\mathbf{Q}_c(t) = [\mathbf{P}_0(t)|\mathbf{P}_1(t)|\cdots|\mathbf{P}_{n-1}(t)]\,, \text{ where} \tag{3.16a}$$

$$\mathbf{P}_0(t) = \mathbf{B}(t), \quad \mathbf{P}_{k+1}(t) = \mathbf{A}(t)\mathbf{P}_k(t) + \dot{\mathbf{P}}_k(t), \quad \forall k \geq 1. \tag{3.16b}$$

In the general case, Brunovský's seminal paper (Brunovský 1969) proved that a LTP system of order n is controllable if and only if it is controllable over n periods. Bittanti, Colaneri, and Guardabassi (1984) showed that this reduced to m periods, where m is the degree of the minimal polynomial of the monodromy matrix. Finally, Kabamba (1986) showed that the number of periods is at most the controllability index of the pair $\{\mathbf{\Phi}(T_A,0), \mathbf{W}(T_A,0)\}$, which is known (Chen 1984, Theorem 5.8) to be less than m. This can be summarized by the following result:

Theorem 3.5.1. *Consider system (2.1) of order n, its corresponding monodromy matrix $\mathbf{\Phi}(T_A,0)$, and its Gramian $\mathbf{W}(T_A,0)$ at $t = T_A$ ($\mathbf{W} = \mathbf{W}_C$ or $\mathbf{W} = \mathbf{W}_R$). Furthermore, let ν be the controllability index of the pair $\{\mathbf{\Phi}(T_A,0), \mathbf{W}(T_A,0)\}$.*

The following statements are equivalent:

- *System (2.1) is controllable;*
- *The pair $\{\mathbf{\Phi}(T_A,0), \mathbf{W}(T_A,0)\}$ is controllable;*
- *System (2.1) is controllable over $(0, \nu T_A)$;*

- $\mathbf{W}(\nu T_A, 0)$ *is positive definite.*

The relationship between controllability and invariant assignment for LTP systems was provided by Brunovský (1969):

THEOREM 3.5.2. *A LTP system is controllable if and only if there exists a T_A-periodic $r \times n$ matrix $\mathbf{K}(\cdot)$ which can assign the eigenvalues of the monodromy matrix in such a way that they appear as complex conjugate pairs, and their product is positive.*

NOTE 3.5.1. The restriction on the sign of the product of the eigenvalues is explained by Liouville's Theorem (Bellman 1970a):

$$\det \mathbf{\Phi}(t, 0) = \exp\left(\int_0^t \mathrm{tr}\mathbf{A}(\sigma)d\sigma\right), \tag{3.17}$$

illustrating that $\det \mathbf{\Phi}(t, 0) > 0,\ \forall t$. This has to hold in particular at $t = T_A$ when $\det \mathbf{\Phi}(T_A, 0)$ is the product of the eigenvalues of the monodromy matrix.

NOTE 3.5.2. Recalling the duality between controllability and observability (e.g., Chen 1984, Theorem 5-10), in Theorem 3.5.1 all the results are preserved if 'controllable' is replaced by 'observable' and $\mathbf{W}(t_2, t_1)$ is replaced by the observability Gramian

$$\mathbf{W}_O(t_2, t_1) \triangleq \int_{t_1}^{t_2} \mathbf{\Phi}^T(t, t_1)\mathbf{C}^T(t)\mathbf{C}(t)\mathbf{\Phi}(t, t_1)dt. \tag{3.18}$$

CHAPTER 4

Real Floquet Factors

4.1. Introduction

As we saw in the previous chapter, the STM of a LTP system is unique but its Floquet factors are not. In the case of real factors derived from Corollary 3.2.2, Note 3.2.1 pointed out that the period of $\mathbf{L}(\cdot)$ can then only be assumed to be $2T_A$. This leads to a loss of information on its primary periodicity, as illustrated via a simple example.

EXAMPLE 4.1.1. Consider the T_A-periodic system proposed by Wu (1978):

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t),$$

$$\mathbf{A}(t) = \begin{bmatrix} a + \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & a + \cos(\omega t) \end{bmatrix}, \quad a \in \mathbb{R}, \quad T_A = \frac{2\pi}{\omega}, \tag{4.1a}$$

whose STM is known in closed form:

$$\mathbf{\Phi}(t,0) = \exp\left(\begin{bmatrix} at + \frac{1}{\omega}\sin(\omega t) & \frac{1-\cos(\omega t)}{\omega} \\ -\frac{1-\cos(\omega t)}{\omega} & at + \frac{1}{\omega}\sin(\omega t) \end{bmatrix}\right). \tag{4.1b}$$

Both $2T_A\mathbf{F}_0$ and $2T_A\mathbf{F}_1$ are real logarithms of $\mathbf{\Phi}(2T_A, 0)$, where

$$\mathbf{F}_0 = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \quad \mathbf{F}_1 = \begin{bmatrix} a & \frac{\omega}{2} \\ -\frac{\omega}{2} & a \end{bmatrix}. \tag{4.1c}$$

The corresponding $\mathbf{L}_{F_0}(t) \stackrel{\Delta}{=} \mathbf{\Phi}(t,0)\exp(-t\mathbf{F}_0)$ and $\mathbf{L}_{F_1}(t) \stackrel{\Delta}{=} \mathbf{\Phi}(t,0)\exp(-t\mathbf{F}_1)$ are

$$\mathbf{L}_{F_0}(t) = \mathrm{e}^{\beta_2} \begin{bmatrix} \cos(\beta_1) & -\sin(\beta_1) \\ \sin(\beta_1) & \cos(\beta_1) \end{bmatrix} \begin{bmatrix} \cos(1/\omega) & \sin(1/\omega) \\ -\sin(1/\omega) & \cos(1/\omega) \end{bmatrix}, \tag{4.1d}$$

$$\beta_1 = \frac{\cos(\omega t)}{\omega}, \quad \beta_2 = \frac{\sin(\omega t)}{\omega},$$

$$\mathbf{L}_{F_1}(t) = \exp\left(\begin{bmatrix} \frac{1}{\omega}\sin\omega t & \frac{1-\cos\omega t}{\omega} - \frac{\omega}{2}t \\ -\frac{1-\cos\omega t}{\omega} + \frac{\omega}{2}t & \frac{1}{\omega}\sin\omega t \end{bmatrix} \right). \tag{4.1e}$$

$\mathbf{L}_{F_0}(\cdot)$ and $\mathbf{L}_{F_1}(\cdot)$ are real and $2T_A$-periodic. However, while $\mathbf{L}_{F_0}(\cdot)$ is primarily T_A-periodic, $\mathbf{L}_{F_1}(\cdot)$ is T_A-antiperiodic and primarily $2T_A$-periodic. The variations of these matrices are illustrated in Fig. 4.1, p.36.

This example shows that Corollary 3.2.2 is useful to prove the existence of real factorizations, but fails to provide all pertinent pieces of information relative to the periodicity of the Floquet factors. To keep consistent with the complex Floquet factors and the other general properties of LTP systems, a new result should be based on the monodromy matrix $\mathbf{\Phi}(T_A, 0)$ rather than $\mathbf{\Phi}(2T_A, 0)$. This result should be able to indicate *a priori* the periodicity of $\mathbf{L}(\cdot)$. Moreover, we notice that to a primarily $2T_A$-periodic $\mathbf{L}_{F_1}(\cdot)$ corresponds a primarily T_A-periodic system, whereas one may have expected only a primarily $2T_A$-periodic $\mathbf{A}(\cdot)$. Therefore, a converse statement would also be welcome.

In this chapter, we formulate a novel result which achieves both objectives while highlighting the pitfalls of existing results. Henceforth, the proofs of continuity do not vary from the standard theorems, while we focus on periodicity.

4.2. Previous Work

Yakubovich (1970) and Yakubovich and Starzhinskii (1975, Chapter II §2.3) were the first to propose a result (stated here with our current notation) which generalized Corollary 3.2.2.

THEOREM 4.2.1. *In the equation*

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \tag{4.2a}$$

let $\mathbf{A}(\cdot)$ *be a real matrix function, where* $\mathbf{A}(\cdot)$ *is integrable and piecewise continuous on* $(0, T_A)$*, and* $\mathbf{A}(t+T_A) = \mathbf{A}(t)$*. An arbitrary real matrix* $\mathbf{X}(\cdot, 0)$ *that is a fundamental solution of (4.2a) may be expressed as*

$$\mathbf{X}(t, 0) = \mathbf{L}(t)\exp(t\mathbf{F}), \tag{4.2b}$$

where $\mathbf{F}$ *is a real constant matrix,* $\mathbf{L}(\cdot)$ *is a real matrix function such that*

$$\mathbf{L}(t + T_A) = \mathbf{L}(t)\mathbf{Y}, \tag{4.2c}$$

and $\mathbf{Y}$ *some real matrix such that*

$$\mathbf{Y}^2 = \mathbf{I}, \tag{4.2d}$$

$$\mathbf{FY} = \mathbf{YF}. \tag{4.2e}$$

In particular,

$$\mathbf{L}(t + 2T_A) = \mathbf{L}(t) \quad \forall t.$$

The function $\mathbf{L}(\cdot)$ *is continuous with an integrable piecewise-continuous derivative.*

Conversely, let $\mathbf{L}(\cdot)$*,* $\mathbf{F}$*, and* $\mathbf{Y}$ *be arbitrary real matrices satisfying conditions (4.2c) and (4.2d),* $\forall t \ \det \mathbf{L}(t) \neq 0$*, and let* $\mathbf{L}(\cdot)$ *have an integrable piecewise-continuous derivative. Then (4.2b) is a fundamental matrix for some equation of the form (4.2a) with a real* T_A*-periodic matrix* $\mathbf{A}(\cdot)$*.*

Unfortunately this result does not cover all possible real Floquet factorizations. Indeed if constant real matrices $\mathbf{K}$ and $\mathbf{M}$ commute, and if $\log(\mathbf{M})$ is a *primary matrix function* (PMF) of $\mathbf{M}$, then $\mathbf{K}$ and $\log(\mathbf{M})$ will also commute (Horn and Johnson 1994, Theorem 6.2.9 (d)). However, when $\log(\mathbf{M})$ is not a PMF, then the commutativity may not always be preserved. In this case, there may exist $\mathbf{F}$ and $\mathbf{Y}$ such that

$\exp(T\mathbf{F}) = \mathbf{Y}\mathbf{\Phi}(T_A, 0)$, $\mathbf{Y}^2 = \mathbf{I}$, and $\mathbf{Y}\exp(T\mathbf{F}) = \exp(T\mathbf{F})\mathbf{Y}$ but Eqs. (4.2d) and (4.2e) are not satisfied, as illustrated in the following counterexample.

EXAMPLE 4.2.1. Consider the primarily T_A-periodic matrix with $\alpha \neq 0$ so $T_A = 1/2$:

$$\mathbf{A}(t) = 2\pi \begin{bmatrix} -1 + \alpha\cos^2(2\pi t) & 1 - \alpha\sin(2\pi t)\cos(2\pi t) & 0 \\ -1 - \alpha\sin(2\pi t)\cos(2\pi t) & -1 + \alpha\sin^2(2\pi t) & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{4.3a}$$

If we define the rotation matrix

$$\mathbf{R}(\theta) \triangleq \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \tag{4.3b}$$

it can be verified that the state-transition matrix of this system is

$$\mathbf{\Phi}(t, 0) = \begin{bmatrix} \mathbf{R}(2\pi t) & \mathbf{0}_2 \\ \mathbf{0}_2^T & 1 \end{bmatrix} \mathrm{diag}\{\mathrm{e}^{2\pi(\alpha-1)t}, \mathrm{e}^{-2\pi t}, \mathrm{e}^{-2\pi t}\}. \tag{4.3c}$$

Note that

$$\mathbf{\Phi}(T_A, 0) = \mathrm{diag}\left\{-\mathrm{e}^{\pi(\alpha-1)}, -\mathrm{e}^{-\pi}, \mathrm{e}^{-\pi}\right\} \tag{4.3d}$$

does not have a real logarithm (cf. Theorem 2.3.2). One suitable choice for $\mathbf{Y}$ is

$$\mathbf{Y}_0 = \mathrm{diag}\left\{-1, -1, 1\right\}, \tag{4.3e}$$

giving a logarithm such that

$$\mathbf{F}_0 \triangleq \frac{1}{T_A}\log(\mathbf{Y}_0\mathbf{\Phi}(T_A, 0)) = \mathrm{diag}\left\{2\pi(\alpha-1), -2\pi, -2\pi\right\}, \tag{4.3f}$$

where since $\exp(t\mathbf{F}_0) = \mathrm{diag}\{\mathrm{e}^{2\pi(\alpha-1)t}, \mathrm{e}^{-2\pi t}, \mathrm{e}^{-2\pi t}\}$,

$$\mathbf{L}_0(t) = \mathbf{\Phi}(t, 0)\exp(-t\mathbf{F}_0) = \begin{bmatrix} \mathbf{R}(2\pi t) & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.3g}$$

In this case, it is readily verified that all the aspects of Theorem 4.2.1 are satisfied, notably $\mathbf{F}_0\mathbf{Y}_0 = \mathbf{Y}_0\mathbf{F}_0$ — both $\mathbf{F}_0$ and $\mathbf{Y}_0$ are diagonal.

If instead of $\mathbf{Y}_0$ we take $\mathbf{Y}_1 = \operatorname{diag}\{-1, 1, -1\}$, we can obtain one logarithm of $\mathbf{Y}_1\mathbf{\Phi}(T_A, 0)$ giving

$$\mathbf{F}_1 \triangleq \frac{1}{T_A}\log(\mathbf{Y}_1\mathbf{\Phi}(T_A,0)) = \frac{1}{T_A}\log(\operatorname{diag}\left\{\mathrm{e}^{\pi(\alpha-1)}, -\mathrm{e}^{-\pi}, -\mathrm{e}^{-\pi}\right\}) \tag{4.3h}$$

$$= \mathbf{F}_0 + \mathbf{G}, \quad \mathbf{G} \triangleq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2\pi \\ 0 & -2\pi & 0 \end{bmatrix}, \quad \exp(t\mathbf{G}) = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{R}(2\pi t) \end{bmatrix}, \tag{4.3i}$$

and the periodic matrix

$$\mathbf{L}_1(t) \triangleq \mathbf{\Phi}(t,0)\exp(-t\mathbf{F}_1) \tag{4.3j}$$

$$= \mathbf{\Phi}(t,0)\exp(-t\mathbf{F}_0)\exp(-t\mathbf{G}) \tag{4.3k}$$

$$= \mathbf{L}_0(t)\exp(-t\mathbf{G}) \tag{4.3l}$$

$$= \begin{bmatrix} \cos(2\pi t) & \sin(2\pi t)\cos(2\pi t) & -\sin^2(2\pi t) \\ -\sin(2\pi t) & \cos^2(2\pi t) & -\sin(2\pi t)\cos(2\pi t) \\ 0 & \sin(2\pi t) & \cos(2\pi t) \end{bmatrix}, \tag{4.3m}$$

again has period $2T_A = 1$. We see that $\mathbf{Y}_1 = \mathbf{L}_1(T_A)$, but for $t \neq kT_A$, $k \in \mathbb{Z}^{+*}$,

$$\mathbf{L}_1(t + T_A) \neq \mathbf{L}_1(t)\mathbf{Y}_1, \tag{4.3n}$$

and

$$\mathbf{Y}_1\mathbf{F}_1 \neq \mathbf{F}_1\mathbf{Y}_1. \tag{4.3o}$$

$\mathbf{F}_1$ and $2T_A$-periodic $\mathbf{L}_1(\cdot)$ provide a real decomposition of $\mathbf{\Phi}(\cdot, 0)$ without satisfying the conditions of Theorem 4.2.1. Thus, this theorem alone does not cover all the real factorizations provided by Corollary 3.2.2.

4.3. Characterization

4.3.1. $\mathbf{F}_Y$ matrices Let $\mathbf{Y} \in \mathbb{R}^{n\times n}$ such that $T_A\mathbf{F}_Y \in \mathcal{RL}og\left\{\mathbf{Y}\mathbf{\Phi}(T_A,0)\right\}$; we then have

$$\mathbf{Y}\mathbf{\Phi}(T_A,0) = \exp\left(T_A\mathbf{F}_Y\right). \tag{4.4}$$

We can now define matrix $\mathbf{L}_{F_Y}(\cdot)$ as

$$\forall t \in \mathbb{R}^+, \quad \mathbf{L}_{F_Y}(t) \stackrel{\Delta}{=} \mathbf{\Phi}(t,0)\exp(-t\mathbf{F}_Y). \tag{4.5}$$

Note that $\mathbf{L}_{F_Y}(0) = \mathbf{I}$.

The classical results recalled in §3.3 and §3.4 rely on a specific relationship between the eigenvalues μ_i $(i = 1 \ldots n)$ of $\mathbf{\Phi}(T_A,0)$ and the eigenvalues λ_i $(i = 1 \ldots n)$ of $\mathbf{F}$, where $\mathbf{F}$ is any complex matrix obtained from Theorem 3.2.1, namely

$$\mathbf{\Phi}(T_A,0) = \exp(T_A\mathbf{F}). \tag{4.6a}$$

Considering the polar representation of μ_i

$$\mu_i = \rho_i\exp(\iota\theta_i), \qquad \iota^2 = -1, \quad \rho_i \stackrel{\Delta}{=} |\mu_i| \geq 0, \quad -\pi \leq \theta_i < \pi, \tag{4.6b}$$

every eigenvalue λ_i has the form

$$\lambda_i = \frac{\ln(\rho_i)}{T_A} + \iota\frac{\theta_i + 2k\pi}{T_A}, \qquad k \in \mathbb{Z}. \tag{4.6c}$$

When introducing matrix $\mathbf{Y}$ in the definition of $\mathbf{F}_Y$ in Eq. (4.4) it is important to preserve the relation

$$\mathrm{Re}(\lambda_i) = \frac{\ln(\rho_i)}{T_A}. \tag{4.7}$$

Indeed, consider the case of a 2×2 system where

$$\mathbf{\Phi}(T_A,0) = \mathrm{diag}\left\{\mu_1,\mu_2\right\}, \quad \mathbf{Y} = \mathrm{diag}\left\{\gamma_1,\gamma_2\right\}, \tag{4.8a}$$

then the eigenvalues of any $\mathbf{F} \triangleq \log\left(\mathbf{Y}\boldsymbol{\Phi}(T_A, 0)\right)/T_A$ are

$$\lambda_i = \frac{\ln(|\gamma_i|\rho_i)}{T_A} + \iota\frac{\theta_i + (2k+\kappa_i)\pi}{T_A}, \quad k \in \mathbb{Z}, \quad \kappa_i = \begin{cases} 0 & \text{if } \gamma_i > 0, \\ 1 & \text{if } \gamma_i < 0. \end{cases} \tag{4.8b}$$

For $\mathbf{F}$ to be real, Theorem 2.3.2 only requires $\gamma_1\mu_1$ and $\gamma_2\mu_2$ to be both positive, or both negative and equal. However it is possible to satisfy this condition while being in a situation where

- $\rho_i < 1$ but $\mathrm{Re}(\lambda_i) > 0$, or
- $\rho_1 = \rho_2$ but $\lambda_1 \neq \lambda_2$.

Thus, while these factorizations would be mathematically correct, there would be no guarantee that the factors would obey the canonical stability results, the paradigm of the sampling of period T_A, and the fact that μ_i and λ_i have the same multiplicity.

In other words, the s-plane strip image of the unit circle (cf. Fig. 3.1, p.23) would be distorted and the resulting λ_i may lose quantitative and qualitative dynamical significance. All these properties are essential to the control engineer, and can be preserved by requiring that the eigenvalues of $\boldsymbol{\Phi}(T_A, 0)$ and $\mathbf{Y}\boldsymbol{\Phi}(T_A, 0)$ have the same moduli.

4.3.2. Periodicity of $\mathbf{L}(\cdot)$

THEOREM 4.3.1. *Let $\boldsymbol{\Phi}(\cdot, 0)$ be the STM of a system (3.1) where $\mathbf{A}(\cdot) \in \mathcal{A}_{T_A}$. Let $\mathbf{Y}, \mathbf{F}_Y$, and $\mathbf{L}_{F_Y}(\cdot)$ be the matrices defined in Eqs. (4.4) and (4.5), and $k \in \mathbb{Z}^+$.*

$\mathbf{L}_{F_Y}(\cdot)$ is kT_A-periodic if and only if

$$\boldsymbol{\Phi}^k(T_A, 0) = \left[\boldsymbol{\Phi}(T_A, 0)\mathbf{Y}\right]^k. \tag{4.9}$$

$\mathbf{L}_{F_Y}(\cdot)$ is kT_A-antiperiodic if and only if

$$\boldsymbol{\Phi}^k(T_A, 0) = -\left[\boldsymbol{\Phi}(T_A, 0)\mathbf{Y}\right]^k. \tag{4.10}$$

PROOF: Recalling Theorem 3.1.1, $\mathbf{A}(\cdot) \in \mathcal{A}_{T_A}$ if and only if

$$\forall t \in \mathbb{R}^+, \quad \boldsymbol{\Phi}(t + kT_A, 0) = \boldsymbol{\Phi}(t, 0)\boldsymbol{\Phi}^k(T_A, 0). \tag{4.11a}$$

Using Eq. (4.5), Eq. (4.11a) becomes

$$\forall t \in \mathbb{R}^+, \quad \mathbf{L}_{F_Y}(t + kT_A)\exp[(t + kT_A)\mathbf{F}_Y] = \mathbf{\Phi}(t,0)\mathbf{\Phi}^{k-1}(T_A,0)\mathbf{L}_{F_Y}(T_A)\exp(T_A\mathbf{F}_Y), \tag{4.11b}$$

which leads to

$$\begin{aligned}\forall t \in \mathbb{R}^+, \quad \mathbf{L}_{F_Y}(t + kT_A) = \mathbf{L}_{F_Y}(t)\exp(t\mathbf{F}_Y)\mathbf{\Phi}^{k-1}(T_A,0) \times \\ \mathbf{L}_{F_Y}(T_A)\exp[-((k-1)T_A + t)\mathbf{F}_Y].\end{aligned} \tag{4.11c}$$

On the other hand, consider

$$\forall t \in \mathbb{R}^+, \quad \mathbf{L}_{F_Y}(t + kT_A) = \epsilon\mathbf{L}_{F_Y}(t), \tag{4.11d}$$

where $\mathbf{L}_{F_Y}(\cdot)$ is kT_A-periodic if and only if $\epsilon = +1$, kT_A-antiperiodic if and only if $\epsilon = -1$. Taking into consideration Eqs. (4.11c) and (4.11d), this periodicity condition can be rewritten as

$$\forall t \in \mathbb{R}^+, \quad \exp(t\mathbf{F}_Y)\mathbf{\Phi}^{k-1}(T_A,0)\mathbf{L}_{F_Y}(T_A)\exp[-((k-1)T_A + t)\mathbf{F}_Y] = \epsilon\mathbf{I}. \tag{4.11e}$$

Left-multiplication by $\exp(-t\mathbf{F}_Y)$, and right multiplication by $\exp(t\mathbf{F}_Y)$ transforms Eq. (4.11e) into

$$\forall t \in \mathbb{R}^+, \quad \mathbf{\Phi}^{k-1}(T_A,0)\mathbf{L}_{F_Y}(T_A)\exp[-(k-1)T_A\mathbf{F}_Y] = \epsilon\mathbf{I}. \tag{4.11f}$$

Recalling from Eq. (4.5),

$$\mathbf{L}_{F_Y}(T_A) = \mathbf{\Phi}(T_A,0)\exp(-T_A\mathbf{F}) = \mathbf{Y}^{-1}, \tag{4.11g}$$

and by raising Eq. (4.4) to the power $-(k-1)$, Eq. (4.11f) becomes

$$\mathbf{\Phi}^{k-1}(T_A,0)\mathbf{Y}^{-1}\left[\mathbf{Y}\mathbf{\Phi}(T_A,0)\right]^{-(k-1)} = \epsilon\mathbf{I}, \tag{4.11h}$$

or,

$$\mathbf{\Phi}^{k-1}(T_A,0) = \epsilon\left[\mathbf{Y}\mathbf{\Phi}(T_A,0)\right]^{(k-1)}\mathbf{Y}. \tag{4.11i}$$

Right-multiplication of Eq. (4.11i) by $\mathbf{\Phi}(T_A, 0)$ yields Eq. (4.9) for $\epsilon = 1$ and Eq. (4.10) for $\epsilon = -1$. □

COROLLARY 4.3.2. *There always exists a matrix* $\mathbf{Y}$ *such that* $\mathbf{L}_{F_Y}$ *is* $2T_A$*-periodic.*

PROOF: Consider the real Jordan canonical form

$$\mathbf{W}_J^{-1}\mathbf{\Phi}(T_A, 0)\mathbf{W}_J \triangleq \begin{bmatrix} \mathbf{J}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 \end{bmatrix} = \mathbf{J} \tag{4.12a}$$

where the $n_2 \times n_2$ block $\mathbf{J}_2$ contains all the negative real eigenvalues of $\mathbf{\Phi}(T_A, 0)$ and no others. With this partitioning, define matrices $\mathbf{K}$ and $\mathbf{Y}$ as

$$\mathbf{K} \triangleq \begin{bmatrix} \mathbf{I}_{n_1 \times n_1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{n_2 \times n_2} \end{bmatrix}, \quad \mathbf{Y} \triangleq \mathbf{W}_J \mathbf{K} \mathbf{W}_J^{-1}. \tag{4.12b}$$

We see that $\mathbf{KJ}$ has no negative real eigenvalues, so by Theorem 2.3.2 there exists real $T\mathbf{F} = \ln(\mathbf{Y}\mathbf{\Phi}(T, 0))$. Additionally, since $\mathbf{JK} = \mathbf{KJ}$ and $\mathbf{K}^2 = \mathbf{I}$,

$$\begin{aligned} [\mathbf{\Phi}(T_A, 0)\mathbf{Y}]^2 &= [\mathbf{W}_J \mathbf{J} \mathbf{W}_J^{-1} \mathbf{W}_J \mathbf{K} \mathbf{W}_J^{-1}]^2 \\ &= \mathbf{W}_J \mathbf{JKJK} \mathbf{W}_J^{-1} \\ &= \mathbf{W}_J \mathbf{J}^2 \mathbf{K}^2 \mathbf{W}_J^{-1} \\ &= \mathbf{W}_J \mathbf{J}^2 \mathbf{W}_J^{-1} \\ &= \mathbf{\Phi}^2(T_A, 0). \end{aligned} \tag{4.12c}$$

Equation (4.9) is satisfied for $k = 2$, thus proving the $2T_A$-periodicity of $\mathbf{L}_{F_Y}(\cdot)$. □

4.3.3. General Real Factorizations The results from the two previous subsections are summarized in two new theorems.

THEOREM 4.3.3. *In Eq. (3.1), let* $\mathbf{A}(\cdot) \in \mathcal{A}_{T_A}$ *and* $\mathbf{\Phi}(\cdot, 0)$ *be its (real) state-transition matrix. Let* $\mathbf{Y} \in \mathbb{R}^{n \times n}$ *such that*

- $\mathbf{Y}\mathbf{\Phi}(T_A, 0)$ *has a real logarithm;*
- *the eigenvalues of* $\mathbf{Y}\mathbf{\Phi}(T_A, 0)$ *and* $\mathbf{\Phi}(T_A, 0)$ *have same moduli.*

For any $\mathbf{F}_Y$ *satisfying* $T\mathbf{F}_Y \in \mathcal{RL}og(\mathbf{Y\Phi}(T_A,0))$, *the real factor*

$$\mathbf{L}_{F_Y}(t) \triangleq \mathbf{\Phi}(t,0)\exp(-t\mathbf{F}_Y) \tag{4.13a}$$

is continuous with a piecewise continuous derivative, $\mathbf{L}_{F_Y}(T_A,0) = \mathbf{Y}^{-1}$, *and:*

- kT_A*-periodic if and only if*

$$\mathbf{\Phi}^k(T_A,0) = [\mathbf{\Phi}(T_A,0)\mathbf{Y}]^k\,; \tag{4.13b}$$

- kT_A*-antiperiodic if and only if*

$$\mathbf{\Phi}^k(T_A,0) = -\,[\mathbf{\Phi}(T_A,0)\mathbf{Y}]^k\,. \tag{4.13c}$$

Any $\mathbf{Y}$ *satisfying the condition of this theorem will be henceforth called a* Yakubovich matrix. *It is always possible to find* $\mathbf{Y}$ *such that* $\mathbf{L}_{F_Y}(\cdot)$ *is* $2T_A$*-periodic.*

Note 4.3.1. $\mathbf{Y}$ can be constructed using the Jordan decomposition of $\mathbf{\Phi}(T_A,0)$ as illustrated in the proof of Corollary 4.3.2.

Theorem 4.3.4 (Converse). *Let* $k \in \mathbb{Z}^{+*}, \mathbf{L}(\cdot) \in \mathcal{L}^{\mathbb{R}}_{kT_A}$ *and* $\mathbf{F} \in \mathbb{R}^{n\times n}$. *Then*

$$\mathbf{\Phi}(t,0) \triangleq \mathbf{L}(t)\exp(t\mathbf{F}) \tag{4.14a}$$

is the STM of some system of the form (3.1) where $\mathbf{A}(\cdot) \in \mathcal{A}_{T_A}$ *if*

$$\mathbf{\Phi}^k(T_A,0) = [\mathbf{\Phi}(T_A,0)\mathbf{Y}]^k \tag{4.14b}$$

and $\mathbf{Y}$ *is a Yakubovich matrix.*

Example 4.3.1. Building on Example 4.1.1, the value of the monodromy matrix (cf. Eq. (4.1b)) is

$$\mathbf{\Phi}(T_A,0) = \exp\left(\frac{2a\pi}{\omega}\right)\mathbf{I}. \tag{4.15a}$$

Now consider the matrix

$$\mathbf{Y}_{F_2} \triangleq \gamma\begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}, \quad \gamma = \exp\left(\frac{\omega - 2a\pi}{\omega}\right), \tag{4.15b}$$

and verify that

$$\mathbf{\Phi}^3(T_A,0) = \left[\mathbf{\Phi}(T_A,0)\mathbf{Y}_{F_2}\right]^3. \tag{4.15c}$$

Define

$$\mathbf{F}_2 \stackrel{\Delta}{=} \log\left(\mathbf{Y}_{F_2}\mathbf{\Phi}(T_A,0)\right) = \begin{bmatrix} 1 & \omega/3 \\ -\omega/3 & 1 \end{bmatrix}, \text{and} \tag{4.15d}$$

$$\mathbf{L}_{F_2}(t) \stackrel{\Delta}{=} \mathbf{\Phi}(t,0)\exp(-t\mathbf{F}_2). \tag{4.15e}$$

Theorem 4.3.3 and Eq. (4.15c) indicate that $\mathbf{L}_{F_2}(\cdot)$ is $3T_A$ periodic. We can further notice that

$$\mathbf{\Phi}(T_A,0) \neq \left[\mathbf{\Phi}(T_A,0)\mathbf{Y}_{F_2}\right], \quad \mathbf{\Phi}^2(T_A,0) \neq \left[\mathbf{\Phi}(T_A,0)\mathbf{Y}_{F_2}\right]^2. \tag{4.15f}$$

In other words, $\mathbf{L}_{F_2}(\cdot)$ is $3T_A$ periodic without being either T_A- or $2T_A$-periodic, and thus is primarily $3T_A$-periodic (cf. Fig. 4.1, p.36).

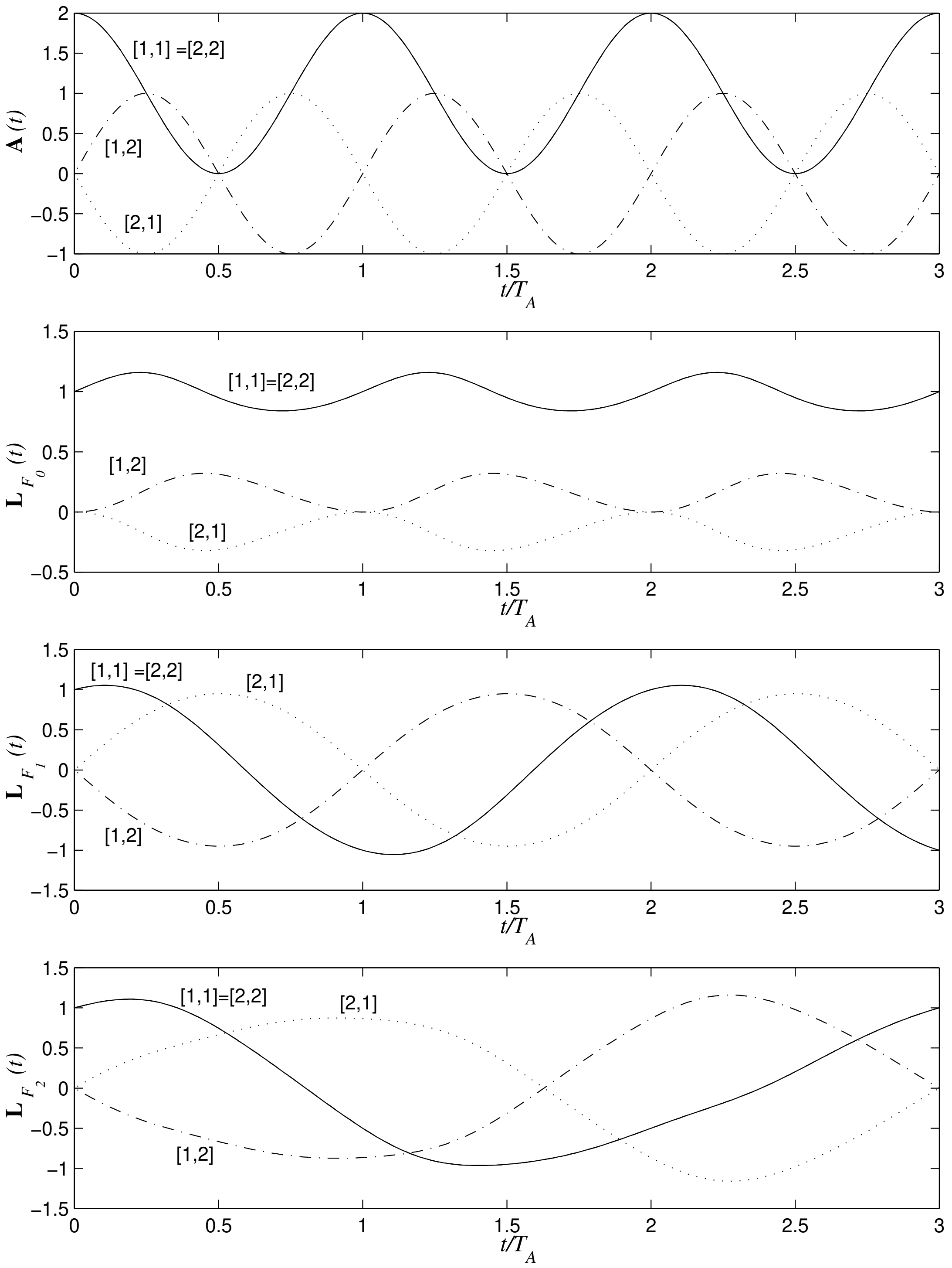


Figure 4.1: Periodic matrix entries in Examples 4.1.1 and 4.3.1: $[i,j]$ denotes the (i,j) entry of the matrix.

CHAPTER 5

Derivation of the Floquet Factors

In this chapter we review approaches for the derivation of the Floquet factors $\mathbf{L}(\cdot)$ and $\mathbf{F}$ of a given LTP system of the form (3.1). Bellman (1970b) best summarized the issues involved when he deemed the representation provided by Floquet-Lyapunov theory "an elegant and important one but not easy to use for quantitative purposes. There is no readily available way of determining [it]."

Existing methods consist of two steps and are based on the initial computation of the STM followed by the derivation of its Floquet factors. They were until now the only approach available; we describe them in Section 5.1. We then propose a novel method which follows the converse approach.

5.1. Two-Step Methods

With the STM being known, a constant matrix $\mathbf{F}$ can be derived using the results of the previous chapter. $\mathbf{L}(\cdot)$ is then obtained from

$$\mathbf{L}(t) \triangleq \mathbf{\Phi}(t,0)\exp(-t\mathbf{F}). \tag{5.1}$$

$\mathbf{\Phi}(\cdot,0)$ is known in closed form only in specific cases (Erugin 1966, §4-5), for instance when the equation is commutative (Lukes 1982, §8.4). In the general case, the STM has to be computed numerically. Knowledge of $\mathbf{\Phi}(\cdot,0)$ is only required over the interval $(0,T_A)$, since other values may be obtained from Eq. (3.4).

5.1.1. Direct Numerical Integration A popular technique is based on the integration of the differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \quad \mathbf{x}(\cdot) \in \mathbb{R}^n \tag{5.2a}$$

for n different initial conditions

$$\mathbf{x}_k(0) = \mathbf{e}_k, \quad k \in \{1, \cdots, n\}, \tag{5.2b}$$

where $\mathbf{e}_k = [\delta_{ki}]$ is a vector of dimension n with $(n-1)$ zero entries and the k-th entry equal to one. Let $\mathbf{x}_i(t),\ i \in \{1, \cdots, n\}$ be the n independent solutions obtained for each initial condition. Then, the STM is

$$\mathbf{\Phi}(t, 0) \triangleq [\mathbf{x}_1(t)|\mathbf{x}_2(t)| \cdots |\mathbf{x}_n(t)] \,. \tag{5.2c}$$

Gaonkar, Prasad, and Sastry (1981) provide a comparison of the efficiency of different integration schemes.

5.1.2. Spectral Methods An alternative to direct numerical integration schemes consists in computing the expansion of the STM on a spectral basis $\boldsymbol{\sigma}(\cdot)$ of order m, $\boldsymbol{\sigma}(\cdot) = \begin{bmatrix}\sigma_0(\cdot) & \sigma_1(\cdot) & \cdots & \sigma_{m-1}(\cdot)\end{bmatrix}^T$, which could be, for instance, a Fourier basis or a polynomial basis (e.g. Chang, Yang, and Wang 1986; Wu 1991). The expansion $\mathbf{C}_\Phi$ of the STM is obtained by solving a linear system

$$\mathbf{M}\mathbf{C}_\Phi = \mathbf{N}, \tag{5.3}$$

where $\mathbf{M}$ and $\mathbf{N}$ are known constant matrices based on the spectral expansion of $\mathbf{A}(\cdot)$. This procedure was reviewed by Montagnier, Spiteri, and Angeles (1998).

5.2. Direct Derivation

The two previous approaches are based on an initial-value problem (IVP) formulation. There is no guarantee that solutions will remain independent after numerical integration. On the other hand, it is well known (Ascher, Mattheij, and Russell 1988;

Keller 1976) that the solution to a boundary-value problem (BVP) is better conditioned than the solution to an IVP. This leads us to consider the following BVP formulation.

5.2.1. BVP Formulation Recalling Eq. (3.7d), $\mathbf{L}(\cdot)$ satisfies the following first-order matrix differential equation:

$$\forall t \in \mathbb{R}^+, \quad \dot{\mathbf{L}}(t) = \mathbf{A}(t)\mathbf{L}(t) - \mathbf{L}(t)\mathbf{F}. \tag{5.4a}$$

Moreover, we can add the boundary conditions (BCs) using Corollary 3.2.2:

$$\mathbf{L}(0) = \mathbf{L}(2T_A) = \mathbf{I}. \tag{5.4b}$$

Finally, as $\mathbf{F}$ is constant, we can add the equation

$$\dot{\mathbf{F}} = \mathbf{0} \tag{5.4c}$$

so that the problem is *well defined.*

Solving this system for $\mathbf{L}(\cdot)$ and $\mathbf{F}$ is done by using a standard BVP sofware package, e.g. `COLDAE`, developed by Ascher and Spiteri (1994), or `bvp4c` in `MATLAB`™ (Kierzenka 1998; Kierzenka and Shampine 2001).

NOTE 5.2.1. As Eq. (5.4a) is not linear, a numerical solution first requires providing an initial guess. In our approach, we have opted for producing this initial guess with a zero-order-hold-like algorithm (Friedmann, Hammond, and Woo 1977; Hsu 1974).

The main idea lies in discretizing the period in a sequence of intervals and approximating the initial LTP system by a LTI system on each of these intervals. Hsu (1974) showed that the monodromy matrix of the discretized system always converges to that of the LTP system as the interval length diminishes.

NOTE 5.2.2. A BC at $t = T_A$ rather than $t = 2T_A$ can be derived using a Yakubovich matrix $\mathbf{Y}$ introduced in Theorem 4.3.3: Equation (5.4b) becomes

$$\mathbf{L}(0) = \mathbf{I}, \quad \mathbf{L}(T_A) = \mathbf{Y}^{-1}. \tag{5.4d}$$

Independent of its periodicity, values of $\mathbf{L}(\cdot)$ outside $(0, T_A)$ are obtained via Eq. (4.11c), repeated below:

$$\forall (t,k) \in \mathbb{R}^+ \times \mathbb{Z}^{+*},$$
$$\mathbf{L}(t + kT_A) = \mathbf{L}(t) \exp(t\mathbf{F}) \boldsymbol{\Phi}^{k-1}(T_A, 0) \mathbf{L}(T_A) \exp[-((k-1)T_A + t)\mathbf{F}]. \tag{5.4e}$$

5.2.2. Obtaining $\mathbf{L}^{-1}(.)$ Matrix $\mathbf{L}^{-1}(\cdot)$ plays a major role in the applications of the theory. Inverting $\mathbf{L}(\cdot)$ at each desired value may be computationally expensive and may lead to round-off error amplification. A better approach resorts to a BVP formulation for $\mathbf{L}^{-1}(\cdot)$ that mirrors Eq. (5.4b). Differentiating the identity

$$\forall t \in \mathbb{R}^+, \quad \mathbf{L}(t)\mathbf{L}^{-1}(t) = \mathbf{I}, \tag{5.5}$$

and using the expression of $\dot{\mathbf{L}}(\cdot)$ given in Eq. (5.4a) leads to the new BVP:

$$\forall t \in \mathbb{R}^+, \quad \dot{\mathbf{L}}^{-1}(t) = -\mathbf{L}^{-1}(t)\mathbf{A}(t) + \mathbf{F}\mathbf{L}^{-1}(t), \tag{5.6a}$$

$$\mathbf{L}^{-1}(0) = \mathbf{I}, \quad \mathbf{L}^{-1}(T_A) = \mathbf{Y}, \tag{5.6b}$$

$$\dot{\mathbf{F}} = \mathbf{0}. \tag{5.6c}$$

CHAPTER 6

Gramian-Based Controller

6.1. Review of Previous Work

When designing feedback controllers, periodic systems have often only been considered as general time-varying systems. The rationale behind those controllers was to take advantage of intrinsic properties of each particular system (e.g., Fisher and Bryson Jr. 1995; Friedmann 1983; Ilić-Spong and Mak 1986; Ilić-Spong, Miller, MacMinn, and Thorp 1985; Kern 1980), rather than trying to develop a general framework based upon Floquet-Lyapunov theory. The few instances of the latter approach have mainly focused on the classical control method of invariant assignment: It helps build a better understanding of the issues at stake in Floquet-Lyapunov theory by relying upon stability concepts which are common to both time-periodic systems and time-invariant systems (cf. Sections 3.3 and 3.4), since the eigenvalues of a LTI system are also its Floquet exponents.

Assigning the invariants of linear time-invariant systems is widely described in the literature, and was also achieved for discrete-time LTP systems (e.g., Willems, Kučera, and Brunovský 1984). However, the same task has proved far more challenging in the case of continuous-time LTP systems. The different attempts usually fall into two main categories, whether the (continuous-time) feedback is based on the discrete or continuous state.

6.1.1. Discrete-State Feedback Brunovský (1969) proposed a feedback control consisting of impulsive feedbacks of the instantaneous state at n suitably chosen times within a period. The limitations of this approach were discussed by Willems et al. (1984). Kabamba (1986) introduced the concept of "sampled state periodic hold" and gave an explicit expression for a piecewise continuous periodic feedback. He showed that when the controllability index is less than n, the whole monodromy matrix can be assigned. Al-Rahmani and Franklin (1989) adopted a similar framework and, using *generalized reachability Gramians*, designed a periodic piecewise constant feedback control to assign the Floquet multipliers arbitrarily, and under conditions similar to those first derived by Kabamba (1986), the whole monodromy matrix.

6.1.2. Continuous-State Feedback The task of producing a continuous-state periodic feedback (CPF) control has been arduous. Attempts can be separated into two categories, whether

(i) the controller is built based on the transformed system of Note 3.2.2, viz.

$$\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t) + \mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{u}(t), \tag{6.1}$$

and the Floquet exponents are assigned by a feedback $\mathbf{u}(t) = -\mathbf{K}(t)\mathbf{z}(t)$ (Calico and Wiesel 1984, 1986; Joseph 1993); or,

(ii) one considers the closed-loop system

$$\dot{\mathbf{x}}(t) = [\mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}(t)]\,\mathbf{x}(t) \tag{6.2}$$

and Theorem 3.2.1 (Representation Theorem) is invoked to find $\mathbf{K}(t)$ (Kern 1980, 1986; Laptinsky 1988).

6.1.2.1. *Calico and Wiesel* Calico and Wiesel (1984, 1986) produced a "modal control" based on Eq. (6.1) which can move Floquet exponents one at a time. Placing complex-conjugate pairs is problematic, the case of exponents appearing in higher-order Jordan blocks was not discussed, and the authors later noted (Webb, Calico, and Wiesel 1991) numerical instabilities in their scheme. Despite its limitations, their approach offered a lot of insight into the problem and provided solutions to simple cases.

6.1.2.2. *Joseph* Joseph (1993) (see also Sinha and Joseph 1994) tried to overcome the periodicity of the input matrix $\mathbf{L}^{-1}(t)\mathbf{B}(t)$ in Eq. (6.1) by introducing an "auxiliary" LTI system which would asymptotically converge towards the original LTP system. Unfortunately, this scheme was based on a misuse of the pseudo-inverse of matrix $\mathbf{B}(t)$ and of some results from nonlinear analysis. Moreover, Montagnier, Spiteri, and Angeles (2001, cf. App. A) proved that stability of the auxiliary system did not guarantee stability of the LTP system. Lee and Balas (1999) proposed to repair the feedback design, but failed to notice that their new scheme was based on the knowledge of the closed-form of the STM, which is not available in the general case for LTP systems.

6.1.2.3. *Kern and Laptinsky* These schemes are based on the system represented by Eq. (6.2), which is itself a Floquet system and has a pair, initially unknown, of Floquet factors $\mathbf{L}_K(\cdot)$ and $\mathbf{F}_K$. Kern (1980, 1986) established a differential system of nonlinear equations in $\mathbf{L}_K(\cdot)$ and $\mathbf{K}(\cdot)$. Having assigned the constant Floquet factor $\mathbf{F}_K$ for the closed-loop system, one solved first for the corresponding Lyapunov transformation $\mathbf{L}_K(\cdot)$ and then for $\mathbf{K}(\cdot)$. Laptinsky (1988), using results from Zoubov (1978), adopted a similar approach in which the feedback matrix $\mathbf{K}(\cdot)$ was assumed to be the sum of a periodic $n \times n$ matrix $\tilde{\mathbf{K}}(.)$ and a constant $n \times n$ matrix $\bar{\mathbf{K}}$. Unfortunately, both schemes are impractical when the input matrix $\mathbf{B}(t)$ is not a square matrix of rank n, which is the most common case.

6.2. Motivation

Willems, Kučera, and Brunovský (1984) stressed the fundamental restrictions of LTI feedback control systems: Eigenvalues can be assigned, but not their algebraic and geometric multiplicities, nor their eigenvectors. In the same paper, the authors conjectured that a periodic time-varying feedback would be able to relax these design restrictions. They provided an example of a completely unrestricted assignment of the invariant factors (i.e. eigenvectors and eigenvalues) for discrete-time periodic systems, but only partial results were obtained in the continous-time case.

We introduce below a novel continuous-time, continuous-state periodic controller which can assign the whole monodromy matrix, in other words both its eigenvalues and eigenvectors. The technique can be applied to any system since it only relies upon intrinsic properties derived from the Floquet-Lyapunov theory.

The approach builds on the works of Kern (1980) and Laptinsky (1988). However, whereas their work was dependent on the invertibility of the input matrix $\mathbf{B}(\cdot)$, we base our design on the invertibility of the Gramian, which is guaranteed from the controllability of the system. Properties of linear state-feedback strategies based on the Gramian have been previously investigated for LTI systems (Pearson and Kwon 1976), and, more recently, for LTP systems (Nicolao and Strada 1997). This new scheme can be put in parallel with the results obtained by Al-Rahmani and Franklin (1989) with discrete-state continous-time controllers.

6.3. Novel Formulation

Consider the LTP system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \tag{6.3}$$

and let ν be the controllability index of the pair $\{\mathbf{\Phi}(T_A, 0), \mathbf{W}_R(T_A, 0)\}$, where $\mathbf{W}_R(\cdot, 0)$ was defined in Eq. (2.3b). From Theorem 3.5.1, the system is controllable if and only if

$$\det \mathbf{W}_R(\nu T_A, 0) \neq 0. \tag{6.4}$$

If this is the case, Theorem 2.2.3 leads to

$$\forall t > \nu T_A, \quad \det \mathbf{W}_R(t, 0) \neq 0. \tag{6.5}$$

Choosing a CPF control law of the form

$$\mathbf{u}(t) = -\mathbf{K}(t)\mathbf{x}(t), \qquad \mathbf{K}(t + pT_A) = \mathbf{K}(t), \qquad p \in \mathbb{Z}^+, \quad p \geq \nu, \tag{6.6}$$

the closed-loop system becomes (cf. Fig. 6.1)

$$\dot{\mathbf{x}}(t) = \mathbf{A}_K(t)\mathbf{x}(t), \qquad \mathbf{A}_K(t) = \mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}(t). \tag{6.7}$$

Because this system is T_{A_K}-periodic ($T_{A_K} = pT_A$), we can apply the Representation

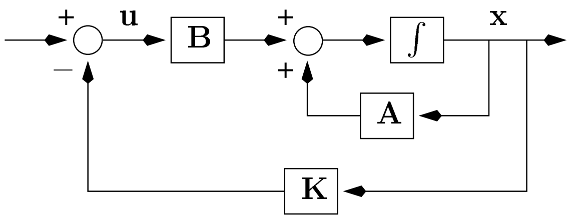


Figure 6.1: Closed-loop system

Theorem 3.2.1 and its state-transition matrix $\mathbf{\Phi}_K(t, 0)$ is of the form

$$\forall t \in \mathbb{R}^+, \quad \mathbf{\Phi}_K(t, 0) \triangleq \mathbf{L}_K(t) \exp(t\mathbf{F}_K), \tag{6.8}$$

where $\mathbf{L}_K(\cdot) \in \mathcal{L}^{\mathbb{R}}_{kT_{A_K}}$ $(k \in \mathbb{Z}^{+*})$ and $\mathbf{F}_K \in \mathbb{R}^{n\times n}$. Recalling the results of Chapter 4 and Theorem 4.3.3, the BCs imposed on $\mathbf{L}_K(\cdot)$ are

$$\mathbf{L}_K(0) = \mathbf{I}, \quad \mathbf{L}_K(T_{A_K}) = \mathbf{Y}_K^{-1}, \tag{6.9a}$$

for some Yakubovich matrix $\mathbf{Y}_K$ that determines the periodicity T_{L_K} of $\mathbf{L}_K(\cdot)$. On the other hand, matrix $\mathbf{L}_K(\cdot)$ satisfies Eq. (3.7d), which here becomes:

$$\forall t \in \mathbb{R}^+, \quad \dot{\mathbf{L}}_K(t) = \mathbf{A}_K(t)\mathbf{L}_K(t) - \mathbf{L}_K(t)\mathbf{F}_K. \tag{6.9b}$$

Note 6.3.1. At this point, it is important to stress the correspondence in the formulation with §5.2. When considering the open-loop problem, the system matrix $\mathbf{A}(\cdot)$ is usually known but its STM and Floquet factors are not. Here, we have the converse objective: Given a desired STM, determine first the Floquet factors and then obtain the feedback matrix $\mathbf{K}(\cdot)$ and, thus, $\mathbf{A}_K(\cdot) = \mathbf{A}(\cdot) - \mathbf{B}(\cdot)\mathbf{K}(\cdot)$.

NOTE 6.3.2. This approach is a straight application of the converse statements in Theorems. 3.1.1 and 3.2.1 limited to the real field thanks to the new results of Theorem 4.3.3. One should note that the canonical Floquet theory assumes the system matrix $\mathbf{A}(\cdot)$ to be piecewise continuous in order to obtain those converse statements. Therefore one should expect the feedback matrix $\mathbf{K}(\cdot)$ to be piecewise continuous as well.

6.4. T_{A_K}-Periodic Feedback

We provide below a proof for a lemma first proposed by Laptinsky (1988).

LEMMA 6.4.1. $\mathbf{L}_K(\cdot)$ *is the solution of Eq. (6.9b) if and only if* $\mathbf{L}_K(\cdot)$ *satisfies*

$$\forall t \in \mathbb{R}^+, \quad \mathbf{L}_K(t) = \mathbf{\Phi}(t,0) \left\{ \mathbf{I} - \int_0^t \mathbf{\Phi}(0,\tau) \left[\mathbf{B}(\tau)\mathbf{K}(\tau)\mathbf{L}_K(\tau) + \mathbf{L}_K(\tau)\mathbf{F}_K \right] d\tau \right\}. \tag{6.10}$$

PROOF: From Eq. (6.9b), we have

$$\forall t \in \mathbb{R}^+, \qquad \dot{\mathbf{L}}_K(t) = \left[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}(t)\right] \mathbf{L}_K(t) - \mathbf{L}_K(t)\mathbf{F}_K. \tag{6.11a}$$

Recalling Def. 2.2.3, $\mathbf{\Phi}(\cdot, 0)$ satisfies

$$\forall t \in \mathbb{R}^+, \qquad \dot{\mathbf{\Phi}}(t,0) = \mathbf{A}(t)\mathbf{\Phi}(t,0). \tag{6.11b}$$

Noting also that the derivative of

$$\forall t \in \mathbb{R}^+, \qquad \mathbf{\Phi}^{-1}(t,0)\mathbf{\Phi}(t,0) = \mathbf{I} \tag{6.11c}$$

leads to

$$\forall t \in \mathbb{R}^+, \qquad \mathbf{\Phi}^{-1}(t,0)\dot{\mathbf{\Phi}}(t,0) + \dot{\mathbf{\Phi}}^{-1}(t,0)\mathbf{\Phi}(t,0) = 0, \tag{6.11d}$$

we can write

$$\forall t \in \mathbb{R}^+, \qquad \mathbf{\Phi}^{-1}(t,0)\mathbf{A}(t)\mathbf{\Phi}(t,0) + \dot{\mathbf{\Phi}}^{-1}(t,0)\mathbf{\Phi}(t,0) = 0. \tag{6.11e}$$

After right-multiplying Eq. (6.11e) by $\mathbf{\Phi}(0,t)$ and rearranging, we obtain

$$\forall t \in \mathbb{R}^+, \qquad \dot{\mathbf{\Phi}}(0,t) = -\mathbf{\Phi}(0,t)\mathbf{A}(t). \tag{6.11f}$$

Pre-multiplying Eq. (6.11a) by $\mathbf{\Phi}(0,t)$ yields

$$\begin{aligned} &\forall t \in \mathbb{R}^+, \\ \mathbf{\Phi}(0,t)\dot{\mathbf{L}}_K(t) &= \mathbf{\Phi}(0,t)\left\{\left[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}(t)\right]\mathbf{L}_K(t) - \mathbf{L}_K(t)\mathbf{F}_K\right\} \\ &= \mathbf{\Phi}(0,t)\mathbf{A}(t)\mathbf{L}_K(t) - \mathbf{\Phi}(0,t)\left[\mathbf{B}(t)\mathbf{K}(t)\mathbf{L}_K(t) + \mathbf{L}_K(t)\mathbf{F}_K\right]. \end{aligned} \tag{6.11g}$$

Using Eq. (6.11f) we obtain

$$\begin{aligned} &\forall t \in \mathbb{R}^+, \\ &\mathbf{\Phi}(0,t)\dot{\mathbf{L}}_K(t,0) = -\dot{\mathbf{\Phi}}(0,t)\mathbf{L}_K(t) - \mathbf{\Phi}(0,t)\left[\mathbf{B}(t)\mathbf{K}(t)\mathbf{L}_K(t) + \mathbf{L}_K(t)\mathbf{F}_K\right], \end{aligned} \tag{6.11h}$$

which can also be written as

$$\begin{aligned} &\forall t \in \mathbb{R}^+, \\ &\mathbf{\Phi}(0,t)\dot{\mathbf{L}}_K(t,0) + \dot{\mathbf{\Phi}}(0,t)\mathbf{L}_K(t) = -\mathbf{\Phi}(0,t)\left[\mathbf{B}(t)\mathbf{K}(t)\mathbf{L}_K(t) + \mathbf{L}_K(t)\mathbf{F}_K\right]. \end{aligned} \tag{6.11i}$$

Integration of this equation leads to Eq. (6.10).

Conversely, taking the derivative of Eq. (6.10) with respect to t results in Eq. (6.10).

□

Now, recalling the second BC in Eq. (6.9b), $\mathbf{L}_K(T_{A_K}) = \mathbf{Y}_K^{-1}$, Eq. (6.10) can be rewritten at $t = T_{A_K}$ as

$$\int_0^{T_{A_K}} \mathbf{\Phi}(0,\tau)\mathbf{B}(\tau)\mathbf{K}(\tau)\mathbf{L}_K(\tau)d\tau = -\left\{\int_0^{T_{A_K}} \mathbf{\Phi}(0,\tau)\mathbf{L}_K(\tau)d\tau\mathbf{F}_K + \mathbf{\Phi}(0,T_{A_K})\mathbf{Y}_K^{-1} - \mathbf{I}\right\}. \quad (6.12)$$

Multiplying both sides of the above equation by $\mathbf{\Phi}(T_{A_K},0)$ leads to

$$\int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K},\tau)\mathbf{B}(\tau)\mathbf{K}(\tau)\mathbf{L}_K(\tau)d\tau = -\left\{\int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K},\tau)\mathbf{L}_K(\tau)d\tau\mathbf{F}_K + \mathbf{Y}_K^{-1} - \mathbf{\Phi}(T_{A_K},0)\right\}. \quad (6.13)$$

Equation (6.13) leads to a new lemma:

LEMMA 6.4.2. *If $\mathbf{K}(\cdot)$ is a $r \times n$ T_{A_K}-periodic feedback matrix satisfying Eqs. (6.7) and (6.8), then there exists a constant matrix $\mathbf{K}_c \in \mathbb{R}^{n\times n}$ such that*

$$\forall t,\ 0 \le t < T_{A_K}, \qquad \mathbf{K}(t) = \mathbf{B}^T(t)\mathbf{\Phi}^T(T_{A_K},t)\mathbf{K}_c\mathbf{L}_K^{-1}(t), \quad (6.14a)$$

$$\forall k \in \mathbb{Z}^{+*}, \qquad \mathbf{K}(t + kT_{A_K}) = \mathbf{K}(t). \quad (6.14b)$$

NOTE 6.4.1. As anticipated in Note 6.3.2, $\mathbf{K}(\cdot)$ may have a discontinuity at $t = kT_{A_K}$.

PROOF: The existence of $\mathbf{L}_K(.)$ is known from Eqs. (6.7) and (6.8) and, since $\mathbf{L}_K(.) \in \mathcal{L}^{\mathbb{R}}_{k_1 T_{A_K}}$ $(k_1 \in \mathbb{Z}^{+*})$, $\mathbf{L}_K(t)$ is invertible for all t. Using $\mathbf{K}(t)$ given by Eq. (6.14a), Eq. (6.13) takes the form

$$\int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K},\tau)\mathbf{B}(\tau)\mathbf{B}^T(\tau)\mathbf{\Phi}^T(T_{A_K},\tau)d\tau\mathbf{K}_c = -\left\{\mathbf{Y}_K^{-1} - \mathbf{\Phi}(T_{A_K},0) + \int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K},\tau)\mathbf{L}_K(\tau)d\tau\mathbf{F}_K\right\}. \quad (6.15a)$$

Furthermore, noting that

$$\mathbf{W}_R(T_{A_K},0) = \int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K},\tau)\mathbf{B}(\tau)\mathbf{B}^T(\tau)\mathbf{\Phi}^T(T_{A_K},\tau)d\tau, \tag{6.15b}$$

and recalling that Eq. (6.5) leads to

$$\det \mathbf{W}_R(T_{A_K},0) \neq 0, \tag{6.15c}$$

Eq. (6.15a) can be rewritten as

$$\mathbf{K}_c = -\mathbf{W}_R{}^{-1}(T_{A_K},0)\left\{\mathbf{Y}_K^{-1} - \mathbf{\Phi}(T_{A_K},0) + \int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K},\tau)\mathbf{L}_K(\tau)d\tau\mathbf{F}_K\right\}, \tag{6.15d}$$

thus proving the existence of $\mathbf{K}_c$. □

Substituting Eq. (6.15d) for $\mathbf{K}_c$ into Eq. (6.14a) yields

$$\forall t,\ 0 \leq t < T_{A_K},\quad \mathbf{K}(t) = \Big[-\mathbf{B}^T(t)\mathbf{\Phi}^T(T_{A_K},t)\mathbf{W}_R{}^{-1}(T_{A_K},0)\big\{\mathbf{Y}_K^{-1} - \mathbf{\Phi}(T_{A_K},0) + \int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K},\tau)\mathbf{L}_K(\tau)d\tau\mathbf{F}_K\Big\}\Big]\mathbf{L}_K^{-1}(t). \tag{6.16}$$

In the next section we use Eq. (6.16) to obtain an integral equation.

6.5. Derivation of an Integral Equation for $\mathbf{L}_K(\cdot)$

THEOREM 6.5.1. $\{\mathbf{L}_K(\cdot),\mathbf{F}_K\}$, *with* $\mathbf{L}_K(0) = \mathbf{I}$, $\mathbf{L}_K(T_{A_K}) = \mathbf{Y}_K^{-1}$, *is a pair of Floquet factors of the closed-loop system (6.7) for* $\mathbf{K}(\cdot)$ *given in Eq. (6.14) if and only if*

$$\forall t,\ 0 \leq t < T_{A_K},$$

$$\mathbf{L}_K(t) = \left[\mathbf{W}_R(t,0)\mathbf{\Phi}^T(T_{A_K},t)\mathbf{W}_R{}^{-1}(T_{A_K},0)\int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K},\tau)\mathbf{L}_K(\tau)d\tau - \int_0^t \mathbf{\Phi}(t,\tau)\mathbf{L}_K(\tau)d\tau\right]\mathbf{F}_K + \mathbf{L}_{K_0}(t), \tag{6.17a}$$

where

$$\mathbf{L}_{K_0}(t) = \mathbf{\Phi}(t,0) + \mathbf{W}_R(t,0)\mathbf{\Phi}^T(T_{A_K},t){\mathbf{W}_R}^{-1}(T_{A_K},0)\left\{\mathbf{Y}_K^{-1} - \mathbf{\Phi}(T_{A_K},0)\right\} \quad (6.17b)$$

is the value of $\mathbf{L}_K(\cdot)$ *when* $\mathbf{F}_K = \mathbf{0}$.

PROOF (NECESSITY): Lemma 6.4.1 showed that if $\mathbf{L}_K(\cdot)$ is the solution of BVP (6.9), then $\mathbf{L}_K(\cdot)$ satisfies Eq. (6.10). Starting from the expression of $\mathbf{L}_K(\cdot)$ in Eq. (6.10),

$$\mathbf{L}_K(t) = \mathbf{\Phi}(t,0)\left\{\mathbf{I} - \int_0^t \mathbf{\Phi}(0,\tau)\left[\mathbf{B}(\tau)\mathbf{K}(\tau)\mathbf{L}_K(\tau) + \mathbf{L}_K(\tau)\mathbf{F}_K\right]d\tau\right\}, \quad (6.18a)$$

this can be rewritten as

$$\mathbf{L}_K(t) = \mathbf{L}_{K_1}(t) + \mathbf{L}_{K_2}(t) + \mathbf{L}_{K_3}(t), \quad (6.18b)$$

where

$$\mathbf{L}_{K_1}(t) \triangleq \mathbf{\Phi}(t,0), \quad (6.18c)$$

$$\mathbf{L}_{K_2}(t) \triangleq -\int_0^t \mathbf{\Phi}(t,\tau)\mathbf{B}(\tau)\mathbf{K}(\tau)\mathbf{L}_K(\tau)d\tau, \quad (6.18d)$$

and

$$\mathbf{L}_{K_3}(t) \triangleq -\int_0^t \mathbf{\Phi}(t,\tau)\mathbf{L}_K(\tau)d\tau\mathbf{F}_K. \quad (6.18e)$$

Upon consideration of $\mathbf{L}_{K_2}(t)$ using the expression for $\mathbf{K}(t)$ derived in Eq. (6.16), we obtain

$$\mathbf{L}_{K_2}(t) = -\int_0^t \mathbf{\Phi}(t,\tau)\mathbf{B}(\tau)\left[-\mathbf{B}^T(\tau)\mathbf{\Phi}^T(t,\tau)\mathbf{\Phi}^T(T_{A_K},t){\mathbf{W}_R}^{-1}(T_{A_K},0)\ \times\right.$$
$$\left.\left\{\mathbf{Y}_K^{-1} - \mathbf{\Phi}(T_{A_K},0) + \int_0^{T_{A_K}} \mathbf{\Phi}(pT,s)\mathbf{L}_K(s)ds\mathbf{F}_K\right\}\right]\mathbf{L}_K^{-1}(\tau)\mathbf{L}_K(\tau)d\tau. \quad (6.18f)$$

After simplification, this reduces to

$$\mathbf{L}_{K_2}(t) = \mathbf{W}_R(t,0)\mathbf{\Phi}^T(T_{A_K},t)\mathbf{W}_R{}^{-1}(T_{A_K},0) \times \left\{\mathbf{Y}_K^{-1} - \mathbf{\Phi}(T_{A_K},0) + \int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K},s)\mathbf{L}_K(s)ds\mathbf{F}_K\right\} \tag{6.18g}$$

$$= \mathbf{L}_{K\,2a}(t) + \mathbf{L}_{K\,2b}(t), \tag{6.18h}$$

where

$$\mathbf{L}_{K\,2a}(t) \stackrel{\triangle}{=} \mathbf{W}_R(t,0)\mathbf{\Phi}^T(T_{A_K},t)\mathbf{W}_R{}^{-1}(T_{A_K},0)\left\{\mathbf{Y}_K^{-1} - \mathbf{\Phi}(T_{A_K},0)\right\}, \tag{6.18i}$$

and

$$\mathbf{L}_{K\,2b}(t) \stackrel{\triangle}{=} \mathbf{W}_R(t,0)\mathbf{\Phi}^T(T_{A_K},t)\mathbf{W}_R{}^{-1}(T_{A_K},0)\int_0^{T_{A_K}} \mathbf{\Phi}(T_{A_K},s)\mathbf{L}_K(s,0)ds\ \mathbf{F}_K. \tag{6.18j}$$

We can now readily identify

$$\mathbf{L}_{K_0}(t) = \mathbf{L}_{K_1}(t) + \mathbf{L}_{K_{2a}}(t), \tag{6.18k}$$

$$\mathbf{L}_K(t) = \mathbf{L}_{K_{2b}}(t) + \mathbf{L}_{K_3}(t) + \mathbf{L}_{K_0}(t). \tag{6.18l}$$

□

Proof (Sufficiency): $\mathbf{L}_K(\cdot)$ given in Eq. (6.17) corresponds to $\mathbf{L}_K(\cdot)$ in Eq. (6.10) for $\mathbf{K}(\cdot)$ given in Eq. (6.16). Lemma 6.4.1 already proved the equivalence between Eqs. (6.9b) and (6.10) for any $\mathbf{K}(\cdot)$. We need to show that the BCs given in Eq. (6.9a) are satisfied for this choice of $\mathbf{K}(\cdot)$. At $t = 0$, recalling that $\mathbf{W}_R(0,0) = \mathbf{0}$, Eq. (6.17) simplifies to

$$\begin{aligned}\mathbf{L}_K(0) &= \mathbf{L}_{K_0}(0)\\ &= \mathbf{\Phi}(0,0)\\ &= \mathbf{I}.\end{aligned} \tag{6.19a}$$

On the other hand, at $t = T_{A_K}$, since $\mathbf{\Phi}(T_{A_K}, T_{A_K}) = \mathbf{I}$, Eq. (6.17) becomes

$$\begin{aligned}\mathbf{L}_K(T_{A_K}) &= \mathbf{L}_{K_0}(T_{A_K}) \\ &= \mathbf{\Phi}(T_{A_K}, 0) + \left\{\mathbf{Y}_K^{-1} - \mathbf{\Phi}(T_{A_K}, 0)\right\} \\ &= \mathbf{Y}_K^{-1}. \end{aligned} \tag{6.19b}$$

□

NOTE 6.5.1. Eq. (6.17) can be rewritten as an integral equation by noting that

$$\mathbf{L}_K(t, 0) = \int_0^{T_{A_K}} \mathbf{P}(t, \tau)\mathbf{L}_K(\tau) d\tau \mathbf{F}_K + \mathbf{L}_{K_0}(t), \tag{6.20}$$

where

$$\mathbf{P}(t,\tau) = \begin{cases} \mathbf{W}_R(t,0)\mathbf{\Phi}^T(T_{A_K},0)\mathbf{W}_R{}^{-1}(T_{A_K},0)\mathbf{\Phi}(T_{A_K},\tau) - \mathbf{\Phi}(t,\tau), \\ \qquad\qquad\qquad\qquad 0 \le \tau \le t \le T_{A_K}, \\ \mathbf{W}_R(t,0)\mathbf{\Phi}^T(T_{A_K},0)\mathbf{W}_R{}^{-1}(T_{A_K},0)\mathbf{\Phi}(T_{A_K},\tau), \\ \qquad\qquad\qquad\qquad 0 \le t \le \tau \le T_{A_K}. \end{cases}$$

NOTE 6.5.2. It is sufficient to determine $\mathbf{L}_K(\cdot)$ over $[0, T_{A_K}]$ since only values within this interval are used for $\mathbf{K}(\cdot)$. If desired, values outside this interval can then be obtained by invoking Eq. (4.11c).

NOTE 6.5.3. It is of course possible to use the controllability Gramian $\mathbf{W}_C(t_2, t_1)$ instead of the reachability Gramian $\mathbf{W}_R(t_2, t_1)$ in the equations. In this case, it can be verified that

$$
\begin{aligned}
&\forall t, \quad 0 \le t < T_{A_K}, \quad \forall k \in \mathbb{Z}^{+*} \\
&\mathbf{K}(t) = \mathbf{B}^T(t)\mathbf{\Phi}^T(0,t)\bar{\mathbf{K}}_c\mathbf{L}_K^{-1}(t), \quad \mathbf{K}(t + kT_{A_K}) = \mathbf{K}(t), && \text{(6.21a)} \\
&\bar{\mathbf{K}}_c = -{\mathbf{W}_C}^{-1}(T_{A_K},0)\left\{\mathbf{\Phi}(0,T_{A_K})\mathbf{Y}_K^{-1} - \mathbf{I} + \int_0^{T_{A_K}} \mathbf{\Phi}(0,\tau)\mathbf{L}_K(\tau)d\tau\mathbf{F}_K\right\}, && \text{(6.21b)} \\
&\mathbf{L}_K(t) = \mathbf{\Phi}(t,0)\left[\mathbf{W}_C(t,0){\mathbf{W}_C}^{-1}(T_{A_K},0)\int_0^{T_{A_K}} \mathbf{\Phi}(0,\tau)\mathbf{L}_K(\tau)d\tau\right. \\
&\qquad\qquad \left. - \int_0^t \mathbf{\Phi}(0,\tau)\mathbf{L}_K(\tau)d\tau\right]\mathbf{F}_K + \mathbf{L}_{K_0}(t), && \text{(6.21c)} \\
&\mathbf{L}_{K_0}(t) = \mathbf{\Phi}(t,0)\left[\mathbf{I} + \mathbf{W}_C(t,0){\mathbf{W}_C}^{-1}(T_{A_K},0)\left\{\mathbf{\Phi}(0,T_{A_K})\mathbf{Y}_K^{-1} - \mathbf{I}\right\}\right]. && \text{(6.21d)}
\end{aligned}
$$

6.6. Solving the Integral Equation

Although the solution of scalar integral equations has been investigated (e.g., Delves and Mohamed 1985, and references therein), to the best of our knowledge there are no standard techniques for solving matrix integral equations. In this section, we introduce an approach based on a spectral method.

Consider a spectral basis $\boldsymbol{\sigma}(\cdot)$ of order s defined over the interval $[0,1]$:

$$\boldsymbol{\sigma}(u) = \begin{bmatrix}\sigma_1(u) & \sigma_2(u) & \cdots & \sigma_s(u)\end{bmatrix}^T, \quad 0 \le u \le 1. \tag{6.22}$$

We introduce the changes of variable

$$t = vT_{A_K}, \quad \tau = uT_{A_K}, \quad 0 \le t,\tau \le T_{A_K}, \quad 0 \le u,v \le 1, \tag{6.23}$$

in order to express Eq. (6.17) over $[0,1]$:

$$
\begin{aligned}
&\mathbf{L}_K(vT_{A_K}) = T_{A_K}\left[\widetilde{\mathbf{W}}(v)\int_0^{T_{A_K}} \mathbf{\Phi}(0,uT_{A_K})\mathbf{L}_K(uT_{A_K})du\right. \\
&\qquad\qquad \left. - \int_0^v \mathbf{\Phi}(vT_{A_K},uT_{A_K})\mathbf{L}_K(uT_{A_K})du\right]\mathbf{F}_K + \mathbf{L}_{K_0}(vT_{A_K}), && \text{(6.24a)} \\
&\widetilde{\mathbf{W}}(v) \triangleq \mathbf{W}_R(vT_{A_K},0)\mathbf{\Phi}^T(T_{A_K},vT_{A_K}){\mathbf{W}_R}^{-1}(T_{A_K},0)\mathbf{\Phi}(T_{A_K},0). && \text{(6.24b)}
\end{aligned}
$$

Recalling Def. 2.3.2, let $\mathbf{l}(v)$ and $\mathbf{l}_{K_0}(v)$ be the n^2-dimensional vectors defined as:

$$\mathbf{l}(v) \triangleq \mathrm{vec}\left(\mathbf{L}_K(vT_{A_K})\right), \tag{6.25a}$$

$$\mathbf{l}_{K_0}(v) \triangleq \mathrm{vec}\left(\mathbf{L}_{K_0}(vT_{A_K})\right). \tag{6.25b}$$

Using Lemma 2.3.1, Eq. (6.24a) becomes a n^2-dimensional vector equation, namely,

$$\mathbf{l}(v) = T_{A_K}\left[\int_0^1 \mathbf{\Gamma}(v,u)\mathbf{l}(u)du - \int_0^v \mathbf{\Theta}(v,u)\mathbf{l}(u)du\right] + \mathbf{l}_{K_0}(v), \tag{6.26a}$$

$$\mathbf{\Gamma}(v,u) = \mathbf{F}_K^T \otimes \left(\widetilde{\mathbf{W}}(v)\mathbf{\Phi}(T_{A_K}, uT_{A_K})\right), \tag{6.26b}$$

$$\mathbf{\Theta}(v,u) = \mathbf{F}_K^T \otimes \mathbf{\Phi}(vT_{A_K}, uT_{A_K}). \tag{6.26c}$$

We will now expand this equation on the spectral basis $\boldsymbol{\sigma}(\cdot)$.

6.6.1. Expansion of $\mathbf{\Theta}$ The (p,q) entry of the $n^2 \times n^2$ matrix $\mathbf{\Theta}$ introduced in Eq. (6.26c) is such that

$$\Theta_{p,q}(v,u) = F_{j_0,i_0}\Phi_{i_1,j_1}(vT_{A_K}, uT_{A_K}) = F_{j_0,i_0}\sum_{k=1}^{n}\Phi_{i_1,k}(vT_{A_K},0)\Phi_{k,j_1}(0,uT_{A_K})$$
$$1 \le m,p,q \le n^2, \quad 1 \le i_0,i_1,j_0,j_1 \le n, \quad p = (i_1-1)n + i_0, \quad q = (j_1-1)n + j_0. \tag{6.27}$$

First, consider the expansions of $\Phi_{k,j_1}(0,uT_{A_K})$ and of $l_m(u)$, the m-th entry of $\mathbf{l}(u)$:

$$\Phi_{k,j_1}(0,uT_{A_K}) \approx \boldsymbol{\sigma}^T(u)\boldsymbol{\phi}^s_{k,j_1}(0,\cdot), \quad \boldsymbol{\phi}^s_{k,j_1}(0,\cdot) \in \mathbb{R}^s, \tag{6.28a}$$

$$l_m(u) \approx \boldsymbol{\sigma}^T(u)\mathbf{l}^s_m, \quad \mathbf{l}^s_m \in \mathbb{R}^s. \tag{6.28b}$$

This leads to the expansion of $\Theta_{p,q}(v,u)l_m(u)$ with respect to u :

$$\begin{aligned}\Theta_{p,q}(v,u)l_m(u) &\approx F_{j_0,i_0}\left[\boldsymbol{\sigma}^T(u)\sum_{k=1}^{n}\Phi_{i_1,k}(vT_{A_K},0)\boldsymbol{\phi}^s_{k,j_1}(0,\cdot)\right]\left[\boldsymbol{\sigma}^T(u)\mathbf{l}^s_m\right] \\ &= \boldsymbol{\sigma}^T(u)F_{j_0,i_0}\sum_{k=1}^{n}\Phi_{i_1,k}(vT_{A_K},0)\mathbf{Q}\left(\boldsymbol{\phi}^s_{k,j_1}(0,\cdot)\right)\mathbf{l}^s_m,\end{aligned} \tag{6.29}$$

where $\mathbf{Q}(\cdot)$ is the product matrix defined in Eq. (2.11). Using the integration matrix $\mathbf{G}$ defined in Eq. (2.10), the integral is approximated by

$$\int_0^v \Theta_{p,q}(v,u) l_m(u) du \approx \boldsymbol{\sigma}^T(v) F_{j_0,i_0} \mathbf{G}^T \sum_{k=1}^{n} \Phi_{i_1,k}(vT_{A_K},0) \mathbf{Q}\left(\boldsymbol{\phi}^s_{k,j_1}(0,\cdot)\right) \mathbf{l}^s_m, \quad (6.30)$$

which is a function of v. Taking into account the expansion of $\Phi_{i_1,k}(vT_{A_K},0)$:

$$\Phi_{i_1,k}(vT_{A_K},0) \approx \boldsymbol{\sigma}^T(v)\boldsymbol{\phi}^s_{i_1,k}(\cdot,0), \quad \boldsymbol{\phi}^s_{i_1,k}(\cdot,0) \in \mathbb{R}^s, \quad (6.31)$$

leads to

$$\begin{aligned}\int_0^v \Theta_{p,q}(v,u) l_m(u) du &\approx F_{j_0,i_0} \sum_{k=1}^{n} \left[\boldsymbol{\sigma}^T(v)\boldsymbol{\phi}^s_{i_1,k}(\cdot,0)\right] \boldsymbol{\sigma}^T(v) \mathbf{G}^T \mathbf{Q}\left(\boldsymbol{\phi}^s_{k,j_1}(0,\cdot)\right) \mathbf{l}^s_m \\ &= \boldsymbol{\sigma}^T(v) \mathbf{Q}_{\Theta_{p,q}} \mathbf{l}^s_m, \qquad (6.32a)\end{aligned}$$

where matrix $\mathbf{Q}_{\Theta_{p,q}} \in \mathbb{R}^{s\times s}$ is such that

$$\mathbf{Q}^{\Theta}_{p,q} = F_{j_0,i_0} \sum_{k=1}^{n} \mathbf{Q}\left(\boldsymbol{\phi}^s_{i_1,k}(\cdot,0)\right) \mathbf{G}^T \mathbf{Q}\left(\boldsymbol{\phi}^s_{k,j_1}(0,\cdot)\right). \quad (6.32b)$$

The expansion of the vector integral is of the form

$$\int_0^v \boldsymbol{\Theta}(v,u)\mathbf{l}(u)du \approx \left\{\mathbf{I}_{n^2} \otimes \boldsymbol{\sigma}^T(v)\right\} \mathbf{C}_{\Theta}\mathbf{l}^s, \quad (6.33a)$$

where $\mathbf{C}_{\Theta} \in \mathbb{R}^{n^2 s\times n^2 s}$ is a block matrix whose (p,q) block is $\mathbf{Q}_{\Theta_{p,q}}$:

$$\mathbf{C}_{\Theta} = \begin{bmatrix} \mathbf{Q}_{\Theta_{1,1}} & \mathbf{Q}_{\Theta_{1,2}} & \cdots & \mathbf{Q}_{\Theta_{1,n^2}} \\ \vdots & \vdots & & \vdots \\ \mathbf{Q}_{\Theta_{n^2,1}} & \mathbf{Q}_{\Theta_{n^2,2}} & \cdots & \mathbf{Q}_{\Theta_{n^2,n^2}} \end{bmatrix}. \quad (6.33b)$$

6.6.2. Expansion of $\boldsymbol{\Gamma}$ We follow a similar approach to expand Eq. (6.26b). The (p,q) entry of the $n^2 \times n^2$ matrix $\boldsymbol{\Gamma}$ is such that

$$\Gamma_{p,q}(v,u) = F_{j_0,i_0} \sum_{k=1}^{n} \widetilde{W}_{i_1,k}(v) \Phi_{k,j_1}(0, uT_{A_K})$$
$$1 \le m,p,q \le n^2, \quad 1 \le i_0, i_1, j_0, j_1 \le n, \quad p = (i_1-1)n + i_0, \quad q = (j_1-1)n + j_0. \tag{6.34}$$

Recalling the expansions of $\Phi_{k,j_1}(0, uT_{A_K})$ and of $l_m(u)$, defined in Eq. (6.28), we obtain the s-order expansion with respect to u :

$$\begin{aligned}\Gamma_{p,q}(v,u) l_m(u) &\approx F_{j_0,i_0} \boldsymbol{\sigma}^T(u) \left[\sum_{k=1}^{n} \widetilde{W}_{i_1,k}(v) \boldsymbol{\phi}^s_{k,j_1}(0,\cdot) \right] \left[\boldsymbol{\sigma}^T(u) \mathbf{l}^s_m \right] \\ &= \boldsymbol{\sigma}^T(u) F_{j_0,i_0} \sum_{k=1}^{n} \widetilde{W}_{i_1,k}(v) \mathbf{Q} \left(\boldsymbol{\phi}^s_{k,j_1}(0,\cdot) \right) \mathbf{l}^s_m. \end{aligned} \tag{6.35}$$

The integral of Eq. (6.35) with respect to u thus becomes

$$\int_0^1 \Gamma_{p,q}(v,u) l_m(u) du \approx F_{j_0,i_0} \sum_{k=1}^{n} \widetilde{W}_{i_1,k}(v) \boldsymbol{\sigma}^T(1) \mathbf{G}^T \mathbf{Q} \left(\boldsymbol{\phi}^s_{k,j_1}(0,\cdot)) \right) \mathbf{l}^s_m, \tag{6.36}$$

which is a scalar function of v. Now, considering the expansion of $\widetilde{W}_{i_1,k}(v)$ with respect to v,

$$\widetilde{W}_{i_1,k}(v) \approx \boldsymbol{\sigma}^T(v) \widetilde{\mathbf{w}}^s_{i_1,k}, \tag{6.37}$$

we obtain:

$$\int_0^1 \Gamma_{p,q}(v,u) l_m(u) du \approx \boldsymbol{\sigma}^T(v) \mathbf{Q}_{\Gamma_{p,q}} \mathbf{l}^s_m, \tag{6.38a}$$

$$\mathbf{Q}_{\Gamma_{p,q}} = F_{j_0,i_0} \sum_{k=1}^{n} \widetilde{\mathbf{w}}^s_{i_1,k} \boldsymbol{\sigma}^T(1) \mathbf{G}^T \mathbf{Q} \left(\boldsymbol{\phi}^s_{k,j_1}(0,\cdot) \right). \tag{6.38b}$$

Finally, the expansion of the vector integral is

$$\int_0^1 \boldsymbol{\Gamma}(v,u) \mathbf{l}(u) du \approx \left\{ \mathbf{I}_{n^2} \otimes \boldsymbol{\sigma}^T(v) \right\} \mathbf{C}_\Gamma \mathbf{l}^s, \tag{6.39}$$

where $\mathbf{C}_\Gamma \in \mathbb{R}^{n^2 s \times n^2 s}$ is a block matrix whose (p,q) block is $\mathbf{Q}_{\Gamma_{p,q}}$.

6.6.3. Solving Taking into account the expansion of $\mathbf{l}_{K_0}(\cdot)$,

$$\mathbf{l}_{K_0}(v) = \left\{\mathbf{I}_{n^2} \otimes \boldsymbol{\sigma}^T(v)\right\} \mathbf{l}^s_{K_0}, \quad \mathbf{l}^s_{K_0} \in \mathbb{R}^{n^2 r}, \tag{6.40}$$

and using Eqs. (6.33) and (6.39) from the two previous subsections, Eq. (6.24a) can be approximated by

$$\left\{\mathbf{I}_{n^2} \otimes \boldsymbol{\sigma}^T(v)\right\} \mathbf{l}^s = \left\{\mathbf{I}_{n^2} \otimes \boldsymbol{\sigma}^T(v)\right\} \left[T_{A_K}\left(\mathbf{C}_\Gamma - \mathbf{C}_\Theta\right) \mathbf{l}^s + \mathbf{l}^s_{K_0}\right]. \tag{6.41}$$

The expansion coefficients of the entries of $\mathbf{L}_K(vT_{A_K})$ contained in $\mathbf{l}^s$ are the solution of the $n^2 s \times n^2 s$ linear system

$$\left[\mathbf{I}_{n^2 s} - T_{A_K}\left(\mathbf{C}_\Gamma - \mathbf{C}_\Theta\right)\right] \mathbf{l}^s = \mathbf{l}^s_{K_0} \tag{6.42}$$

which can be obtained using standard techniques (e.g. Golub and Van Loan 1996).

6.7. Design Rationale

6.7.1. Summary The synthesis of a controller for a LTP system described in the previous sections can be performed in six steps:

(i) Assign a new period T_{A_K} for the closed loop-system such that

$$\det \mathbf{W}_R(T_{A_K}, 0) \neq 0; \tag{6.43}$$

(ii) Assign the desired monodromy matrix $\boldsymbol{\Phi}_K(T_{A_K}, 0)$ of the closed-loop system satisfying the relation (cf. Note 3.5.1)

$$\det \boldsymbol{\Phi}_K(T_{A_K}, 0) > 0; \tag{6.44}$$

(iii) Choose a pair $\{\mathbf{Y}_K, \mathbf{F}_K\}$ such that

$$\mathbf{Y}_K \boldsymbol{\Phi}(T_{A_K}, 0) = \exp(T_{A_K} \mathbf{F}_K); \tag{6.45}$$

(iv) Solve Eq. (6.24a) for $\mathbf{L}_K(\cdot)$;

(v) Compute constant matrix $\mathbf{K}_c$ from Eq. (6.15d);

(vi) Derive the feedback matrix $\mathbf{K}(t)$ from Eq. (6.14).

NOTE 6.7.1. Should the condition $\det \mathbf{\Phi}_K(T_{A_K}, 0) > 0$ not be satisfied, a spurious solution $\mathbf{L}_K(\cdot)$ of the integral equation will still exist but will be singular for some t_s. The matrix $\mathbf{K}(\cdot)$ will not be defined at t_s, and thus the closed-loop system cannot be synthesized.

6.7.2. Computing $\mathbf{L}_K^{-1}(\cdot)$ If desired, the spectral solution of §6.6 can be used as an initial guess to a BVP approach similar to the open-loop case discussed in §5.2. Noting that $\mathbf{K}(\cdot)$ makes only use of $\mathbf{L}_K(\cdot)^{-1}$, it is computationally more efficient to obtain the inverse directly: For the closed-loop system, Eq. (5.6) becomes

$$\forall t \in \mathbb{R}^+,$$

$$\dot{\mathbf{L}}_K^{-1}(t) = -\mathbf{L}_K^{-1}(t)\mathbf{A}_K(t) + \mathbf{F}_K\mathbf{L}_K^{-1}(t), \tag{6.46a}$$

$$\mathbf{L}_K^{-1}(0) = \mathbf{I}, \quad \mathbf{L}_K^{-1}(T_{A_K}) = \mathbf{Y}_K, \tag{6.46b}$$

$$\dot{\mathbf{K}}_c = 0. \tag{6.46c}$$

Note that $\mathbf{F}_K$ has been assigned and we are now solving for the constant matrix $\mathbf{K}_c$ in the BVP. Recalling Eqs. (6.7) and (6.14a), we finally obtain

$$\forall t \in \mathbb{R}^+,$$

$$\dot{\mathbf{L}}_K^{-1}(t) = -\mathbf{L}_K^{-1}(t)\left[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{B}^T(t)\mathbf{\Phi}^T(T_{A_K}, t)\mathbf{K}_c\mathbf{L}_K^{-1}(t)\right] + \mathbf{F}_K\mathbf{L}_K^{-1}(t), \tag{6.47a}$$

$$\mathbf{L}_K^{-1}(0) = \mathbf{I}, \quad \mathbf{L}_K^{-1}(T_{A_K}) = \mathbf{Y}_K, \tag{6.47b}$$

$$\dot{\mathbf{K}}_c = 0. \tag{6.47c}$$

6.7.3. Feedback Synthesis on the Transformed System The synthesis of the controller can alternatively be performed on the transformed system

$$\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t) + \mathbf{L}^{-1}\mathbf{B}(t)\mathbf{u}(t), \tag{6.48}$$

where $\{\mathbf{L}(\cdot), \mathbf{F}\}$ is a pair of Floquet factors of the STM $\mathbf{\Phi}(\cdot, 0)$. Using the approach described in §6.7.1, we can obtain a gain matrix $\mathbf{K}_z(\cdot)$ which can assign the monodromy matrix of

$$\dot{\mathbf{z}}(t) = \left[\mathbf{F} - \mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{K}_z(t)\right] \mathbf{z}(t). \tag{6.49}$$

Using the change of variables $\mathbf{x}(t) = \mathbf{L}(t)\mathbf{z}(t)$, the original system becomes

$$\dot{\mathbf{x}}(t) = \left[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}(t)\right] \mathbf{x}(t), \quad \mathbf{K}(t) = \mathbf{K}_z(t)\mathbf{L}^{-1}(t). \tag{6.50}$$

While there is no change in the assignment of the invariants of the resulting closed-loop system, there is a computational advantage to this technique:

(i) when finding the spectral solution for $\mathbf{L}_K(\cdot)$;

(ii) when assigning only the unstable multipliers of the closed-loop system.

6.7.3.1. *Efficient Expansions* Finding the spectral expansion of $\mathbf{L}_K(\cdot)$ requires, among others, expanding $\mathbf{\Phi}(\cdot, 0)$ and $\mathbf{\Phi}(0, \cdot)$. These expansions can be obtained more efficiently for the STM $\mathbf{\Psi}(t, 0) \triangleq \exp(t\mathbf{F})$ of the transformed system, since matrix $\mathbf{F}$ is constant. Note that

$$\mathbf{F} = \left\{\mathbf{I} \otimes \boldsymbol{\sigma}^T(t)\right\} \mathbf{C}_F, \quad \mathbf{C}_F \triangleq \begin{bmatrix} \mathbf{f}^s_{1,1} & \mathbf{f}^s_{1,2} & \cdots & \mathbf{f}^s_{1,n} \\ \vdots & \vdots & & \vdots \\ \mathbf{f}^s_{n,1} & \mathbf{f}^s_{n,2} & \cdots & \mathbf{f}^s_{n,n} \end{bmatrix} \tag{6.51}$$

where the vector $\mathbf{f}^s_{i,j}$ of the expansion coefficients of F_{ij} can usually be obtained as follows, without any expansions.

Integrating the differential equation

$$\dot{\boldsymbol{\psi}}_i(t) = \mathbf{F}\boldsymbol{\psi}_i(t), \tag{6.52}$$

where $\boldsymbol{\psi}(\cdot)$ is the i-th column vector of matrix $\mathbf{\Psi}(\cdot, 0)$, yields

$$\boldsymbol{\psi}(t) - \mathbf{e}_i = \int_0^t \mathbf{F}\boldsymbol{\psi}_i(u, 0) du, \quad \mathbf{e}_i = \left[\delta_{ij}\right]. \tag{6.53}$$

Using the operational matrix of integration $\mathbf{G}$ (cf. Eq. (2.10)), the integral is approximated by

$$\int_0^t \mathbf{F}\boldsymbol{\psi}_i(u,0)du \approx \boldsymbol{\sigma}^T(t)\mathbf{F}\otimes\mathbf{G}^T\boldsymbol{\psi}_i^s, \tag{6.54}$$

$$\mathbf{e}_i \approx \boldsymbol{\sigma}^T(t)\mathbf{e}_i^s. \tag{6.55}$$

The spectral approximation of Eq. (6.53) then becomes

$$\boldsymbol{\sigma}^T(t)\left[\mathbf{I}_{ns} - \mathbf{F}\otimes\mathbf{G}^T\right]\boldsymbol{\psi}_i^s = \boldsymbol{\sigma}^T(t)\mathbf{e}_i^s \tag{6.56}$$

and the expansion of the i-th column of $\boldsymbol{\Psi}(\cdot,0)$ is the solution of the linear system of order ns :

$$\left[\mathbf{I}_{ns} - \mathbf{F}\otimes\mathbf{G}^T\right]\boldsymbol{\psi}_i^s = \mathbf{e}_i^s. \tag{6.57}$$

We thus derived the expansion of the exponential without having to evaluate the matrix exponential itself, adding to the nineteen ways reviewed by Moler and Loan (1978). Similarly, by replacing $\mathbf{C}_F$ by $(-\mathbf{C}_F)$ in Eq. (6.57), we obtain the expansion of the inverse $\boldsymbol{\Psi}(0,\cdot)$.

6.7.3.2. *Efficient Multiplier Assignment* If one aims at assigning only the unstable multipliers of the closed-loop system, the whole approach can be scaled down by constructing a lower-order controller. Consider the real Jordan decomposition of matrix $\mathbf{F}$

$$\mathbf{F}\triangleq\mathbf{W}_J\begin{bmatrix}\mathbf{J}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{J}_2\end{bmatrix}\mathbf{W}_J^{-1}, \tag{6.58}$$

where the $p\times p$ block $\mathbf{J}_1$ contains all the eigenvalues of $\mathbf{F}$ with positive real part. The change of variable $\mathbf{x}(t) = \mathbf{L}(t)\mathbf{W}_J\mathbf{w}(t)$ leads to the new transformed system

$$\dot{\mathbf{w}}(t) = \begin{bmatrix}\mathbf{J}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{J}_2\end{bmatrix}\mathbf{w}(t) + \begin{bmatrix}\mathbf{D}_1(t)\\ \mathbf{D}_2(t)\end{bmatrix}\mathbf{u}(t), \tag{6.59}$$

where the $p \times r$ matrix $\mathbf{D}_1(\cdot)$ contains the p first rows of $\mathbf{W}_J^{-1}\mathbf{L}^{-1}(\cdot)\mathbf{B}(\cdot)$, and $(n-p) \times r$ matrix $\mathbf{D}_2(\cdot)$ its last $(n-p)$ rows. We can now synthesize a $r \times p$ controller $\mathbf{K}_1(\cdot)$ for the p-order system

$$\dot{\mathbf{w}}_1(t) = \mathbf{J}_1\mathbf{w}_1(t) + \mathbf{D}_1(t)\mathbf{u}(t), \tag{6.60a}$$

such that

$$\dot{\mathbf{w}}_1(t) = \left[\mathbf{J}_1 - \mathbf{D}_1(t)\mathbf{K}_1(t)\right]\mathbf{w}_1(t) \tag{6.60b}$$

has a desired monodromy matrix. Choosing the feedback law

$$\mathbf{u}(t) = -\mathbf{K}_w(t)\mathbf{w}(t) \tag{6.61a}$$

with

$$\mathbf{K}_w(t) = \begin{bmatrix}\mathbf{K}_1(t) & \mathbf{0}_{r\times(n-p)}\end{bmatrix}, \tag{6.61b}$$

system (6.59) becomes

$$\dot{\mathbf{w}}(t) = \begin{bmatrix}\mathbf{J}_1 - \mathbf{D}_1(t)\mathbf{K}_1(t) & \mathbf{0}_{p\times(n-p)} \\ -\mathbf{D}_2(t)\mathbf{K}_1(t) & \mathbf{J}_2\end{bmatrix}\mathbf{w}(t) \tag{6.61c}$$

Due to the block-triangular nature of system (6.61c), its STM has a specific form (see also O'Malley 1997, Example 7, p.149). Let $\mathbf{\Psi}_{11}(\cdot, 0)$ be the STM of system (6.60b), and note that

$$\mathbf{\Psi}_{22}(t, 0) \stackrel{\Delta}{=} \exp(t\mathbf{J}_2) \tag{6.62a}$$

is the STM of homogeneous system

$$\dot{\mathbf{w}}_{2h}(t) = \mathbf{J}_2\mathbf{w}_{2h}(t). \tag{6.62b}$$

Using variation of parameters, the solution of

$$\dot{\mathbf{w}}_2(t) = \mathbf{J}_2\mathbf{w}_2(t) - \mathbf{D}_2(t)\mathbf{K}_1(t)\mathbf{w}_1(t) \tag{6.62c}$$

is

$$\mathbf{w}_2(t) = \mathbf{\Psi}_{22}(t,0)\mathbf{w}_2(0) + \mathbf{\Psi}_{21}(t,0)\mathbf{w}_1(0) \tag{6.62d}$$

$$\mathbf{\Psi}_{21}(t,0) \triangleq -\int_0^t \mathbf{\Psi}_{22}(s,0)\mathbf{D}_2(s)\mathbf{K}_1(s)\mathbf{\Phi}_{11}(s,0)ds. \tag{6.62e}$$

In other words, the STM of system (6.61c) is

$$\mathbf{\Psi}(t,0) \triangleq \begin{bmatrix} \mathbf{\Psi}_{11}(t,0) & \mathbf{0}_{p\times(n-p)} \\ \mathbf{\Psi}_{21}(t,0) & \mathbf{\Psi}_{22}(t,0) \end{bmatrix}. \tag{6.62f}$$

Note that the eigenvalues of the monodromy matrix are those of $\mathbf{\Psi}_{11}(T_{A_K},0)$ and $\mathbf{\Psi}_{22}(T_{A_K},0)$: we assigned the unstables Floquet exponents without modifying the stable Floquet exponents of $\mathbf{J}_2$.

Finally, the controller for the original closed-loop system has the form

$$\dot{\mathbf{x}}(t) = [\mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}(t)]\,\mathbf{x}(t), \tag{6.63a}$$

$$\mathbf{K}(t) = \mathbf{K}_z(t)\mathbf{W}_J^{-1}\mathbf{L}^{-1}(t). \tag{6.63b}$$

6.7.3.3. *Flops Savings* Working on this scaled down system may offer a substantial improvement in the time required for the computations:

- A controller based on the original system would require the expansions of the n^2 entries of $\mathbf{\Phi}(\cdot,0)$ and of its inverse, and of the nr elements of $\mathbf{B}(\cdot)$.
- Recalling the results of §6.7.3.1 to obtain the expansion coefficients of the $\exp(t\mathbf{J}_1)$ without any expansion, the only expansions required are those of the pr entries of $\mathbf{D}(\cdot)$.
- There is an additional saving in the number of flops involved when manipulating matrices of order sp^2 instead of sn^2 (cf. Eqs. (6.33) and (6.39)).

6.7.4. State Reconstruction The previous sections have so far assumed a full-state feedback. We can take advantage of the duality of the optimal regulator and optimal observer problem (Kwakernaak and Sivan 1972, §4.4) by considering the dual system of system (2.1):

$$\dot{\mathbf{x}}(t) = \mathbf{A}^T(t)\mathbf{x}(t) + \mathbf{C}^T(t)\mathbf{u}(t), \tag{6.64a}$$

$$\mathbf{y}(t) = \mathbf{B}^T(t)\mathbf{x}(t). \tag{6.64b}$$

From the duality between controllability and observability (e.g., Chen 1984, Theorem 5-10), system (2.1) is observable if and only if system (6.64) is controllable. In this case, we can design a controller $\mathbf{K}_d(\cdot)$ for the dual system following the steps described in §6.7.1. Defining the observer equation as

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}_O(t)\hat{\mathbf{x}}(t) + \mathbf{K}_O(t)\mathbf{y}(t) + \mathbf{B}(t)\mathbf{u}(t), \tag{6.65a}$$

where $\hat{\mathbf{x}}(\cdot)$ is the m-dimensional observer state vector, and matrix $\mathbf{A}_O(\cdot)$ is chosen such that

$$\mathbf{A}_O(t) \triangleq \mathbf{A}(t) - \mathbf{K}_O(t)\mathbf{C}(t), \tag{6.65b}$$

Kwakernaak and Sivan (1972, Theorem 4.8) showed that the observer gain matrix is $\mathbf{K}_O(\cdot) = \mathbf{K}_d^T(\cdot)$, and the observer-based closed-loop system (cf. Fig. 6.2) becomes

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\hat{\mathbf{x}}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}(t) & -\mathbf{B}(t)\mathbf{K}(t) \\ \mathbf{K}_O(t)\mathbf{C}(t) & \mathbf{A}_O(t) - \mathbf{B}(t)\mathbf{K}(t) \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix}. \tag{6.65c}$$

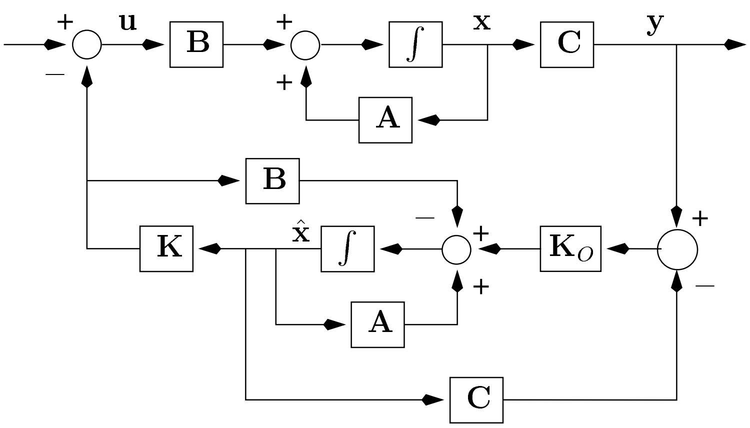


Figure 6.2: Observer-based controller

CHAPTER 7

Application: A Quick-Return Mechanism

7.1. Introduction

A quick-return mechanism is a mechanical transmission that produces a slow feed motion under a load in one direction, followed by a fast return stroke under no load in the opposite direction. Quick-return mechanisms are quite common in manufacturing processes, e.g. in pick-and-place operations, metal-cutting, and metal-forming. The mechanical design of such mechanisms was studied by Wu (1998); we build on one example of his thesis.

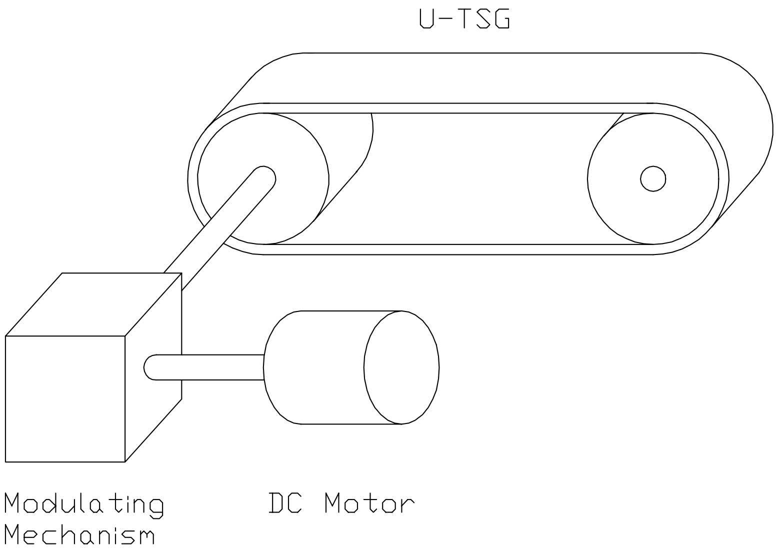


Figure 7.1: Layout of the quick-return mechanism

During the production of veneer, a quick-return mechanism (cf. Fig. 7.1) is needed to transport and glue veneer strips. The uniform speed of the motor is modulated by the modulating mechanism to produce the required motion, and the motion is then transmitted to the uniform-transmission stroke generator (U-TSG) to move the workpiece. The U-TSG can be a ballscrew, a belt-pulley, or a rack-and-pinion mechanism. The stroke of the mechanism is three metres and the period required for each cycle is about two seconds. Wu (1998) recommended that the long-stroke motion be implemented by means of a slider driven by the belt of a belt-pulley mechanism.

The system (cf. Fig. 7.2) is driven by a DC motor operating at its nominal speed of 3000 rpm; the shaft between the speed amplifier and the belt-pulley mechanism has a stiffness k_s. The values of the parameters given in Table 7.1 are taken from the design studies of Wu (1998) and of Asselin and Doléac (1997). Those studies investigated only the mechanical aspects of the system, based upon their findings we now look at the open-loop and closed-loop stability of the system: We assume that a steady-state has been reached, our aim being to design a controller which would reject any incoming perturbations.

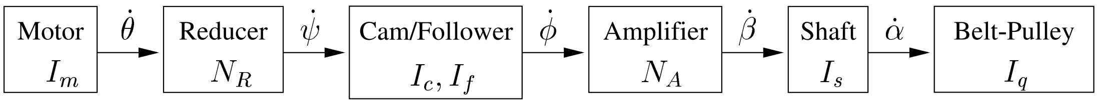


Figure 7.2: Modules of the quick-return mechanism

7.2. Mathematical Model

7.2.1. Motor Model The electro-mechanical model of the motor dynamics is derived from Fig. 7.3.

$$R_m\left[i_s(t) - i(t)\right] = K_m\dot{\theta}(t), \tag{7.1a}$$

$$I_m\ddot{\theta}(t) + C_m\dot{\theta}(t) = K_m i(t) - \tau(t), \tag{7.1b}$$

Belt-pulley mechanism		
D_p	pulley diameter	$0.1m$
D_s	diameter of the flexible shaft	$7\ 10^{-3}m$
l	stroke length	$3m$
l_s	length of the flexible shaft	$3.5\ 10^{-2}m$
m_p	mass of the transported object	$0.72kg$
m_f	mass of the fixtures	$1kg$
N_A	amplification ratio	16
N_R	reduction ratio	1/100
t_p	pulley thickness	$0.012m$
Motor		
ω_0	operating speed	3000 rpm
R_m	motor resistance	9.65Ω
K_m	motor electro-mechanical constant	$0.484V\ s\ rad^{-1}$
C_m	motor viscous damping	$5.25\ 10^{-5}N\ m\ s\ rad^{-1}$
T_A	non-dimensional open-loop period	146
T_s	constant in Eq. (7.18c)	$13.6\ 10^{-2}s$
T_c	cycle time	$2s$

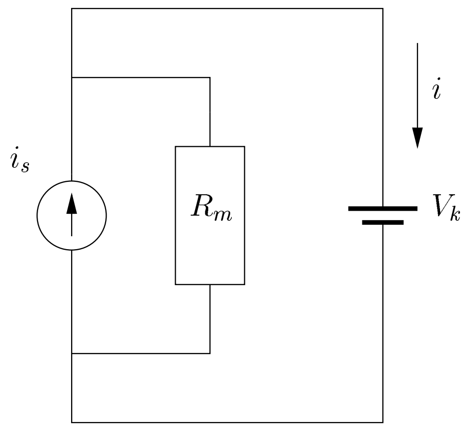
Table 7.1: Parameters of system

where the inductance of the armature has been neglected and τ is the torque applied on the reducer. Equations (7.1a) and (7.1b) can be recast into:

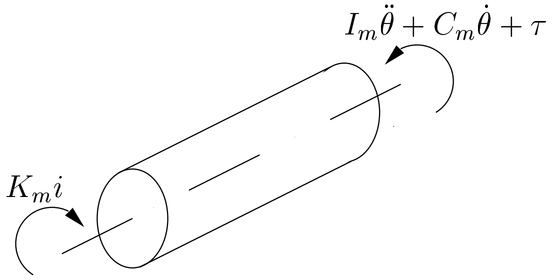
$$K_m i_s(t) - \left[C_m + \frac{{K_m}^2}{R_m}\right]\dot{\theta}(t) - I_m\ddot{\theta}(t) = \tau(t). \tag{7.2}$$

7.2.2. Mechanical System Model Let ψ and ϕ be the angular displacement of the cam and the follower of the system as shown in Fig. 7.2. The reducer of ratio $N_R = 1:16$ and the amplifier of ratio $N_A = 100:1$ provide the relations:

$$\dot{\psi}(t) = N_R\dot{\theta}(t), \qquad \dot{\beta}(t) = N_A\dot{\phi}(t). \tag{7.3}$$



(a) Electrical model



(b) Mechanical model

Figure 7.3: Model of the motor

From the chain rule, we have

$$\dot{\phi}(t) = \phi'(\theta)\dot{\theta}(t), \qquad \phi'(\theta) \triangleq \frac{\partial\phi}{\partial\theta}(\theta). \tag{7.4}$$

The cam was synthesized by Wu (1998) to produce the quick-return motion illustrated in Figs. 7.4 and 7.5, so that each operation takes exactly $T_c = 2$ seconds to perform.

7.2.3. Lagrange Equations The kinetic energy E_k of the system (excluding the motor) can be written as

$$E_k \triangleq \frac{1}{2}I_c\dot{\psi}^2 + \frac{1}{2}I_f\dot{\phi}^2 + \frac{1}{2}I_s\dot{\beta}^2 + \frac{1}{2}I_q\dot{\alpha}^2, \tag{7.5}$$

where I_c, I_f, I_s, and I_q are the moments of inertia of the cam, the follower, the shaft and the belt-pulley mechanism, respectively. Recalling Eq. (7.3), this simplifies to

$$E_k = \frac{1}{2}I_e(\theta)\dot{\theta}^2 + \frac{1}{2}I_q\dot{\alpha}^2, \quad I_e(\theta) \triangleq I_c{N_R}^2 + \left[I_f + I_s{N_A}^2\right]\phi'^2(\theta). \tag{7.6}$$

On the other hand, the potential energy of the same system comes solely from the elastic shaft:

$$E_p \triangleq \frac{1}{2}k_s(\beta - \alpha)^2, \tag{7.7}$$

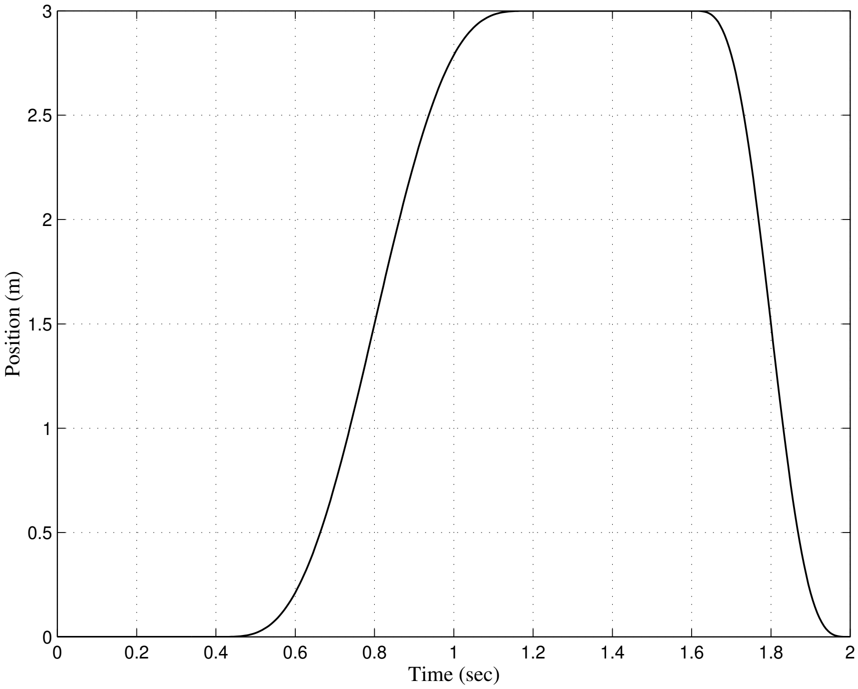


Figure 7.4: Quick-return motion

or, taking into account Eqs. (7.3) and (7.4),

$$E_p = \frac{1}{2} k_s \left[N_A \phi'(\theta)\theta - \alpha \right]^2 . \tag{7.8}$$

Choosing θ and α as generalized coordinates, the generalized forces are

$$Q_\theta = \tau(t), \quad Q_\alpha = 0. \tag{7.9}$$

Define

$$C_e(\theta) \triangleq \frac{1}{2}\frac{dI_e}{d\theta} = \left[I_f + I_s {N_A}^2 \right] \phi'(\theta)\phi''(\theta), \tag{7.10}$$

$$C_e{}'(\theta) \triangleq \frac{dC_e}{d\theta} = \left[I_f + I_s {N_A}^2 \right] \left[\phi''^2(\theta) + \phi'(\theta)\phi^3(\theta) \right], \tag{7.11}$$

$$K(\theta) \triangleq k_s N_A \left[\phi''(\theta)\theta + \phi'(\theta) \right]. \tag{7.12}$$

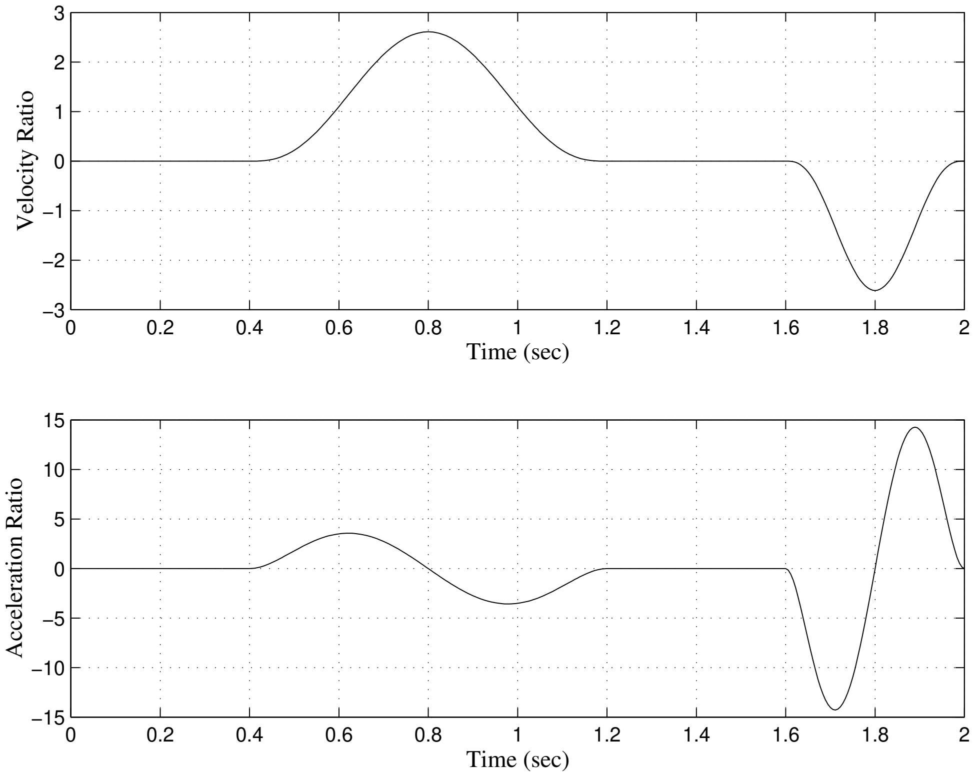


Figure 7.5: Velocity and acceleration of the quick-return motion

Using the Lagrangian formulation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q_q, \quad L \triangleq E_k - E_p, \quad q \in \{\theta, \alpha\}, \tag{7.13}$$

we obtain the system of nonlinear equations

$$I_e(\theta)\ddot{\theta} + C_e(\theta)\dot{\theta}^2 + K(\theta)\left[N_A\phi'(\theta)\theta - \alpha\right] = \tau(\theta), \tag{7.14a}$$

$$I_q\ddot{\alpha} + k_s\left[\alpha - N_A\phi'(\theta)\theta\right] = 0. \tag{7.14b}$$

7.2.4. Linearization We first linearize the system at the operating point,

$$\theta = \theta_0 + \delta\theta, \qquad \theta_0 = \omega_0 t, \tag{7.15a}$$

$$\alpha = \alpha_0 + \delta\alpha, \qquad \alpha_0 = N_A\phi'(\theta_0)\omega_0 t, \tag{7.15b}$$

which leads to the following first-order approximations:

$$I_e(\theta) = I_e(\theta_0) + \frac{dI_e}{d\theta}(\theta_0)\delta\theta = I_e(\theta_0) + 2C_e(\theta_0)\delta\theta, \tag{7.15c}$$

$$C_e(\theta) = C_e(\theta_0) + \frac{dC_e}{d\theta}(\theta_0)\delta\theta = C_e(\theta_0) + {C_e}'(\theta_0)\delta\theta, \tag{7.15d}$$

$$\begin{aligned}\tau(\theta) &= \frac{\partial\tau}{\partial\theta}(\theta_0)\delta\theta + \frac{\partial\tau}{\partial\dot{\theta}}(\theta_0)\delta\dot{\theta} + \frac{\partial\tau}{\partial\ddot{\theta}}(\theta_0)\delta\ddot{\theta} + \frac{\partial\tau}{\partial i_s}(\theta_0)\delta i_s \\ &= K_m\delta i_s - \left[C_m + \frac{{K_m}^2}{R_m}\right]\delta\dot{\theta} - I_m\delta\ddot{\theta}. \end{aligned} \tag{7.15e}$$

System (7.14) becomes, after simplification:

$$\begin{aligned}&[I_e(\theta_0) + I_m]\,\delta\ddot{\theta} + \left[2C_e(\theta_0)\omega_0 + C_m + \frac{{K_m}^2}{R_m}\right]\delta\dot{\theta} \\ &\quad + K(\theta_0)\left(N_A\left\{\phi'(\theta_0)\theta_0 + [\phi'(\theta_0) + \phi''(\theta_0)\theta_0]\,\delta\theta\right\} - [\alpha_0 + \delta\alpha]\right) \\ &\quad + K'(\theta_0)\,[N_A\phi'(\theta_0)\theta_0\delta\theta - \alpha_0\delta] \\ &\quad = \tau(\theta_0) + K_m\delta i_s - [C_e(\theta_0) + {C_e}'(\theta_0)\delta\theta]\,\omega_0^2, \end{aligned} \tag{7.16a}$$

$$I_q\delta\ddot{\alpha} + k_s\left\{\alpha_0 + \delta\alpha - N_A\,[\phi'(\theta_0)\theta_0 + \phi''(\theta_0)\theta_0]\,\delta\theta\right\} = 0. \tag{7.16b}$$

Recalling that the conditions at the equilibrium are

$$K(\theta_0)\,[N_A\phi'(\theta_0)\theta_0 - \alpha_0] = \tau(\theta_0) - C_e(\theta_0)\omega_0^2, \tag{7.17a}$$

$$k_s\alpha_0 - N_A\phi'(\theta_0)\theta_0 = 0, \tag{7.17b}$$

system (7.16) yields

$$\begin{aligned}[I_e(\Omega_s v) + I_m]\,\delta\ddot{\theta}(v) &= -{T_s}^2\left\{K(\Omega_s v)N_A\,[\phi'(\Omega_s v) + \phi''(\Omega_s v)\Omega_s v]\right. \\ &\quad \left. + K'(\Omega_s v)N_A\phi'(\Omega_s v) - \alpha_0(T_s v) + {C_e}'(\Omega_s v)\omega_0^2\right\}\delta\theta(v) \\ &\quad - T_s\left[2C_e(\Omega_s v)\omega_0 + C_m + \frac{{K_m}^2}{R_m}\right]\delta\dot{\theta}(v) \\ &\quad + {T_s}^2K(\Omega_s v)\delta\alpha(v) + K_m\delta i_s(v), \end{aligned} \tag{7.18a}$$

$$I_q\delta\ddot{\alpha}(v) = {T_s}^2 k_s N_A \left[\phi'(\Omega_s v) + \phi''(\Omega_s v)\Omega_s v\right]\delta\theta(v) - {T_s}^2 k_s \delta\alpha(v), \tag{7.18b}$$

where we have introduced non-dimensional time v through the change of variable

$$t = T_s v, \quad T_s = \sqrt{\frac{I_q}{k_s}}, \quad \Omega_s = \omega_0 T_s. \tag{7.18c}$$

7.2.5. State-Space Representation Choosing the state vector

$$\mathbf{x}(t) \triangleq \begin{bmatrix}\delta\theta & \delta\dot{\theta} & \delta\alpha & \delta\dot{\alpha}\end{bmatrix}^T, \tag{7.19a}$$

and the current δi_s as the input variable $u(v)$, system (7.18) can be rewritten in the form

$$\dot{\mathbf{x}}(v) = \mathbf{A}(t)\mathbf{x}(v) + \mathbf{b}(v)u(v), \tag{7.19b}$$

where

$$\mathbf{A}(v) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ A_{21}(v) & A_{22}(v) & A_{23}(v) & 0 \\ 0 & 0 & 0 & 1 \\ A_{41}(v) & 0 & A_{43} & 0 \end{bmatrix}, \tag{7.19c}$$

$$\begin{aligned} A_{21}(v) = -{T_s}^2 \{ & K(\Omega_s v) N_A \left[\phi'(\Omega_s v) + \phi''(\Omega_s v)\Omega_s v\right] + K'(\Omega_s v) N_A \phi'(\Omega_s v)\Omega_s v \\ & -\alpha_0 + {C_e}'(\Omega_s v)\omega_0^2 \} / \left[I_e(\Omega_s v) + I_m\right], \end{aligned} \tag{7.19d}$$

$$A_{22}(v) = -T_s \left[2C_e(\Omega_s v)\omega_0 + C_m + \frac{{K_m}^2}{R_m}\right] / \left[I_e(\Omega_s v) + I_m\right], \tag{7.19e}$$

$$A_{23}(v) = {T_s}^2 K(\Omega_s u) / \left[I_e(\Omega_s v) + I_m\right], \tag{7.19f}$$

$$A_{41}(v) = {T_s}^2 k_s N_A \left[\phi'(\Omega_s v) + \phi''(\Omega_s v)\Omega_s v\right] / I_q, \tag{7.19g}$$

$$A_{43} = -\frac{{T_s}^2 k_s}{I_q}, \tag{7.19h}$$

$$\mathbf{b}(v) = \begin{bmatrix} 0 & b_{21}(v) & 0 & 0 \end{bmatrix}^T, \tag{7.19i}$$

$$b_{21}(v) = K_m / \left[I_e(\Omega_s v) + I_m \right]. \tag{7.19j}$$

Note that $\mathbf{A}(\cdot)$ and $\mathbf{b}(\cdot)$ are periodic with primary period $T_A = T_c/T_s$.

7.3. System Dynamics

7.3.1. System Stability Using the method described in §5.2, we obtain a real pair of Floquet factors $\{\mathbf{L}(\cdot.), \mathbf{F}\}$, from which we derive the state-transition matrix of the open-loop system

$$\mathbf{\Phi}(t, 0) \stackrel{\Delta}{=} \mathbf{L}(t) \exp(t\mathbf{F}).$$

The Floquet exponents λ_i, the eigenvalues of $\mathbf{F}$, are

$$\lambda_1 = -6.9137\ 10^{-7}, \quad \lambda_2 = \bar{\lambda}_3 = -5.5222\ 10^{-8} + 2.0258\iota, \qquad \lambda_4 = -3.6806, \tag{7.20}$$

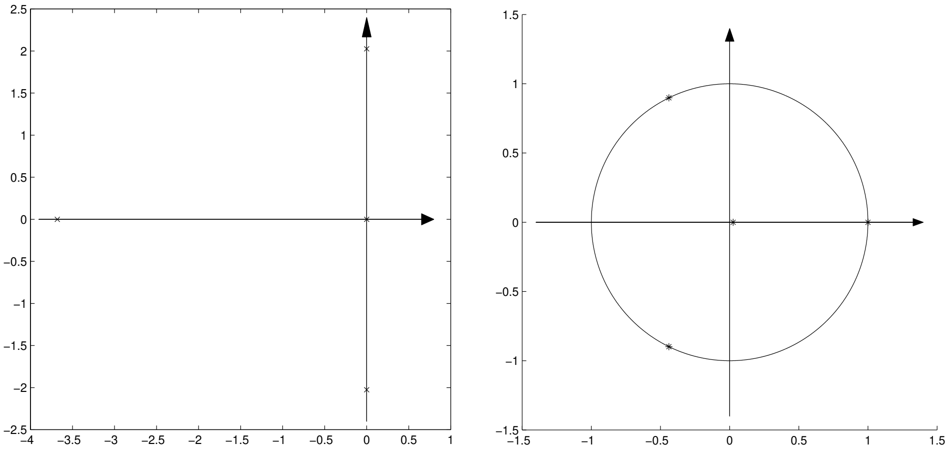
corresponding to the following Floquet multipliers

$$\mu_1 = 1.0000, \qquad \mu_2 = \bar{\mu}_3 = -4.3946\ 10^{-1} + 8.9826\ 10^{-1}\iota, \quad \mu_4 = 2.5207\ 10^{-2}. \tag{7.21}$$

Note that all these eigenvalues are dimensionless since matrix $\mathbf{A}(\cdot)$ is. The system has 3 modes which are oscillatory, that correspond to the Floquet exponents lying on the imaginary axis in the s-plane or the Floquet multipliers lying on the unit circle in the z-plane (cf. Fig. 7.6). The potential pitfalls of the open-loop mechanical-system performance are best illustrated with a simple example. Consider an initial perturbation of one degree on the variable $\delta\alpha$, i.e. the initial state vector is

$$\mathbf{x}_0 = \begin{bmatrix} 0 & \frac{\pi}{180} & 0 & 0 \end{bmatrix}^T. \tag{7.22}$$

Figure 7.7 displays the resulting oscillations of the system, shown here in radians for $\delta\alpha$ and $\delta\theta$, and in radians per second for $\delta\dot{\alpha}$ and $\delta\dot{\theta}$. Thus, the system requires a closed-loop controller in order to eliminate incoming perturbations.



(a) Floquet Exponents (b) Floquet Multipliers

Figure 7.6: System stability in the s- and z-planes

7.3.2. Controller Synthesis We follow the design rationale described in § 6.7. Starting with the transformed system

$$\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t) + \mathbf{L}^{-1}(t)\mathbf{b}(t)u(t), \tag{7.23}$$

we consider the real Jordan form of $\mathbf{F}$:

$$\mathbf{F} \triangleq \mathbf{W}_J \begin{bmatrix} \mathbf{J} & 0 \\ \mathbf{0}_{1\times 2} & \lambda_4 \end{bmatrix} \mathbf{W}_J^{-1}, \quad \mathbf{J} \triangleq \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \mathrm{Re}(\lambda_2) & \mathrm{Im}(\lambda_2) \\ 0 & -\mathrm{Im}(\lambda_2) & \mathrm{Re}(\lambda_2) \end{bmatrix}. \tag{7.24}$$

We focus now on the 3×3 subsystem including the three oscillatory modes of Floquet exponents λ_1, λ_2, and λ_3 :

$$\dot{\mathbf{w}}(t) = \mathbf{J}\mathbf{w}(t) + \mathbf{d}(t)u(t), \tag{7.25}$$

where $\mathbf{d}(\cdot)$ contains the first three rows of $\mathbf{W}_J^{-1}\mathbf{L}^{-1}(\cdot)\mathbf{b}(\cdot)$. In this example the Gramian $\mathbf{W}_R(\cdot, 0)$ of the system is invertible after three periods, and thus, we build a controller $\mathbf{k}_3(\cdot)$ of period $3T_A$ which assigns the monodromy matrix of the subsystem to

$\exp(3T_A\mathbf{F}_{K_3})$, where the constant matrix $\mathbf{F}_{K_3}$ is chosen as

$$\mathbf{F}_{K_3} = \begin{bmatrix} -0.8 & 0 & 0 \\ 0 & -0.8 & 0.8 \\ 0 & -0.8 & -0.8 \end{bmatrix}. \tag{7.26}$$

Note that any matrix $\mathbf{F}_{K_3}$ with eigenvalues in the right-hand side plane would produce a stable system. The values in Eq. (7.26) are only an example chosen so that quantities stay within the framework of the small perturbation hypotheses.

Finally we synthesize the controller for the original system by building the feedback matrix

$$\forall t \in [0, 3T_A[\,, \quad \forall k \in \mathbb{Z}^+,$$

$$\mathbf{k}^T(t) = \mathbf{k}_w^T(t)\mathbf{W}_J^{-1}\mathbf{L}^{-1}(t), \tag{7.27a}$$

$$\mathbf{k}(t + 3kT_A) = \mathbf{k}(t), \tag{7.27b}$$

$$\mathbf{k}_w(t) = \begin{bmatrix}\mathbf{k}_3^T(t) & 0\end{bmatrix}^T. \tag{7.27c}$$

The variations of entries of the gain matrices $\mathbf{k}_3(\cdot)$ and $\mathbf{k}(\cdot)$ are displayed in Figs. 7.9 and 7.10.

Figure 7.8 displays the variations of the state of the closed-loop system $\mathbf{A}(\cdot) - \mathbf{b}(\cdot)\mathbf{k}^T(\cdot)$ for the same initial perturbation as Fig. 7.7. This time the oscillations are eliminated thanks to the controller.

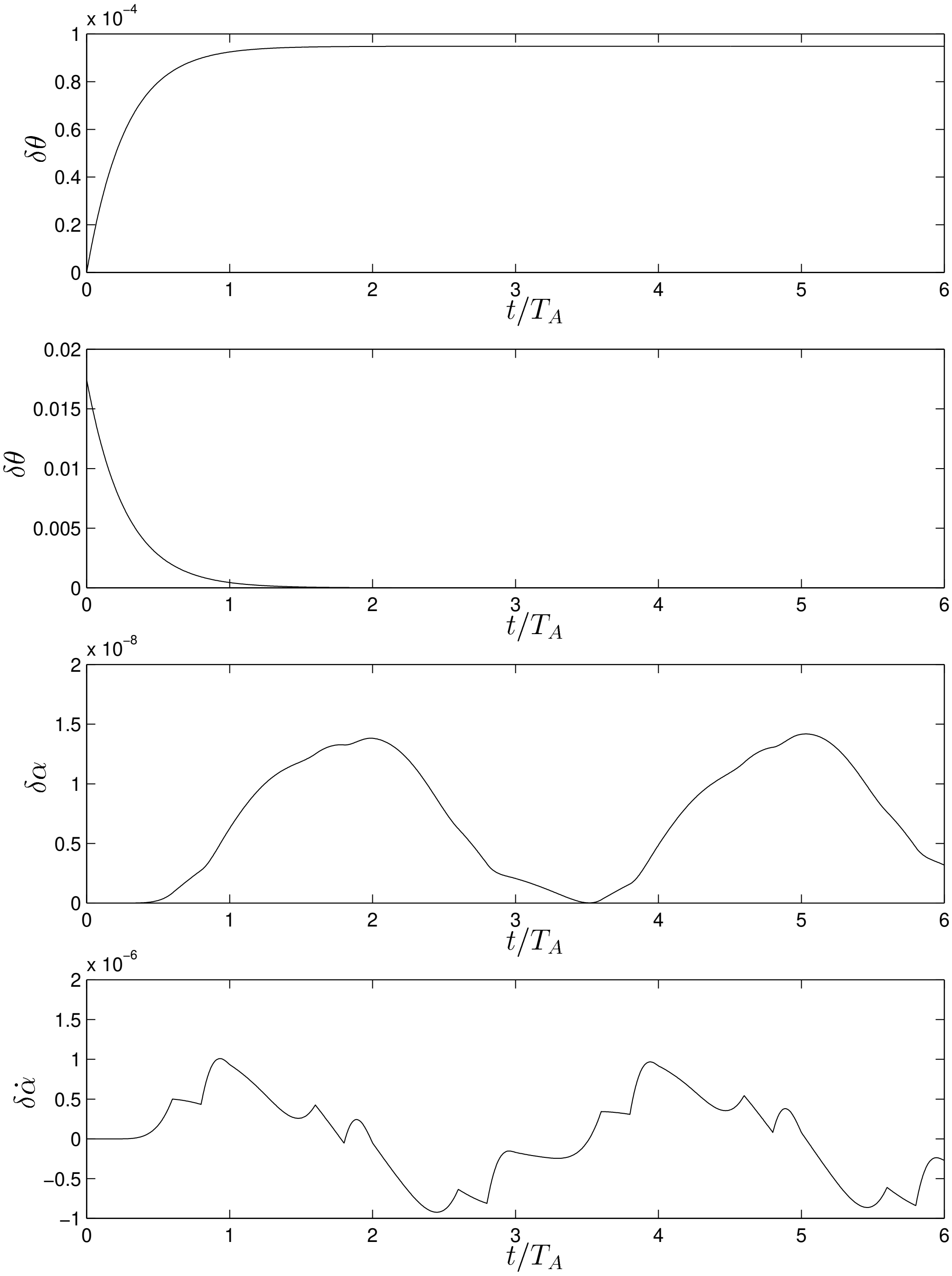


Figure 7.7: Effect of an initial perturbation on the open-loop system

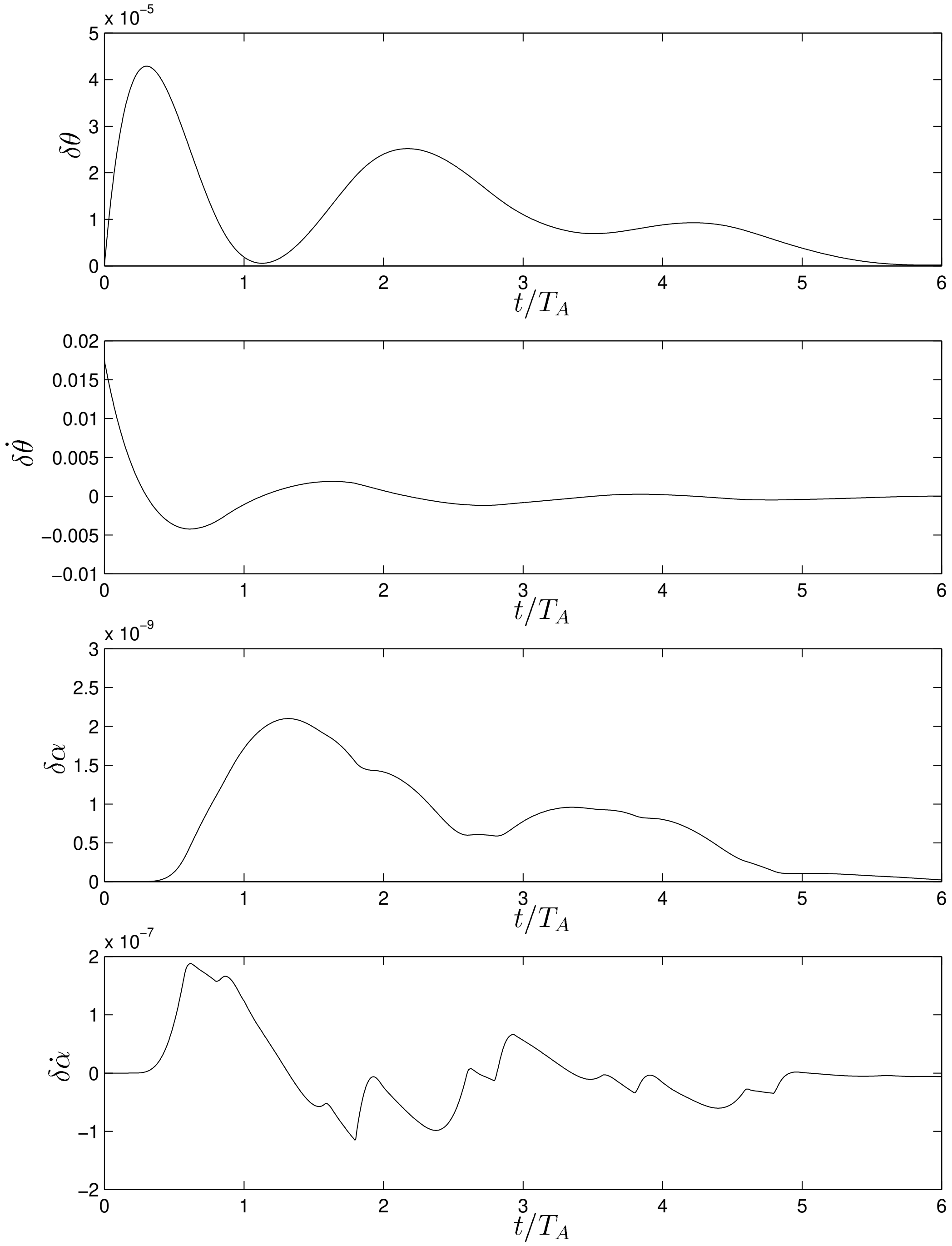


Figure 7.8: Effect of an initial perturbation on the closed-loop system

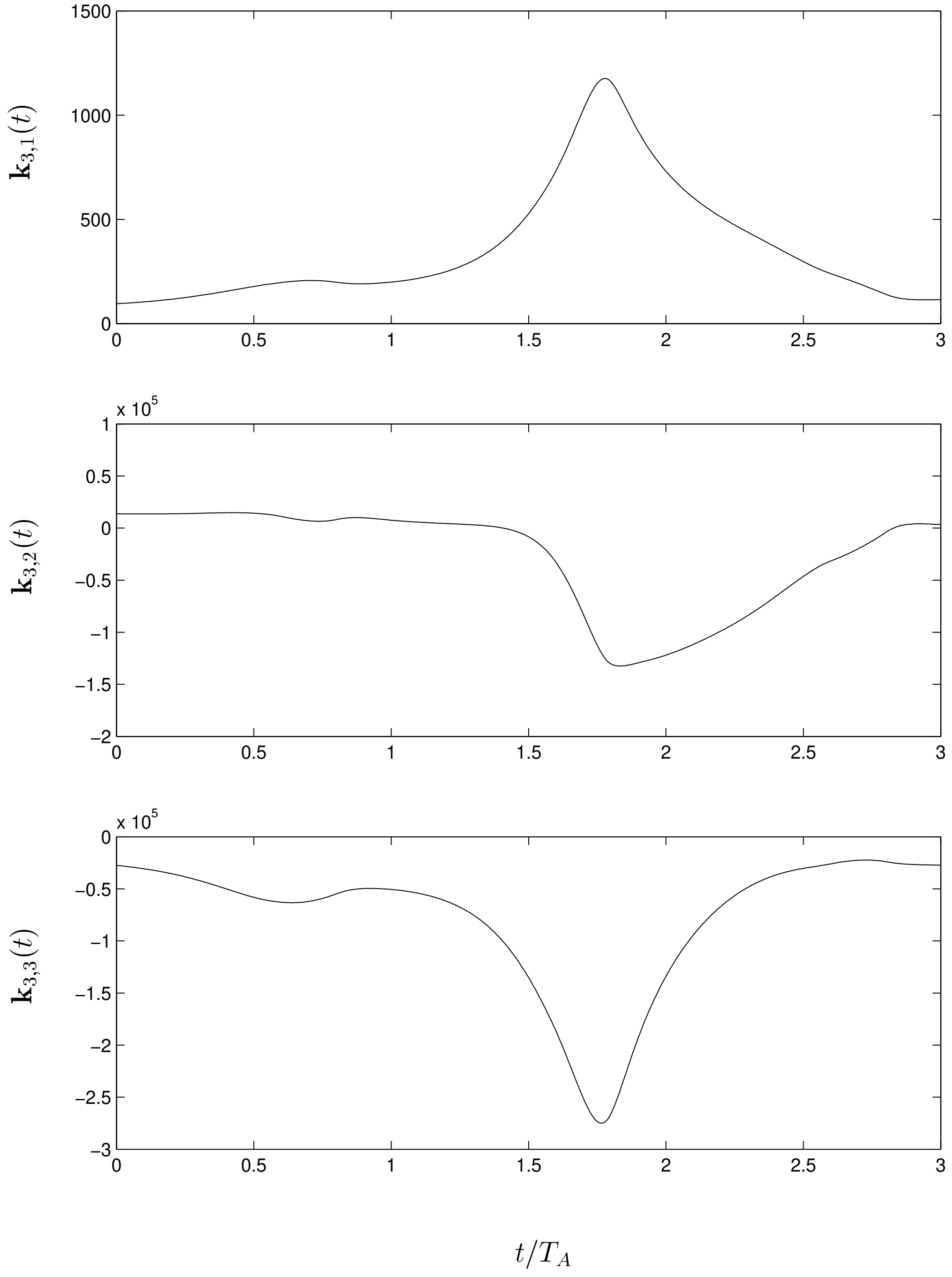


Figure 7.9: Entries of matrix k3(t)

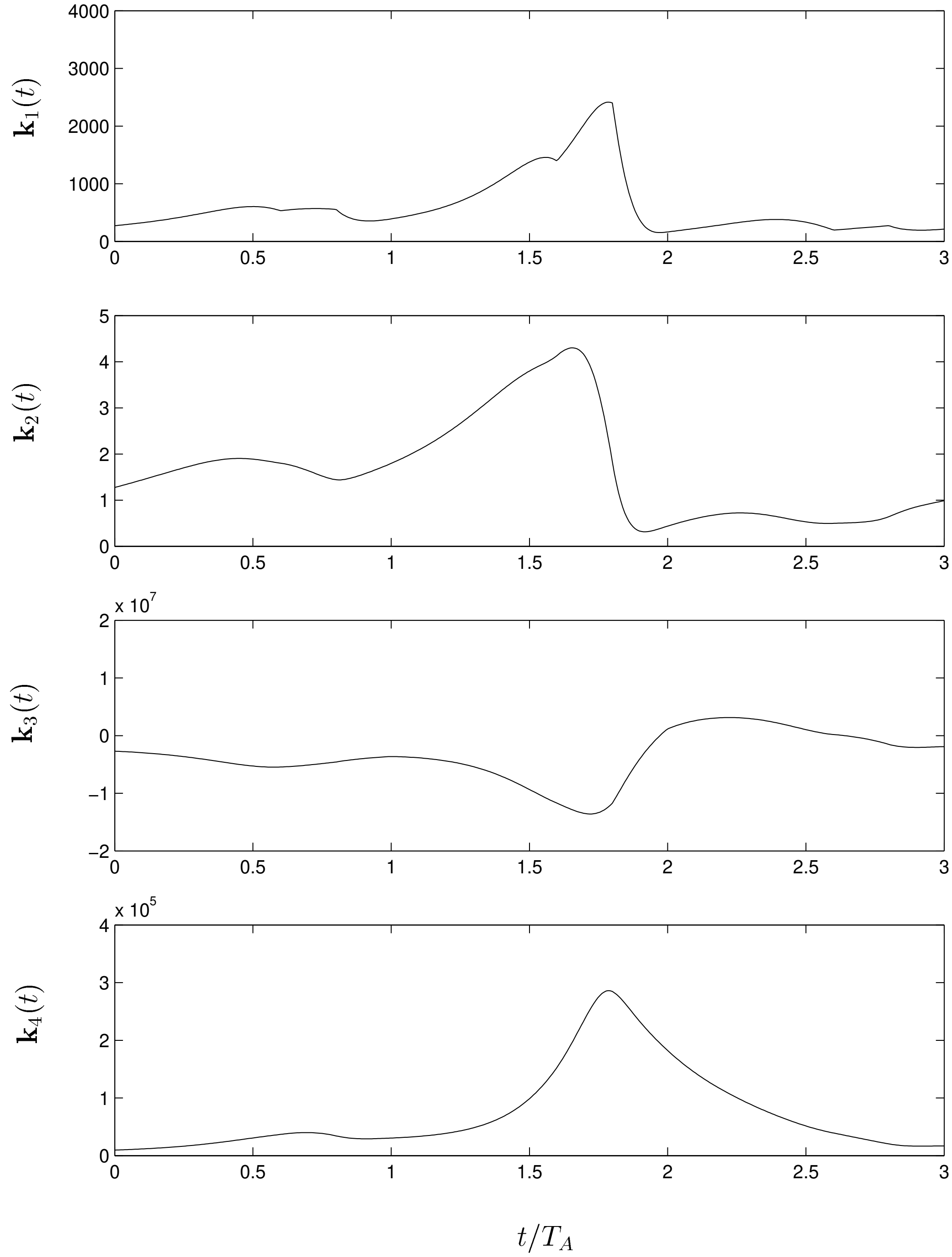


Figure 7.10: Entries of matrix k(t)

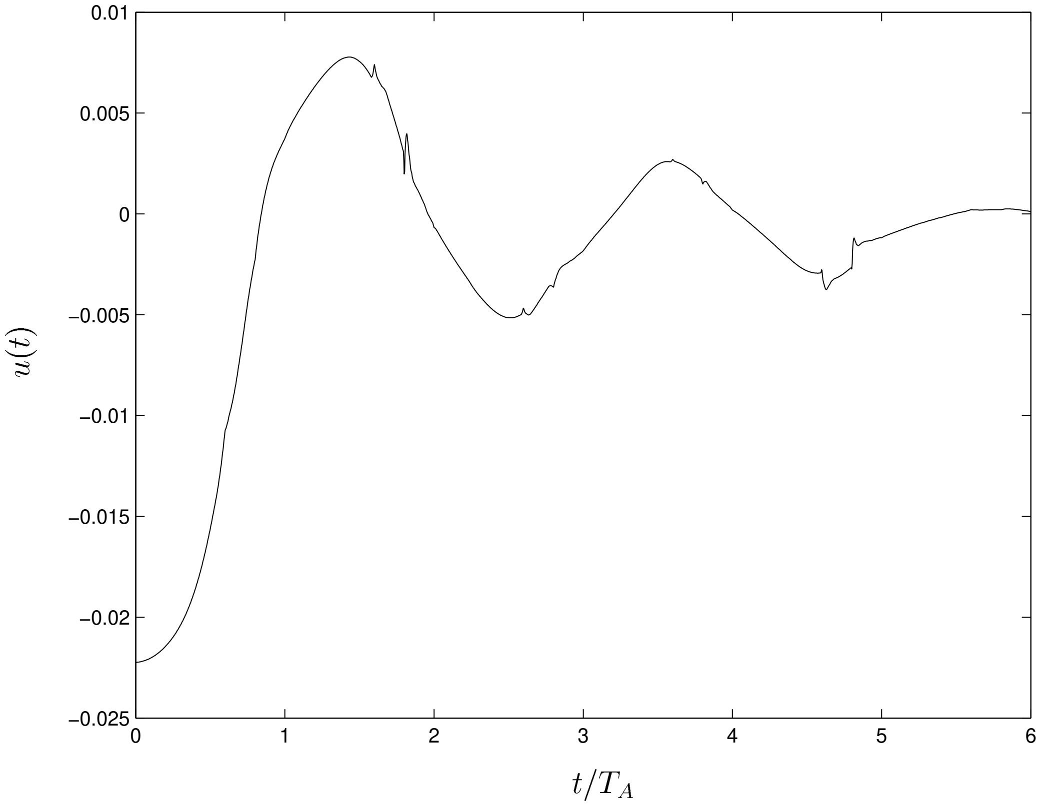


Figure 7.11: u(t)

Conclusions

Summary

This thesis introduced a novel technique for the design of control strategies for linear dynamical systems with periodically varying coefficients. This approach can be implemented using either full-state feedback or observer-based controllers, and has the additional benefit of providing a solution to the long-standing problem of assigning all the invariants of a system. The core of the method hinges on the Floquet-Lyapunov decomposition of the state-transition matrix, and exploits novel results and techniques related to the Floquet factors as well as the invertibility property of the Gramian derived from controllability.

Chapter 4 investigated the real decompositions of the state-transition matrix of a LTP system. We showed that existing results were insufficient: They led to a loss of information on the period of the periodic factor, and failed to identify the whole set of possible real pairs of factors. Using properties of the real matrix logarithm, we introduced a new parameter in obtaining the Floquet factors, namely, a constant matrix that we named the *Yakubovich matrix.* We then established that the existence of periodic factors of periodicity k times the original primary period of the system corresponds to a Yakubovich matrix being a solution of a matrix equation of order k. It was proved that a solution to the equation always exists for $k = 2$, showing consistency with existing results.

In Chapter 5 we first introduced a boundary-value problem formulation for the derivation of the Floquet factors. This approach can be used for obtaining the Floquet factors of the STM of the open-loop system as an alternative to existing methods, all of which are based on the initial computation of the STM. Additionally, this boundary-value formulation is one of the building blocks of the design of the control strategy for the closed-loop system.

Chapter 6 developed the synthesis of a novel controller for LTP systems. The design technique relies on assigning the constant element of the Floquet pair of factors, while solving for the periodic factor. By doing so, this scheme intrinsically allows for the assignment of all the invariants of the system, namely the whole monodromy matrix as opposed to the sole eigenvalue-assignment well-known for linear time-invariant systems. This provided a proof to the old conjecture that continuous periodically varying feedback could achieve what is impossible with a constant gain matrix. The scheme introduced here is the first ever to perform in continuous-time what so far only controllers based on the "sampled-state periodic hold" strategy could.

The method hinges on the specific form introduced in this thesis for the time-varying feedback matrix presented in §6.4. This novel form led to the introduction of the controllability Gramian that is known to be invertible whenever the open-loop system is controllable. This form brought a significant improvement compared to existing work based on approximating the inverse of the input matrix with its generalized inverse. Obtaining the gain matrix first required solving a matrix integral equation, which was accomplished via a spectral method.

Section 6.7 provided a design rationale for the synthesis of the periodic controller. We highlighted how the spectral solution could be used as an initial guess to a boundary-value problem to further refine the solution when necessary. We stressed the computational benefits that could be gained by performing the design on the transformed system obtained by the Reducibility Theorem. Though mainly introduced as a full-state feedback controller, we showed how the approach could be seamlessly transformed into an observer-based controller.

Finally, Chapter 7 studied the dynamics of a quick-return mechanism. After having derived the mathematical model of the system, linearization at the operating point provided a system of equations with periodicially time-varying coefficients (§7.2). After having determined the stability of the system, a closed-loop controller was built in §7.3 using the software toolbox we developed for the `MATLAB`™ environment.

Scope for Further Research

This thesis introduced a new scheme for the control of LTP systems. There exist three main potential research extensions that need further investigation.

- The approach described in the thesis hinges on the Floquet factors for the computation of the STM of the open-loop system as well as for the synthesis of the closed-loop controller. Results in Chapter 4 have emphasised the non-uniqueness of pairs of real Floquet factors. An important aspect that deserves attention is to evaluate their numerical properties with respect to the computational procedures involved in the open-loop and closed-loop problems.
- An advantage of a time-periodic controller is that the scheme allows for the assignment of all the invariants of the system. Such a result had long been called for (Willems, Kučera, and Brunovský 1984). This added degree-of-freedom for the control engineer offers the opportunity for a practical testing of the benefits anticipated by Moore (1976) and comparison with schemes which only allow for eigenvalue-assignment in time-invariant systems terminology, i.e. the Floquet exponents in the case of a LTP system.
- The specific form of the feedback matrix $\mathbf{K}(\cdot)$ in Eq. (6.14) is aimed at the introduction of the Gramian matrix, the invertibility of which allows for the assignment of the whole monodromy matrix. However, since Theorem 3.5.1 indicates that a LTP system may not always be controllable over a single period, the periodicity of the closed-loop may differ from the original system. For those systems where $\mathbf{W}(T_A, 0)$ is not invertible, this restriction could be circumvented by investigating a scheme no longer based on the invertibility of the Gramian, but rather relying upon the second item of the theorem: The controllability of the pair $\{\mathbf{\Phi}(T_A, 0), \mathbf{W}(T_A, 0)\}$. As is well-known from linear

time-invariant system theory, this guarantees the existence of a constant matrix $\mathbf{K}_S$ such that the eigenvalues of $(\mathbf{\Phi}(T_A, 0) + \mathbf{W}(T_A, 0)\mathbf{K}_S)$ may be assigned, as opposed to the whole matrix in the present scheme. The closed-loop system would then be guaranteed to have the same primary period T_A as the open-loop system by trading off full invariant assignment against periodicity.

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APPENDIX A

Pitfalls of a Least-Squares-Equivalent Controller

This appendix is a reprint of the paper by Montagnier, Spiteri, and Angeles (2001), illustrating the shortcomings of an existing control strategy.

Abstract

We review a technique for the design of controllers for linear, time-periodic systems. A major appeal of the technique, first proposed by Sinha and Joseph, is the use of Floquet-Lyapunov theory to transform the periodic system to a form where classical control strategies for time-invariant systems may be employed. However, it is normally impossible to find a completely time-invariant control system that is equivalent to the original time-varying system: Application of the Floquet-Lyapunov transformation in fact yields a time-varying control system that the technique makes equivalent to a time-invariant one in the least-squares sense, in order to subsequently synthesize the controller via pole placement using a constant feedback matrix.

However, classical control and Floquet-Lyapunov theory clearly show that it is erroneous to conclude that the behaviour of the least-squares-equivalent, time-invariant system always matches the behaviour of the original time-periodic system. Using an example found in the original paper, we provide a simple counter-example that illustrates the failure of the proposed strategy and analysis of the reasons for its failure.

A.1. Introduction

Linear, time-periodic (LTP) systems are quite common in scientific and engineering applications, e.g., linkages and cam mechanisms driven at a uniform speed, helicopter rotor blades in forward flight, and the study of bifurcation of periodic orbits. The control of these systems is also of great interest, and is made no less challenging by the fact that the (time-varying) eigenvalues of the periodic matrix in general play no part in determining the stability of the system (Verhulst 1990; Wu 1974). Consider the control system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \tag{A.1a}$$

$$\mathbf{x}(0) = \mathbf{x}_0, \tag{A.1b}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{u} \in \mathbb{R}^m$, and $\mathbf{B} \in \mathbb{R}^{m\times n}$, and the matrices $\mathbf{A}(\cdot)$ and $\mathbf{B}(\cdot)$ are periodic with period T_A. Because the matrix $\mathbf{A}(\cdot)$ of (A.1) is time periodic, the Floquet Representation Theorem (e.g., Brockett 1970) allows us to write the state-transition matrix $\mathbf{\Phi}(\cdot, 0)$ of this system in the form

$$\mathbf{\Phi}(t, 0) \triangleq \mathbf{L}(t) \exp\left(t\mathbf{F}\right), \tag{A.2}$$

where the matrix $\mathbf{L}(t)$ is T-periodic and the matrix

$$\mathbf{F} \triangleq (1/T_A) \ln \mathbf{\Phi}(T_A, 0) \tag{A.3}$$

is constant. The eigenvalues of $\mathbf{F}$ are called the *Floquet exponents* and fully determine the stability of the system (Coddington and Levinson 1955). By means of the Lyapunov Reducibility Theorem (Brockett 1970), the transformation

$$\mathbf{x}(t) = \mathbf{L}(t)\mathbf{z}(t) \tag{A.4}$$

transforms system (A.1) into

$$\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t) + \mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{u}(t), \tag{A.5a}$$

$$\mathbf{z}(0) = \mathbf{x}_0. \tag{A.5b}$$

It is apparent that system (A.5) is not completely time invariant. Hence, classical control methods to relocate the poles of $\mathbf{F}$ are not applicable.

In keeping with classical control theory, we seek a control law for the system of equations (A.1) of the form

$$\mathbf{u}(t) = -\mathbf{K}(t)\mathbf{x}(t). \tag{A.6}$$

Indeed, it is well known (Brunovský 1970) that, if the pair $\{\mathbf{A}(\cdot), \mathbf{B}(\cdot)\}$ is controllable, there exists a periodic feedback $\mathbf{K}(\cdot)$ to assign the Floquet exponents of $\mathbf{A}(\cdot) - \mathbf{B}(\cdot)\mathbf{K}(\cdot)$. The problem of assigning the Floquet invariants of a time-periodic system was discussed in detail by Willems, Kučera, and Brunovský (1984). Several approaches are available for discrete systems (e.g., Colaneri and Kučera 1997), but a general solution has yet to be found for the continuous case. There have been two main approaches to the pole-placement problem for continuous time LTP systems; a distinction can be made on the basis of whether the scheme focuses on placement of the eigenvalues of the state-transition matrix $\mathbf{\Phi}(T_A, 0)$ versus those of the matrix $\mathbf{F}$. For instance, Yoshii and Hakomori (1989) proposed to assign the eigenvalues of the state-transition matrix of system (A.1) by constructing a feedback matrix as a sequence of periodic functions, while Fisher and Bryson Jr. (1995) used an analytical gain-scheduling approach. Al-Rahmani and Franklin (1989) gave a method for pole placement based on multi-rate control. Their work followed on Chammas and Leondes (1978a,b, 1979), Kabamba (1986) and Juan and Kabamba (1989). On the other hand, Calico and Wiesel (1984), and Kern (1980, 1986) built schemes based on the transformed system (A.5).

In this paper, we provide an analysis of the method of pole placement given by Sinha and Joseph (1994) for general continuous-time LTP systems based on the control of a least-squares-equivalent auxiliary system.

A.2. The Auxiliary System

A.2.1. Procedure Sinha and Joseph (1994) propose a new method, which was subsequently used by Pandiyan and Sinha (1995), based on an auxiliary linear, time-invariant (LTI) system, namely,

$$\dot{\tilde{\mathbf{z}}}(t) = \mathbf{F}\tilde{\mathbf{z}}(t) + \tilde{\mathbf{B}}\mathbf{v}(t), \tag{A.7a}$$

$$\tilde{\mathbf{z}}(0) = \mathbf{z}_0, \tag{A.7b}$$

where $\tilde{\mathbf{B}}$ is any constant matrix such that the pair $(\mathbf{F}, \tilde{\mathbf{B}})$ is controllable. A control law for system (A.7) can be written as

$$\mathbf{v}(t) = -\bar{\mathbf{K}}\tilde{\mathbf{z}}(t), \tag{A.8}$$

where $\bar{\mathbf{K}}$ is a constant matrix chosen such that $(\mathbf{F} - \tilde{\mathbf{B}}\bar{\mathbf{K}})$ has a given set of stable eigenvalues.

Now, defining the error vector

$$\mathbf{e}(t) \stackrel{\triangle}{=} \mathbf{z}(t) - \tilde{\mathbf{z}}(t), \tag{A.9}$$

the *error dynamics* of systems (A.5) and (A.7) satisfy

$$\dot{\mathbf{e}}(t) = \left(\mathbf{F} - \tilde{\mathbf{B}}\bar{\mathbf{K}}\right)\mathbf{e}(t) + \mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{u}(t) + \tilde{\mathbf{B}}\bar{\mathbf{K}}\mathbf{z}(t). \tag{A.10}$$

Because $(\mathbf{F} - \tilde{\mathbf{B}}\bar{\mathbf{K}})$ is, by construction, a time-invariant stability matrix, Sinha and Joseph (1994) conclude that the two systems, (A.5) and (A.7), can be made equivalent in the least-squares sense if the gain from the state feedback defined in Eq. (A.6) can be chosen such that

$$\mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{u}(t) = -\tilde{\mathbf{B}}\bar{\mathbf{K}}\mathbf{z}(t), \tag{A.11}$$

which leads to

$$\mathbf{u}(t) = -\mathbf{B}^{\dagger}(t)\mathbf{L}(t)\tilde{\mathbf{B}}\bar{\mathbf{K}}\mathbf{z}(t), \tag{A.12}$$

where $\mathbf{B}^{\dagger}(\cdot)$ is the applicable generalized inverse of $\mathbf{B}(\cdot)$. Recalling that $\mathbf{z}(\cdot) = \mathbf{L}^{-1}(\cdot)\mathbf{x}(\cdot)$, the feedback law for system (A.1) yields:

$$\mathbf{u}(t) = -\mathbf{B}^{\dagger}(t)\mathbf{L}(t)\tilde{\mathbf{B}}\bar{\mathbf{K}}\mathbf{L}^{-1}(t)\mathbf{x}(t), \tag{A.13}$$

Note that, using this feedback law, systems (A.1) and (A.5), respectively, become:

$$\dot{\mathbf{x}}(t) = \left[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{B}^{\dagger}(t)\mathbf{L}(t)\tilde{\mathbf{B}}\bar{\mathbf{K}}\mathbf{L}^{-1}(t)\right]\mathbf{x}(t), \tag{A.14}$$

and

$$\dot{\mathbf{z}}(t) = \left[\mathbf{F} - \mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{B}^{\dagger}(t)\mathbf{L}(t)\tilde{\mathbf{B}}\bar{\mathbf{K}}\right]\mathbf{z}(t), \tag{A.15}$$

while the closed-loop auxiliary system is

$$\dot{\tilde{\mathbf{z}}}(t) = \left(\mathbf{F} - \tilde{\mathbf{B}}\bar{\mathbf{K}}\right)\tilde{\mathbf{z}}(t). \tag{A.16}$$

This scheme is simple to understand and apply, and it has some geometric appeal. Moreover, in cases where $\mathbf{B}(\cdot)$ is invertible, it gives the desired results.

A.2.2. Analysis of Error Dynamics Based upon Eqs. (A.15) and (A.16), the error dynamics defined in (A.9) and (A.10) become

$$\dot{\mathbf{e}}(t) = \left(\mathbf{F} - \tilde{\mathbf{B}}\bar{\mathbf{K}}\right)\mathbf{e}(t) + \left[\tilde{\mathbf{B}}\bar{\mathbf{K}} - \mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{B}^{\dagger}(t)\mathbf{L}(t)\tilde{\mathbf{B}}\bar{\mathbf{K}}\right]\mathbf{z}(t). \tag{A.17}$$

The strategy proposed in Sinha and Joseph (1994) is based on the claim that, since $(\mathbf{F} - \tilde{\mathbf{B}}\bar{\mathbf{K}})$ is stable by construction, this offers a guarantee that the error dynamics are stable and, therefore, the transformed system (A.15) will behave like the auxiliary system (A.16). However, this is not entirely true. In general, Eq. (A.11) will not hold exactly when $\mathbf{B}(\cdot)$ is not invertible. The consequence is that the term in $\mathbf{z}(\cdot)$ in Eq. (A.17) will not vanish. Furthermore, this term cannot be considered simply as a non-homogeneous term because $\mathbf{e}(\cdot)$ itself depends on $\mathbf{z}(\cdot)$.

Looking at the error dynamics in the form of Eq. (A.17) can mislead the control-scheme designer to bounding this non-homogeneous term. Such procedures do exist in control; for instance, Bellman (1953) considers an equation of the form

$$\dot{\mathbf{z}}(t) = \mathbf{A}\mathbf{z}(t) + \mathbf{f}(\mathbf{z}), \quad \mathbf{z}(0) = \mathbf{c}, \tag{A.18}$$

where $\mathbf{A}$ is a constant matrix and $\mathbf{f}(\mathbf{z})$ satisfies the condition

$$||\mathbf{f}(\mathbf{z})||/||\mathbf{z}|| \to 0 \text{ as } ||\mathbf{z}|| \to 0. \tag{A.19}$$

Bellman shows that if,

- Every solution of $\dot{\mathbf{z}}(t) = \mathbf{A}\mathbf{z}(t)$ approaches zero as $t \to \infty$, and
- $\mathbf{f}(\mathbf{z})$ is continuous in some region about $\mathbf{z} = \mathbf{0}$,

then $\mathbf{z} = \mathbf{0}$ is a stable solution of (A.18), and every solution of (A.18) for which $||\mathbf{z}(0)||$ is sufficiently small approaches zero as $t \to \infty$. However, invoking Bellman's theorem does not generally apply in our case because condition (A.19) may not be satisfied.

A.2.3. Floquet Analysis As all the components of Eq. (A.17) are periodic, the most appropriate tool for analysing its stability is Floquet-Lyapunov Theory. Combining Eqs. (A.17) and (A.15), we obtain a linear, time-periodic system of equations for $\mathbf{e}(\cdot)$ and $\mathbf{z}(\cdot)$:

$$\begin{bmatrix} \dot{\mathbf{z}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{F} - \mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{B}^{\dagger}(t)\mathbf{L}(t)\tilde{\mathbf{B}}\bar{\mathbf{K}} & \mathbf{0}_{n\times n} \\ \left(\tilde{\mathbf{B}}\bar{\mathbf{K}} - \mathbf{L}^{-1}(t)\mathbf{B}(t)\mathbf{B}^{\dagger}(t)\mathbf{L}(t)\tilde{\mathbf{B}}\bar{\mathbf{K}}\right) & \mathbf{F} - \tilde{\mathbf{B}}\bar{\mathbf{K}} \end{bmatrix} \begin{bmatrix} \mathbf{z}(t) \\ \mathbf{e}(t) \end{bmatrix}, \tag{A.20}$$

where $\mathbf{0}_{n\times n}$ is the $n \times n$ zero matrix. Therefore, an evaluation of the system stability can be obtained from a straightforward application of classical results.

First, if $\mathbf{B}(\cdot)$ is invertible, then $\mathbf{B}(\cdot)\mathbf{B}^{\dagger}(\cdot) = \mathbf{B}(\cdot)\mathbf{B}^{-1}(\cdot) = \mathbf{I}$, and the system reduces to a linear time-invariant matrix, namely,

$$\begin{bmatrix} \dot{\mathbf{z}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{F} - \tilde{\mathbf{B}}\bar{\mathbf{K}} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{F} - \tilde{\mathbf{B}}\bar{\mathbf{K}} \end{bmatrix} \begin{bmatrix} \mathbf{z}(t) \\ \mathbf{e}(t) \end{bmatrix}, \tag{A.21}$$

whose eigenvalues correspond to the eigenvalues of $(\mathbf{F} - \tilde{\mathbf{B}}\bar{\mathbf{K}})$. In this case, both $\mathbf{z}(\cdot)$ and $\mathbf{e}(\cdot)$ would indeed be stabilized through an appropriate choice of a stabilizing feedback gain matrix $\bar{\mathbf{K}}$, i.e., the algorithm from Sinha and Joseph (1994) would be successful.

However, in the general case, Eq. (A.11) is not satisfied exactly, and system (A.20) is not time invariant; it is time periodic. As is well known from Floquet-Lyapunov Theory, the only way to draw a definitive conclusion on the stability of system (A.20) is to examine all of its Floquet exponents (Verhulst 1990). The block lower-triangular nature of the matrix of (A.20) implies that the Floquet exponents associated with $\mathbf{e}(t)$ correspond to the eigenvalues of $(\mathbf{F} - \tilde{\mathbf{B}}\bar{\mathbf{K}})$; however, nothing can be said a priori of the Floquet exponents associated with $\mathbf{z}(\cdot)$. Therefore, any claim based solely on the eigenvalues of $(\mathbf{F} - \tilde{\mathbf{B}}\bar{\mathbf{K}})$ assuring the convergence of both $\mathbf{e}(\cdot)$ and $\mathbf{z}(\cdot)$ is not justified. Unfortunately, although the algorithm in Sinha and Joseph (1994) is simple, it is not foolproof. We present below a simple counter-example, built by using one of the systems included in the original paper by Sinha and Joseph, and illustrating that $\mathbf{e}(\cdot)$ and $\mathbf{z}(\cdot)$ have different Floquet exponents with different regions of stability.

A.3. Counter-Example

A.3.1. Feedback Synthesis Let

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t)u(t), \tag{A.22a}$$

where

$$\mathbf{A}(t) = 2\pi \begin{bmatrix} -1 + \alpha\cos^2(2\pi t) & 1 - \alpha\sin(2\pi t)\cos(2\pi t) \\ -1 - \alpha\sin(2\pi t)\cos(2\pi t) & -1 + \alpha\sin^2(2\pi t) \end{bmatrix}, \tag{A.22b}$$

$$\mathbf{b}(t) = \begin{bmatrix} 0 \\ \sin(2\pi t) \end{bmatrix}, \tag{A.22c}$$

and α is a parameter. In all simulations, we will set $\alpha = 1.2$, as in Sinha and Joseph (1994).

It is known (Bittanti et al. 1978; Brunovský 1969) that a periodic system of order n is controllable if and only if it is controllable on $[0, nT_A]$. A straightforward check on the rank of the controllability Gramian,

$$\mathbf{W}_C(2T_A, 0) = \int_0^{2T_A} \mathbf{\Phi}(0,\tau)\mathbf{b}(\tau)\mathbf{b}^T(\tau)\mathbf{\Phi}^T(0,\tau)d\tau \tag{A.23}$$

shows that system (A.22a) is controllable.

The Floquet-Lyapunov factors $\mathbf{L}(\cdot)$ and $\mathbf{F}$ of the state-transition matrix of (A.22a) are

$$\mathbf{L}(t) = \begin{bmatrix} \cos(2\pi t) & \sin(2\pi t) \\ -\sin(2\pi t) & \cos(2\pi t) \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} 2\pi(\alpha-1) & 0 \\ 0 & -2\pi \end{bmatrix}. \tag{A.24}$$

It is clear that for $\alpha = 1.2$, one of the Floquet exponents lies in the right-hand plane. We know from Floquet-Lyapunov theory that this is a necessary and sufficient condition for instability of system (A.22a).

Using the transformation $\mathbf{x}(\cdot) = \mathbf{L}(\cdot)\mathbf{z}(\cdot)$, system (A.22a) becomes

$$\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t) + \mathbf{L}^{-1}(t)\mathbf{b}(t)u(t). \tag{A.25}$$

Following Sinha and Joseph (1994), we consider an auxiliary system of the form:

$$\dot{\tilde{\mathbf{z}}}(t) = \mathbf{F}\mathbf{z}(t) + \tilde{\mathbf{b}}(t)v(t). \tag{A.26}$$

In this counter-example, we take

$$\tilde{\mathbf{b}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It can be readily be verified that the pair $(\mathbf{F}, \tilde{\mathbf{b}})$ is controllable, therefore satisfying the criterion for choosing the auxiliary system.

We now seek a feedback of the form $v(\cdot) = -\bar{\mathbf{k}}^T\tilde{\mathbf{z}}(\cdot)$. Applying this feedback, the auxiliary system (A.7) leads to:

$$\dot{\tilde{\mathbf{z}}}(t) = \left(\mathbf{F} - \tilde{\mathbf{b}}\bar{\mathbf{k}}^T\right)\tilde{\mathbf{z}}(t). \tag{A.27a}$$

On the other hand, the corresponding feedback law for the transformed system (A.25), based on least-squares equivalence and given in Eq. (A.12), yields:

$$\dot{\mathbf{z}}(t) = \left[\mathbf{F} - \mathbf{L}^{-1}(t)\mathbf{b}(t)\mathbf{b}^{\dagger}(t)\mathbf{L}(t)\tilde{\mathbf{b}}\bar{\mathbf{k}}^T\mathbf{L}^{-1}(t)\right]\mathbf{z}(t). \tag{A.27b}$$

The next step is to choose a suitable constant feedback gain $\bar{\mathbf{k}}$ for the closed-loop auxiliary system $(\mathbf{F} - \tilde{\mathbf{b}}\bar{\mathbf{k}}^T)$, namely,

$$\mathbf{F} - \tilde{\mathbf{b}}\bar{\mathbf{k}}^T = \begin{bmatrix} 2\pi(\alpha-1) - \bar{k}_1 & -\bar{k}_2 \\ -\bar{k}_1 & -2\pi - \bar{k}_2 \end{bmatrix}. \tag{A.28}$$

Choosing $\bar{k}_2 = 0$, the eigenvalues $\{\xi_i\}_{i=1}^2$ of the above matrix are readily found to be

$$\xi_1 = 2\pi(\alpha-1) - \bar{k}_1, \quad \xi_2 = -2\pi. \tag{A.29}$$

Therefore, any value for $\bar{k}_1$, such that

$$\bar{k}_1 > 2\pi(\alpha - 1) \tag{A.30}$$

guarantees the stability of $(\mathbf{F} - \tilde{\mathbf{b}}\bar{\mathbf{k}}^T)$, i.e., of the closed-loop auxiliary system described by Eq. (A.27a). The claim that this stability is extended to the transformed system (A.27b) is shown to be false by examining the closed-form solution of system (A.27b) for three values of $\bar{k}_1$.

Simplification of Eq. (A.27b) yields:

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} 2\pi(\alpha-1) + \bar{k}_1\sin(2\pi t)(\cos(2\pi t) - \sin(2\pi t)) & 0 \\ \bar{k}_1\cos(2\pi t)(\sin(2\pi\sin(2\pi t) - \cos(2\pi t))) & -2\pi \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}. \tag{A.31}$$

The first equation is readily integrated to yield

$$z_1(t) = z_1(0)\mathrm{e}^{\int_0^t 2\pi(\alpha-1)+\bar{k}_1\sin(2\pi\tau)(\cos(2\pi\tau)-\sin(2\pi\tau))d\tau}, \tag{A.32}$$

which can be written as

$$z_1(t) = Q_1(t)\mathrm{e}^{\lambda_1 t}, \tag{A.33a}$$

with λ_1 constant and equal to

$$\lambda_1 = 2\pi(\alpha - 1) - \bar{k}_1/2, \tag{A.33b}$$

while $Q_1(\cdot)$ is the periodic function given by

$$Q_1(t) = z_1(0)\mathrm{e}^{\bar{k}_1 \sin(2\pi t)(\sin(2\pi t)+\cos(2\pi t))/(4\pi)}. \tag{A.33c}$$

With $z_1(\cdot)$ known, $z_2(\cdot)$ satisfies:

$$\dot{z}_2(t) = -2\pi z_2(t) + z_1(t). \tag{A.34}$$

Solving the homogeneous equation leads immediately to

$$z_{2h}(t) = z_2(0)\mathrm{e}^{\lambda_2 t}, \tag{A.35a}$$

where

$$\lambda_2 = -2\pi. \tag{A.35b}$$

After finding a particular solution, the general solution is

$$z_2(t) = Q_2(t)\mathrm{e}^{-2\pi t}, \tag{A.35c}$$

where

$$Q_2(t) = z_2(0) + z_1(0)\int_0^t \mathrm{e}^{(2\pi\alpha - \bar{k}_1/2)\tau}\mathrm{e}^{\left[\bar{k}_1 \sin(2\pi\tau)(\sin(2\pi\tau)+\cos(2\pi\tau))/(4\pi)\right]}d\tau. \tag{A.35d}$$

A.3.2. Stability Analysis It is easy to recognize λ_1 and λ_2 as the Floquet exponents of system (A.27b). Note that the exponent -2π has not changed, as could have been expected based on the form of Eq. (A.31). On the other hand, the Floquet exponent λ_1 associated with $z_1(t)$ and given in Eq. (A.33b) gives us the stability condition:

$$\bar{k}_1 > 4\pi(\alpha - 1). \tag{A.36}$$

$\bar{k}_1$	Auxiliary system (A.27a)		Transformed system (A.27b)	
	Poles: ξ_1, ξ_2		Floquet Exponents: λ_1, λ_2	
$3\pi(\alpha-1)$	$-\pi(\alpha-1)$	-2π	$\frac{\pi}{2}(\alpha-1)$	-2π
$4\pi(\alpha-1)$	$-2\pi(\alpha-1)$	-2π	0	-2π
$5\pi(\alpha-1)$	$-3\pi(\alpha-1)$	-2π	$-\frac{\pi}{2}(\alpha-1)$	-2π

Table A.1: Floquet exponents of (A.10) depending on placement of unstable pole.

This condition is more restrictive than the stability condition found for the auxiliary system set out in Eq. (A.30). Indeed, choosing to place the first Floquet exponent of the auxiliary system between 0 and $-4\pi(\alpha-1)$ corresponds to

$$2\pi(\alpha-1) < \bar{k}_1 < 4\pi(\alpha-1). \tag{A.37}$$

This will result in a scenario where although the auxiliary system will be stable, the transformed system and, hence, the original system will be unstable.

For instance, consider $\bar{k}_1 = 3\pi(\alpha-1),\ 4\pi(\alpha-1),$ and $5\pi(\alpha-1)$. These three values satisfy condition (A.30) for the stability of the closed-loop auxiliary system (A.27a). Nevertheless, the closed-loop original system (A.14) and its transform (A.27b) are, respectively, unstable, oscillating, and stable. Indeed, the numerical values of the Floquet exponents, $\{\lambda_i\}_{i=1}^{2}$ of system (A.27b) for each chosen $\bar{k}_1$ are provided in Table A.1, showing that the first Floquet exponent is, respectively, positive, zero, and negative. Additionally, we display the system response obtained from the three different pole placements in Fig. A.1.

A.3.3. Remark One might be tempted to correct this scheme by adopting a procedure where the choice for the constant feedback gain depends on the exponents of the closed-loop transformed system, rather than on the auxiliary system. We must stress again that the Floquet exponents in our counter-example were conveniently computed only due to the lower-triangular nature of the system matrix Eq. (A.31),

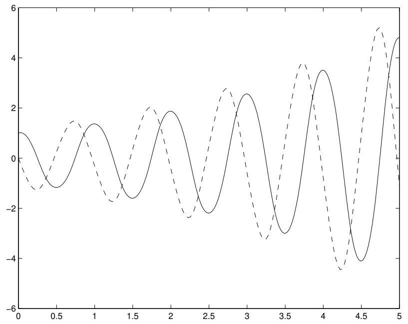
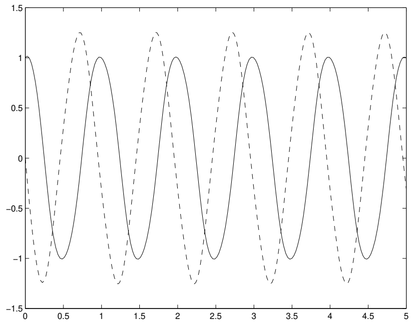
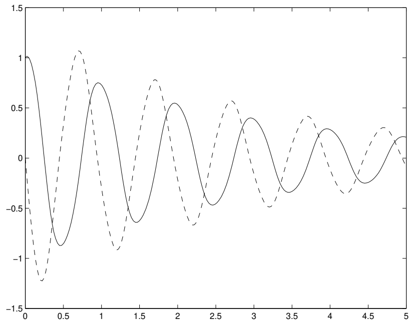
Pole placement of auxiliary system	Corresponding Floquet exponent in original system	Response of original system (x_1 —, x_2 - -)
$\xi_1 = -\pi(\alpha - 1) < 0$ stable	$\lambda_1 = \frac{\pi}{2}(\alpha - 1) > 0$ unstable	
$\xi_1 = -2\pi(\alpha - 1) < 0$ stable	$\lambda_1 = 0$ marginally stable	
$\xi_1 = -3\pi(\alpha - 1) < 0$ stable	$\lambda_1 = -\frac{\pi}{2}(\alpha - 1) < 0$ stable	

Figure A.1: Illustration that stability of the auxiliary system does not guarantee stability of the original system

and the fact that we only needed to move a single exponent. This approach is similar to those in (Calico and Wiesel 1984; Kern 1980, 1986), and has so far proved unsuccessful to provide a general solution to the assignment of Floquet exponents.

A.4. Conclusions

The novelty of the technique for the design of a controller presented by Sinha and Joseph (1994) hinges on the introduction of an 'auxiliary' least-squares-equivalent system to facilitate the pole-placement procedure for periodic systems. The scheme presented is easy to understand and implement, and in cases where the input matrix $\mathbf{B}(\cdot)$ is invertible, it can be effective. However, using the conditions of stability for periodic systems provided by Floquet-Lyapunov theory, we show that stability of this 'auxiliary' system will not necessarily be conveyed to the original system when $\mathbf{B}(\cdot)$ is not invertible. Moreover, attempts to repair this scheme lead back to already existing procedures.

APPENDIX B

Floquet Software Toolbox

This appendix presents the software package in which we implemented the procedures investigated in this thesis. For a given system, the user can study the STM to determine the stability, and afterwards build a closed-loop controller (Fig. B.1).

```
0- Quit
1- Plot variations of A(t)
2- Compute the STM of open-loop system
3- Build a controller
Your choice:
```

Figure B.1: Main Menu

B.1. Computing the STM

The user is offered the choices between different approaches to compute the STM (Fig. B.2).

B.1.1. Zero-Order Hold Algorithm It is possible to study the convergence of the Floquet multipliers vs. dicretisation step using the Zero-Order Hold algorithm (cf. Note 5.2.1), and for a given step size the values of $\mathbf{\Phi}(\cdot, 0)$, $\mathbf{F}$ and $\mathbf{L}(\cdot)$ can be derived (Fig. B.3) for a given Yakubovich matrix $\mathbf{Y}$.

```
   ************************************************
   *      Periodic Systems  Analysis Toolbox       *
   ************************************************

System Currently Analysed: wu01
************************
Wu 1978 IEEE TAC 19(2) -  a=1 omega=2*pi
The closed form solution is available for this system

0 - Back
1 - Closed-Form STM
2 - Zero Order Hold +
3 - Direct Numerical Integration
4 - Spectral Solution
5 - BVP Solution
enter your choice
```

Figure B.2: Computing the STM

B.1.2. Direct Numerical Integration $\mathbf{\Phi}(\cdot,0)$, $\mathbf{L}(\cdot)$ and $\mathbf{F}$ can be computed using any of the ODE solvers available in `MATLAB`$^{\mathrm{TM}}$ following the approach described in § 5.1.1 (cf. Fig. B.4).

B.1.3. Spectral Solution The system matrix $\mathbf{A}(\cdot)$ can be expanded on a given spectral basis using the `COCA`$^{\mathrm{TM}}$ package developed by Bernd Fischer and Jan Modersitzki at the University of Hamburg. Once an expansion of $\mathbf{A}(\cdot)$ is available, $\mathbf{\Phi}(\cdot,0)$, $\mathbf{L}(\cdot)$, and $\mathbf{F}$ can be derived (cf. § 5.1.2). The same expansion of $\mathbf{A}(\cdot)$ can be used to solve the adjoint problem to obtain directly $\mathbf{\Phi}(0,\cdot)$ and $\mathbf{L}^{-1}(\cdot)$.

B.1.4. BVP Solution This is an interface to the `bvp4c` software package to solve the BVP problem formulated in § 5.2.

```
   **************************
   *    Zero Order Hold +   *
   **************************

System Currently Analysed:
*************************
Wu 1978 IEEE TAC 19(2) -  a=1 omega=2*pi
Wu system 02 a=1 omega=2*pi

0 - Back
1 - Convergence of Floquets exponents vs step size
2 - Generate an initial guess for a given step size
enter your choice 2

Guess on 1 or 2 periods? 1

Please give the discretisation step as a % of the period (eg:0.01 for 1%)0.01

The monodromy matrix is
    2.7183    0.0000
   -0.0000    2.7183

its eigenvalue matrix is
    2.7183         0
         0    2.7183

 The Yakubovich matrix will be Y=V*DY*inv(V)
where V=eigenvectors of Phi(T,0)
Give DY
```

Figure B.3: ZOH menu

```
   ***********************************
   *  Direct Numerical Integration    *
   ***********************************

System Currently Analysed: wu01
************************
Wu 1978 IEEE TAC 19(2) -  a=1 omega=2*pi

STM computed with ode15s is available

0 - Back to the MAIN menu
1 - Compute the STM
2 - View a precomputed STM
3 - Compute L & F
4 - Compare STM, L & F to the exact solution
enter your choice 1

0 - Back to the DNI MAIN menu
2 - ODE15s solution
3 - ODE23  solution
4 - ODE23s solution
5 - ODE45  solution
6 - ODE113 solution
Choose the method to compute the STM:
```

Figure B.4: DNI menu

```
   ***************************************************
   *       Controller Synthesis       Toolbox        *
   ***************************************************

Current System: wu01
*************************
Wu 1978 IEEE TAC 19(2) -  a=1 omega=2*pi
The closed form solution is available for this system
Computation of the STM: No choice yet

0 - EXIT
1 - Choose STM method
Enter your choice 1

0 - Closed-form solution
1 - Chebyshev
 The STM of the adjoint is available for this method
2 - BVP4C
 The STM of the adjoint is available for this method
Enter your choice: 2

Loading the results
Use the adjoint for the computation of the inverse?
1-Yes 0-No
```

Figure B.5: Choosing the STM method

B.2. Controller Synthesis

After having chosen how the STM will be evaluated (cf. Fig. B.5), the user indicates whether the system matrix is constant in order to take advantage of the results described in § 6.7.3.1.

Fig. B.6 shows the steps to obtain the feedback matrix. The invertibility of the Gramian is evaluated, and based upon that a period for the closed-loop system can be assigned. The relevant expansions are performed and will remain valid as long as the period of the closed-loop system is not modified. After having assigned the Floquet factor $\mathbf{F}_K$ and a Yakubovich matrix $\mathbf{Y}_K$, Eq. (6.24a) is solved for $\mathbf{L}_K(\cdot)$. With $\mathbf{L}_K(\cdot)$ known, $\mathbf{K}_c$ and, then, $\mathbf{K}(\cdot)$ are derived. This solution can be used as the initial guess

```
   ************************************************
   *       Derivation of Gramian-Based  K(t)        *
   ************************************************

System Currently Analysed: wu01
************************
Wu 1978 IEEE TAC 19(2) -  a=1 omega=2*pi

Computation of the STM: BVP4C for the STM and its adjoint
Fc=
    -1     0
     0    -1

Y=
     1     0
     0     1

The Gramians are available
Primary period of closed-loop system T_Ac = 1
Primary period of closed-loop Lc = 1

0 - Back
1 - Compute monodromy and Gramian
2 - Choose the PRIMARY period of the closed-loop system
3 - Compute the expansions with COCA
4 - Assign Fc & Y
5 - Solve for Lc & Kc
**************************************
10 - Initial guess for Lc/Kc for bvp4c
11 - Solve for Lc/Kc with bvp4c
```

Figure B.6: Deriving $\mathbf{K}(\cdot)$

of the BVP described in § 6.7.2. Once computed, $\mathbf{K}(\cdot)$ is saved and can be used for simulating the closed-loop system.

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